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Theoretical Description of Effective Heat Transfer between Two Viscously Coupled Beads

A. Bérut[1], A. Imparato[2], A. Petrosyan[1], S. Ciliberto [1]
1 − *Université de Lyon, CNRS, Laboratoire de Physique, École Normale Supérieure de Lyon (UMR5672), 46 Allée d'Italie 69364 Lyon Cedex 07, France*

2 − *Department of Physics and Astronomy, University of Aarhus Ny Munkegade, Building 1520, DK–8000 Aarhus C, Denmark*
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We analytically study the role of non-conservative forces, namely viscous couplings, on the statistical properties of the energy flux between two Brownian particles kept at different temperatures. From the dynamical model describing the system, we identify an energy flow that satisfies a Fluctuation Theorem both in the stationary and in the transient state. In particular, for the specific case of a linear non-conservative interaction, we derive an exact Fluctuation Theorem that holds for any measurement time in the transient regime, and which involves the energy flux alone. Moreover, in this regime the system presents an interesting asymmetry between the hot and the cold particle. The theoretical predictions are in good agreement with the experimental results already presented in our previous article [1], where we investigated the thermodynamic properties of two Brownian particles, trapped with optical tweezers, interacting through a dissipative hydrodynamic coupling.

I. INTRODUCTION

The heat transfer between two reservoirs kept at different temperatures, is certainly the simplest and probably the most fundamental out-equilibrium phenomenon that one can study. However, in small systems where the effects of thermal fluctuations on the mean heat flux cannot be neglected, this situation has been analyzed mainly in theoretical models [2–13]. Only a few very recent experiments have studied heat flux fluctuations [1, 14–16], because of the intrinsic difficulties of dealing with large temperature differences at small scales.

In particular, in our recent paper [1] we have presented for the first time some measurements on the energy flux in a system with non conservative interactions. We have reported the statistical properties of the fluctuating energy transfer in a system composed by two trapped Brownian beads kept at different effective temperatures, and interacting only through hydrodynamic coupling.

Motivated by our previous experimental findings, in this article we derive the Fluctuation Theorems (FTs) characterizing the "effective heat" transfer between two Brownian particles interacting with non-conservative linear forces, both in the transient and stationary regime. This is an important problem in the study of the thermodynamics of small motors and Brownian ratchets, because fluctuating energy fluxes have been mostly investigated in systems with conservative couplings [5–7, 11–13], and their statistical properties for dissipative interactions have been less discussed. For example they were addressed in ref. [8], for a very specific dissipative coupling which is, however, difficult to implement in a real experiments. On the contrary, the set-up considered in [1] is rather general, as small devices immersed in a fluid often interact through hydrodynamic coupling (see for example [17–20]).

In the stationary regime, we find that energy fluxes satisfy a stationary exchange fluctuation theorem (xFT):

$$\ln\left(\frac{P(Q_\tau)}{P(-Q_\tau)}\right) \underset{\tau\to\infty}{=} \left(\frac{1}{k_\mathrm{B}T_1} - \frac{1}{k_\mathrm{B}T_2}\right) Q_\tau \qquad (1)$$

where $P(Q_\tau)$ is the probability that an amount of (effective) heat Q_τ is exchanged during a time τ between the two systems at (effective) temperatures T_1 and T_2. In the transient regime we find an asymmetry between the energy fluxes after the sudden application of the effective temperature gradient: while the hot-cold flux satisfies the transient xFT for any integration time, the energy flux between the cold and the hot particle obeys a FT only asymptotically for long times. In particular, for the specific case of the hydrodynamic linear coupling, we derive an exact xFT for the exchanged heat that holds at any time in the transient regime, and which generalizes to the non-conservative case the analogous FT discussed in [21] for conservative interactions. We also show that some results for the conservative coupling case are recovered in the specific configuration where the two traps have the same stiffness. All these theoretical predictions show a good agreement with the experimental results reported in our previous article [1].

The article is organized as follow: we first present the model for the beads dynamics in section (II) and the definitions of the (effective) heat fluxes (III), and we show that the viscous coupling may produce non trivial effects in the energy balance (III A). We then compute, in the framework of stochastic thermodynamics, the statistical properties of the heat flux both in the stationary (IV A) and transient state (IV B), and we show that the transient heat flux has non symmetric statistical properties.

II. THE HYDRODYNAMIC MODEL

Before introducing the stochastic equations describing the system dynamics, we briefly depict the experimental setup studied in [1], to show the connections between the present theoretical analysis and the experimental system. To study a heat flux in presence of a dissipative coupling we used a system (schematically represented in figure 1) composed of two spherical micro-particles, trapped in bi-distilled water by optical tweezers, and interacting only through an hydrodynamic coupling. One of the two particles is submitted to an external white noise, obtained by randomly displacing the position of its trap, which creates an "effective temperature" for the particle. The technical details about the experimental set-up can be found in [1, 22].

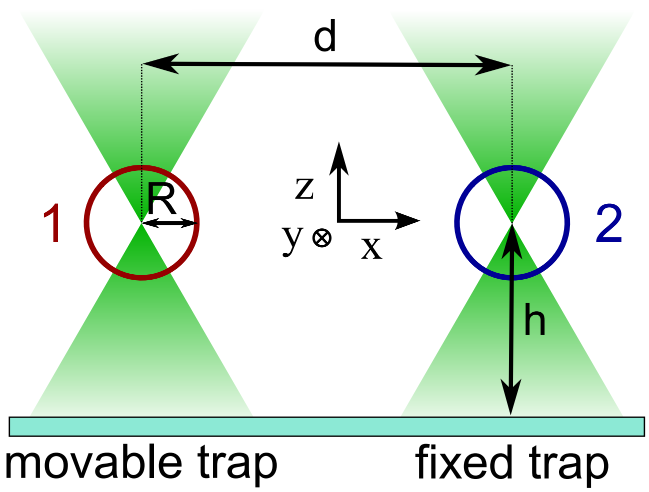


Figure 1: Schematic representation of the two particles trapped by optical tweezers, separated by a distance d along the x axis.

Such a system can be described by a classical hydrodynamic coupling model in low Reynolds-number flow. Following [23–25] the thermally excited motion of two identical particles of radius R trapped at positions separated by a distance d is described by two coupled Langevin equations:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \mathcal{H} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \tag{2}$$

where $\mathcal{H}$ is the hydrodynamic coupling tensor, x_i is the position of the particle i relative to its trapping position and F_i is the force acting on the particle i.

In the case where the displacements are small compared to the mean distance between the particles (i.e. $x \ll d$), the hydrodynamic coupling tensor reads:

$$\mathcal{H} = \begin{pmatrix} 1/\gamma & \epsilon/\gamma \\ \epsilon/\gamma & 1/\gamma \end{pmatrix} \tag{3}$$

where γ is the Stokes friction coefficient ($\gamma = 6\pi R\eta$ where η is the viscosity of water) and ϵ is the coupling coefficient ($\epsilon = \frac{3R}{2d}$ if one takes the first order of the Oseen tensor [26], $\epsilon = \frac{3R}{2d} - \left(\frac{R}{d}\right)^3$ if one takes the Rotne-Prager diffusion tensor [27]).
It is important to notice that in the following theoretical treatment ϵ is considered as a constant because we consider only regimes where the fluctuations of particles positions are smaller than the mean distance between them.

At equilibrium the forces acting on the particles are:

$$F_i = -k_i\, x_i + f_i \tag{4}$$

where k_i is the stiffness of the trap i and f_i are the Brownian random forces which verify:

$$\begin{aligned} \langle f_i(t)\rangle &= 0 \\ \langle f_i(t) f_j(t')\rangle &= 2k_\mathrm{B}T\,(\mathcal{H}^{-1})_{ij}\,\delta(t-t') \end{aligned} \tag{5}$$

where k_B is the Boltzmann constant and T the temperature of the surrounding fluid.

To take account of a gradient of temperature between the two particles, we simply add an external random force f^* on the first one. We make the assumption that this force is completely uncorrelated with the equilibrium Brownian random forces and that it is characterized by an additional temperature ΔT (the particle 1 is then at a temperature $T^* = T + \Delta T$).

$$\begin{aligned} \langle f^*(t)\rangle = 0 \;\;\text{and}\;\; \langle f^*(t) f_i(t')\rangle = 0 \\ \langle f^*(t) f^*(t')\rangle = 2k_\mathrm{B}\Delta T\gamma\delta(t-t') \end{aligned} \tag{6}$$

It follows that the system of equations is:

$$\gamma\dot{x}_1 = -k_1x_1 + \epsilon(-k_2x_2 + f_2) + f_1 + f^*, \tag{7}$$
$$\gamma\dot{x}_2 = -k_2x_2 + \epsilon(-k_1x_1 + f_1 + f^*) + f_2. \tag{8}$$

In Eqs.7 and 8, the equilibrium Brownian noises, induced by the thermal bath verify:

$$\begin{aligned} \langle f_i(t)\rangle &= 0 \\ \langle f_i(t) f_i(t')\rangle &= \frac{2k_\mathrm{B}T\gamma}{1-\epsilon^2}\,\delta(t-t') \\ \langle f_1(t) f_2(t')\rangle &= -\epsilon\frac{2k_\mathrm{B}T\gamma}{1-\epsilon^2}\,\delta(t-t') \end{aligned}$$

while the additional noise verifies:

$$\begin{aligned} \langle f^*(t)\rangle &= 0, \\ \langle f^*(t) f_i(t')\rangle &= 0, \\ \langle f^*(t) f^*(t')\rangle &= 2k_\mathrm{B}\Delta T\gamma\,\delta(t-t'), \end{aligned}$$

It is important to notice that these relations are fully verified in the experimental set-up where f^* is due to an external displacement of the first particle's trap. However, they might be modified if the additional noise on the first particle were correlated with the equilibrium Brownian random forces.

Let us introduce the total normalised noises ξ_i acting on particle i, which read

$$\xi_1 = \frac{1}{\gamma}(\epsilon f_2 + f_1 + f^*), \tag{9}$$

$$\xi_2 = \frac{1}{\gamma}(\epsilon f_1 + \epsilon f^* + f_2). \tag{10}$$

We can thus rewrite eqs. (7)–(8) as

$$\begin{cases} \dot{x}_1 = g_1(x_1, x_2) + \xi_1 \\ \dot{x}_2 = g_2(x_1, x_2) + \xi_2, \end{cases} \tag{11}$$

with the normalised force reading

$$g_i(x_i, x_j) = -\frac{1}{\gamma} k_i x_i - \frac{\epsilon}{\gamma} k_j x_j, \tag{12}$$

and the normalised noises exhibiting the fluctuation-dissipation relations

$$\langle \xi_i(t) \xi_j(t') \rangle = 2\theta_{ij} \delta(t - t'), \tag{13}$$

where we have introduced the diffusion matrix θ_{ij}, whose elements read

$$\theta_{11} = (T + \Delta T)/\gamma, \tag{14}$$

$$\theta_{12} = \epsilon(T + \Delta T)/\gamma, \tag{15}$$

$$\theta_{22} = (T + \epsilon^2 \Delta T)/\gamma. \tag{16}$$

III. THE HEAT FLUXES AND THEIR STATISTICAL PROPERTIES

We now introduce the integrated heat Q_i by analogy with the stochastic heat dissipated by a Brownian particle in a thermal bath [28]. Q_i is defined as the work done by the stochastic forces on the particle i in a time interval τ:

$$\begin{aligned} Q_i &= \int_t^{t+\tau} (\gamma \dot{x}_i - \gamma \xi_i) \dot{x}_i \, \mathrm{d}t' \\ &= -\int_t^{t+\tau} (k_i x_i + \epsilon k_j x_j) \dot{x}_i \, \mathrm{d}t', \end{aligned} \tag{17}$$

where the first equality holds in general, while the second one follows upon substitution of eq. (11). Equation (17) can be rewritten as

$$Q_i = Q_{ii} + Q_{ij}, \tag{18}$$

where the heat Q_i is split into two contributions

$$Q_{ii} = -\int_t^{t+\tau} k_i x_i \dot{x}_i \, \mathrm{d}t', \tag{19}$$

$$Q_{ij} = -\int_t^{t+\tau} \epsilon k_j x_j \dot{x}_i \, \mathrm{d}t', \tag{20}$$

with $i = \{1, 2\}$, $j = \{2, 1\}$. While the first quantity is the work done on the particle i by the quadratic trap, the latter quantity represents the work done on the particle i by the non conservative force alone, due to the fluid-mediated particle-particle interaction.

Note that the integrals appearing in eqs. (19)-(20) are stochastic integrals which have to be interpreted according to the Stratonovich integration scheme.

A. The mean values of the heat fluxes

In this section we consider the total heat as introduced in eq. (18), calculate its average value, and show that only the term involving the non–conservative force Q_{ij} contributes to the average value $\langle Q_i \rangle$.

Indeed from eq. (17) we have

$$\begin{aligned} \dot{Q}_i &= -(k_i x_i + \epsilon k_j x_j)\dot{x}_i = -(k_i x_i + \epsilon k_j x_j)(g_i + \xi_i) \\ &= F_{Q_i} + \xi_{Q_i}, \end{aligned} \tag{21}$$

which is a Langevin-like equation for the stochastic variable $Q_i(t)$. The deterministic force acting on $Q_i(t)$ reads

$$F_{Q_i} = -(k_i x_i + \epsilon k_j x_j) g_i = \frac{1}{\gamma}(k_i x_i + \epsilon k_j x_j)^2, \tag{22}$$

while the stochastic force reads

$$\xi_{Q_i} = -(k_i x_i + \epsilon k_j x_j)\xi_i. \tag{23}$$

In appendix A, starting from the Langevin equations (11) and (21), we derive the Fokker-Plank equation for the probability distribution $P(x_1, x_2, t)$ and for the joint probability distribution $P(x_1, x_2, Q_i, t)$, which allow us to calculate the heat rates

$$q_1 = \partial_t \langle Q_1 \rangle = \frac{k_\mathrm{B} \Delta T k_2^2 \epsilon^2 \left(\epsilon^2 - 1\right)}{\gamma(k_1 + k_2)}, \tag{24}$$

$$q_2 = \partial_t \langle Q_2 \rangle = -\frac{k_\mathrm{B} \Delta T k_1 k_2 \epsilon^2 \left(\epsilon^2 - 1\right)}{\gamma(k_1 + k_2)} \tag{25}$$

with $q_1 = q_2 = 0$ if $\Delta T = 0$, and $q_1 + q_2 = 0$ if $k_1 = k_2$.

Using data from the experiments presented in [1], we can evaluate the experimental values of the heat rate q_i (see figure 2). They show a good agreement with the values predicted by eqs. (24) and (25).

We now turn our attention to the stochastic variables Q_{ij}, and consider the Langevin equation expressing their time evolution, which reads

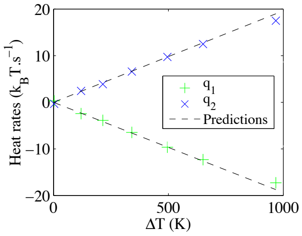
$$\dot{Q}_{ij} = -\epsilon k_j x_j \dot{x}_i = -\epsilon k_j x_j (g_i + \xi_i) = F_{Q_{ij}} + \xi_{Q_{ij}}, \tag{26}$$

where the deterministic force acting on $Q_{ij}(t)$ is

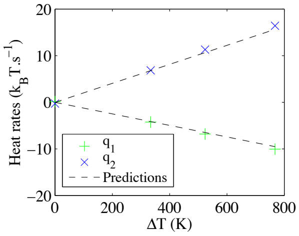
$$F_{Q_{ij}} = -\epsilon k_j x_j g_i = \frac{1}{\gamma}\left[\epsilon k_i x_i k_j x_j + (\epsilon k_j x_j)^2\right], \tag{27}$$

while the stochastic force reads

$$\xi_{Q_{ij}} = -\epsilon k_j x_j \xi_i. \tag{28}$$



(a)



(b)

Figure 2: Experimental heat rates q_1 and q_2. a) In the case $k_1 = k_2$ ($\approx 3.35\,\mathrm{pN/\mu m}$) b) In the case $k_1 \neq k_2$ ($k_1 \approx 4.20\,\mathrm{pN/\mu m}$ and $k_2 \approx 2.55\,\mathrm{pN/\mu m}$). The dashed lines are the theoretical predictions obtained by substituting the corresponding experimental values in equations (24) and (25).

Following the same steps as above (see appendix A for the details) one finds the rates

$$q_{12} = \partial_t \langle Q_{12} \rangle = \frac{k_\mathrm{B} \Delta T k_2^2 \epsilon^2 \left(\epsilon^2 - 1\right)}{\gamma(k_1 + k_2)} \qquad (29)$$

$$q_{21} = \partial_t \langle Q_{21} \rangle = -\frac{k_\mathrm{B} \Delta T k_1 k_2 \epsilon^2 \left(\epsilon^2 - 1\right)}{\gamma(k_1 + k_2)} \qquad (30)$$

which are identical to eqs. (24)-(25). By recalling the definitions of Q_i, Q_{ii} and Q_{ij}, eqs.(18)–(20), we conclude that the heat rates q_1 and q_2 are the work done per time unit by the non-conservative forces alone. This makes sense if one considers the following physical argument: the heat flows because of the term x_2 on the right hand side of the equation for $\dot{x}_1$, and because of the term x_1 on the right hand side of the equation for $\dot{x}_2$ in the Langevin equations (11). Then Q_{ij} can be seen as the work done by the interaction force on each particle. This would be the case even if the coupling forces were conservative.

We also notice that $q_1 = -q_2$ only when $k_1 = k_2$ because in this case the system is perfectly symmetric, and the equations (11) become equivalent to those of a system with a conservative coupling:

$$\begin{cases} \gamma \dot{x}_1 = -\partial_{x_1} U + \xi_1, \\ \gamma \dot{x}_2 = -\partial_{x_2} U + \xi_2, \end{cases} \qquad (31)$$

where $U(x_1, x_2) = k\left(x_1^2 + 2\epsilon x_1 x_2 + x_2^2\right)/2$. In this case, we have the energy conservation $Q_1(t) + Q_2(t) = U(x_1(t), x_2(t)) - U(x_1(0), x_2(0))$ for any time t. On the other hand $q_1 \neq -q_2$ when $k_1 \neq k_2$ because in the general case the forces are not conservative, and thus we cannot invoke the conservation of energy. Thus the fact that the heat dissipated by particle 1 is not the opposite of the heat dissipated by particle 2 is only due to the dissipative nature of the coupling, and it is not induced by any specificity of the random forcing f^*. In the experimental system, if $k_1 = k_2$, the dissipative nature of the coupling cannot be observed and the energy fluxes Q_i due to the "effective temperature difference" behave as conventional heat fluxes in conservative systems.

IV. STEADY STATE AND TRANSIENT FLUCTUATION THEOREMS

In the present section we present the main results concerning the FTs, while the detailed proofs are discussed in appendix B.

Starting from the Langevin equations for the microscopic variables (11), one can calculate the probability of observing a single trajectory in the phase space. Given an initial $\mathbf{x} = (x_1, x_2)$ and a final state $\mathbf{x}' = (x_1', x_2')$, the transition probability $P_F(\mathbf{x} \to \mathbf{x}'|\mathbf{x}, t)$ of a trajectory connecting the points $\mathbf{x}$ and $\mathbf{x}'$ in a finite time interval $[t, t, +\tau]$ and the transition probability $P_B(\mathbf{x}' \to \mathbf{x}|\mathbf{x}', t+\tau)$ of the corresponding backward trajectory obey the detailed FT

$$\frac{P_F(\mathbf{x} \to \mathbf{x}'|\mathbf{x}, t)}{P_B(\mathbf{x}' \to \mathbf{x}|\mathbf{x}', t+\tau)} = \mathrm{e}^{\frac{1}{k_\mathrm{B}}\left(\frac{Q_{11}}{T+\Delta T} + \frac{Q_{22}}{T} - \frac{\Delta T}{T(T+\Delta T)} Q_{12}\right)} \qquad (32)$$

where the quantities Q_{ij} are given by eqs. (19)-(20) and depend on the specific trajectory. It is worth to note that this equation holds for any τ.

A. The steady state

Starting from eq. (32) we can prove the steady state FTs for the quantities Q_i and Q_{ij}, see appendix C for details. Specifically we prove that in the steady state, Q_{12} and Q_{21} verify an FT in the long time limit :

$$\ln\left(\frac{P(Q_{12})}{P(-Q_{12})}\right) \underset{\tau \to \infty}{\simeq} \left(\frac{1}{k_\mathrm{B} T_1} - \frac{1}{k_\mathrm{B} T_2}\right) Q_{12} \qquad (33)$$

and

$$\ln\left(\frac{P(Q_{21})}{P(-Q_{21})}\right) \underset{\tau \to \infty}{\simeq} \frac{k_2}{k_1}\left(\frac{1}{k_\mathrm{B} T_2} - \frac{1}{k_\mathrm{B} T_1}\right) Q_{21}. \qquad (34)$$

Here we have rewritten eqs. C.2, C.4 of appendix C to have the explicit dependence on T_1 and T_2.

These xFT are analogous to the one presented in [3] for the heat exchanged between two heat bath put in contact during a time τ. It is interesting to notice that because of the dissipative coupling the FT for Q_{21} has a prefactor k_2/k_1 which disappears only in the symmetric case when $k_2 = k_1$.

B. The transient regime

Following the protocol discussed in [13, 21], we now assume that we prepare our system such that at $t \to -\infty$ the temperature difference is vanishing $\Delta T = 0$, and then at $t = 0$ we suddenly turn on the temperature difference ΔT, with $T_1 = T + \Delta T$, and start measuring the heat currents for $t \geq 0$. We assume to prepare the system in the same way along the backward trajectories: at $t' = \tau - t \to -\infty$ we take $\Delta T = 0$, and then at

$t' = 0$ we "turn on" the temperature difference ΔT and start measuring the heat currents along the backward trajectories. Since the particle interaction do not change in time, the PDF for the initial position of the forward trajectory is equal to the PDF for the initial position of the backward trajectory. Starting from eq. (32), in appendix D we prove that, for the specific protocol where the system is prepared with $\Delta T = 0$ at the beginning of both the forward and the backward trajectories, the following detailed FT holds for any τ

$$\frac{P(x_1, x_2, t=0) P_F(\mathbf{x} \to \mathbf{x}'|\mathbf{x}, 0)}{P(x_1', x_2', t'=0) P_B(\mathbf{x}' \to \mathbf{x}|\mathbf{x}', \tau)} = \mathrm{e}^{-\frac{\Delta T}{k_\mathrm{B} T(T+\Delta T)} Q_1} \quad (35)$$

This FT generalizes to the non-conservative case the analogous FT discussed in [21] for conservative interactions. As discussed in [21] such a FT involves only the measurable heat currents $\dot{Q}_i$ and no boundary term depending on the final and the initial state $\mathbf{x}'$ and $\mathbf{x}$ appears in its expressions, at variance with the FT discussed in [29]. However, differently from [21], in the present study we consider non-conservative forces, and thus the condition of local detailed balance does not apply here. Had we chosen to change suddenly the temperature T_2 at $t = 0$, with $T_2 = T + \Delta T$, the heat flux Q_2 would appear on the rhs of eq. (35) instead of Q_1.

As a consequence the PDF of Q_1 obeys the integrated FT

$$\ln\left(\frac{P(Q_1, \tau)}{P(-Q_1, \tau))}\right) = \left(\frac{1}{k_\mathrm{B} T_1} - \frac{1}{k_\mathrm{B} T_2}\right) Q_1, \quad (36)$$

which holds for any $\tau > 0$, while the heat Q_2 verifies an FT

$$\ln\left(\frac{P(Q_2, \tau)}{P(-Q_2, \tau))}\right) \underset{\tau \to \infty}{\simeq} \left(\frac{1}{k_\mathrm{B} T_2} - \frac{1}{k_\mathrm{B} T_1}\right) Q_2. \quad (37)$$

which holds only in the limit of large τ, see appendix D for the details.

As already mentioned in introduction, the theoretical predictions for the statistical properties of the heat have already been compared with the experimental results both in the stationary and transient regimes [1].

V. DISCUSSION AND CONCLUSIONS

This article presents several theoretical results on the energy exchanged between two Brownian particles coupled by viscous interactions, and kept at different temperatures.

Starting from the coupled Langevin equations, we have defined a heat flux as the energy exchanged between a particle at equilibrium with the fluid, and a particle submitted to an uncorrelated additional source of noise. For this energy flow, we have theoretically demonstrated the following behaviors:

a) The mean fluxes are linear functions of the effective temperature difference.

b) The exchanged fluxes Q_{ij} satisfy the exchange fluctuation theorem (xFT) in the stationary state.

c) The total flux for the hot particle Q_1 satisfies the transient xFT for any time after the sudden application of the temperature gradient, whereas the total flux for the cold particle Q_2 satisfies it only asymptotically (i.e. for long integration times).

In particular, the last property had been previously predicted only for systems with a conservative coupling [13] and we have proved it also in the case of dissipative linear coupling. For all the theoretical predictions, the experimental results show a very good agreement. Note also that the integrated FTs in points b) and c) follow from more general detailed FTs that hold at the level of single trajectories. Our results show that there are strong analogies between the statistical properties of a dissipatively coupled system with those of a conservatively coupled one (see also ref. [30] for a detailed comparison), and are particularly relevant in all of the cases where an external random forcing is applied to a system which is coupled to another one.

VI. ACKNOWLEDGMENTS

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Appendices

In these appendices we take $k_\mathrm{B} = 1$ to simplify the notation.

A. THE FOKKER-PLANCK EQUATIONS FOR THE PROBABILITY DISTRIBUTION OF THE STOCHASTIC VARIABLES

Starting from the Langevin equation (11) for the the stochastic variables x_i, we introduce the Fokker-Planck (FP) equation for the probability distribution function (PDF) $P(x_1, x_2, t)$, which reads [31]

$$\begin{aligned}\partial_t P(x_1,x_2,t) &= \mathcal{L}_0 P(x_1,x_2,t) \\ &= -\partial_{x_i}\left[g_i(x_2,x_2)P(x_1,x_2,t)\right] + \partial_{x_i}\partial_{x_j}\theta_{ij}P(x_1,x_2,t).\end{aligned} \tag{A.1}$$

The stationary steady state PDF, obeying $\partial_t P(x_1, x_2, t) = 0$, reads

$$P_{ss}(x_1,x_2,\Delta T) = \frac{\sqrt{ac-b^2}}{\pi}\exp\left[-\left(ax_1^2 + 2bx_1x_2 + cx_2^2\right)\right], \tag{A.2}$$

where the dependence on the temperature gradient ΔT has been made explicit, as it will be useful in the following, and with

$$\begin{aligned} a &= \frac{k_1(k_1+k_2)\left[(k_1+k_2)T + \epsilon^2 k_2\Delta T\right]}{\Theta}, \\ b &= -\frac{\epsilon k_1 k_2 (k_1+k_2)\Delta T}{\Theta}, \\ c &= \frac{k_2(k_1+k_2)\left[(k_1+k_2)(T+\Delta T) - \epsilon^2 k_2 \Delta T\right]}{\Theta}, \\ \Theta &= 2\left[(T^2 + T\Delta T)(k_1+k_2)^2 - \epsilon^2(\epsilon^2-1)k_2^2\Delta T^2)\right]. \end{aligned}$$

From eq. (A.2) we also notice that the steady state PDF for a vanishing temperature gradient reads

$$P_{ss}(x_1,x_2,\Delta T=0) \propto \exp\left[-\left(\frac{k_1}{2T}x_1^2 + \frac{k_2}{2T}x_2^2\right)\right]. \tag{A.3}$$

as expected because the coupling is dissipative.

We now consider the Langevin equation for Q_i as given by eq. (21) in the main text. The stochastic forces acting on the variables $x_i(t)$ (eq. (11)) and on $Q_i(t)$ (eq. (21)) obey the fluctuation-dissipation relations

$$\begin{aligned}\langle \xi_i(t)\xi_{Q_i}(t')\rangle &= -(k_ix_i + \epsilon k_jx_j)2\theta_{ii}\delta(t-t'), \\ \langle \xi_j(t)\xi_{Q_i}(t')\rangle &= -(k_ix_i + \epsilon k_jx_j)2\theta_{ij}\delta(t-t'), \\ \langle \xi_{Q_i}(t)\xi_{Q_i}(t')\rangle &= (k_ix_i + \epsilon k_jx_j)^2 2\theta_{ii}\delta(t-t').\end{aligned}$$

The Fokker-Planck (FP) equation for the joint probability distribution $\mathcal{P}(x_1, x_2, Q_i, t)$ reads [12, 13, 32, 33]

$$\begin{aligned}\partial_t\mathcal{P} &= \mathcal{L}_0\mathcal{P} - \partial_{Q_i}\left(F_{Q_i}\mathcal{P}\right) - \theta_{ii}\left[(k_ix_i+\epsilon k_jx_j)\partial_{x_i}\partial_{Q_i}\mathcal{P} + \partial_{x_i}\partial_{Q_i}(k_ix_i+\epsilon k_jx_j)\mathcal{P}\right] \\ &\quad -\theta_{ij}\left[(k_ix_i+\epsilon k_jx_j)\partial_{x_j}\partial_{Q_i}\mathcal{P} + \partial_{x_j}\partial_{Q_i}(k_ix_i+\epsilon k_jx_j)\mathcal{P}\right] + \gamma F_{Q_i}\theta_{ii}\partial^2_{Q_i}\mathcal{P},\end{aligned} \tag{A.4}$$

where $\mathcal{L}_0$ is the FP operator for the variables x_1 and x_2 alone appearing in eq. (A.1). We now introduce the generating function

$$\psi(x_1,x_2,\lambda,t) = \int \mathrm{d}Q_i\mathcal{P}(x_1,x_2,Q_i,t)\mathrm{e}^{\lambda Q_i}, \tag{A.5}$$

and from (A.4) we obtain the FP equation for $\psi(x_1, x_2, \lambda, t)$

$$\begin{aligned}\partial_t\psi &= \mathcal{L}_0\psi + \lambda F_{Q_i}\psi + \lambda\theta_{ii}\left[(k_ix_i+\epsilon k_jx_j)\partial_{x_i}\psi + \partial_{x_i}(k_ix_i+\epsilon k_jx_j)\psi\right] \\ &\quad +\lambda\theta_{ij}\left[(k_ix_i+\epsilon k_jx_j)\partial_{x_j}\psi + \partial_{x_j}(k_ix_i+\epsilon k_jx_j)\psi\right] + \gamma F_{Q_i}\theta_{ii}\lambda^2\psi,\end{aligned} \tag{A.6}$$

From eq. (A.5) we notice that

$$\partial_t\langle Q_i\rangle = \partial_t\int \mathrm{d}x_1\mathrm{d}x_2\ \partial_\lambda\psi(x_1,x_2,\lambda,t)|_{\lambda=0}, \tag{A.7}$$

and thus from eq. (A.6) we obtain

$$\partial_t\langle Q_i\rangle = \partial_t\int \mathrm{d}\mathbf{x}\ \partial_\lambda\psi(\mathbf{x},\lambda,t)|_{\lambda=0} = \langle F_{Q_i}\rangle - \theta_{ii}k_i - \theta_{ij}\epsilon k_j. \tag{A.8}$$

By noticing that $\langle F_{Q_i} \rangle = \left\langle (k_i x_i + \epsilon k_j x_j)^2 \right\rangle / \gamma$ (see eq. (22)), and using eq. (A.2) to calculate the correlations $\left\langle x_i^2 \right\rangle$ and $\langle x_i x_j \rangle$, one obtains the heat rates q_1 and q_2, as given in eqs. (24)-(25) in the main text.

The same procedure is used to obtain the rates for the quantities $\langle Q_{ij} \rangle$ as given by eqs. (29)-(30) in the main text.

B. PROOF OF THE FLUCTUATION THEOREM

In order to proof the Fluctuation Theorem for the heat exchanged between two reservoirs in the case of dissipative coupling , we need to calculate the probability of a single trajectory in the phase space. We first introduce the distribution of the noises ξ_i appearing in eq. (11). Those are gaussian correlated noises with fluctuation-dissipation relations given by eq. (13), and thus their probability distribution reads

$$\phi(\xi_1, \xi_2) = \frac{\Delta t}{\pi} |\mathbf{m}|^{1/2} \exp\left[-\Delta t \left(m_{11}\xi_1 + 2m_{12}\xi_1\xi_2 + m_{22}\xi_2^2\right)\right] \tag{B.1}$$

where $\mathbf{m}$ is the symmetric correlation matrix $\mathbf{m} = \boldsymbol{\theta}^{-1}/4$, with elements

$$\begin{aligned} m_{11} &= \frac{\gamma\left(\Delta T \epsilon^2 + T\right)}{4T\left(1-\epsilon^2\right)\left(T+\Delta T\right)}, && \text{(B.2)} \\ m_{22} &= \frac{\gamma}{4T(1-\epsilon^2)}, && \text{(B.3)} \\ m_{12} &= -\frac{\gamma}{4T(1-\epsilon^2)}, && \text{(B.4)} \end{aligned}$$

and Δt is a small time increment.

We now calculate the transition probability from a state $\mathbf{x} = (x_1, x_2)$ to a new state $\mathbf{x}' = (x_1', x_2')$ in a time interval Δt. Let $\Delta x_i = x_i' - x_i$, we have

$$\begin{aligned} P_F(\mathbf{x} \to \mathbf{x}'|\mathbf{x}, t) &= \int \mathrm{d}\xi_1 \mathrm{d}\xi_2\, \delta(\Delta x_1 - \Delta t(g_1(x_1, x_2) + \xi_1))\delta(\Delta x_2 - \Delta t(g_2(x_1, x_2) + \xi_2))\phi(\xi_1, \xi_2) \\ &= \int \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{e}^{\mathrm{i}q_i \Delta x_i} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2\, \mathrm{e}^{\Delta t[q_i(g_i(x_i,x_j)+\xi_i)+m_{ij}\xi_i\xi_j]} \\ &= \frac{4\pi}{\Delta t}|\mathbf{m}|^{1/2}\, \mathrm{e}^{-\frac{\Delta t}{m_{22}}\left[|\mathbf{m}|(\dot{x}_1 - g_1(x_1,x_2))^2 + (m_{22}(\dot{x}_2 - g_2(x1,x2)) + m_{12}(\dot{x}_1 - g_1(x_1,x_2)))^2\right]}, \end{aligned} \tag{B.5}$$

where the sum over repeated indexes is understood. Similarly, we found for the reverse (backward) transition

$$\begin{aligned} P_B(\mathbf{x}' \to \mathbf{x}|\mathbf{x}', t+\Delta t) &= \int \mathrm{d}\xi_1 \mathrm{d}\xi_2\, \delta(\Delta x_1 + \Delta t(g_1(x_1', x_2') + \xi_1))\delta(\Delta x_2 + \Delta t(g_2(x_1', x_2') + \xi_2))\phi(\xi_1, \xi_2) \\ &= \frac{4\pi}{\Delta t}|\mathbf{m}|^{1/2}\, \mathrm{e}^{-\frac{\Delta t}{m_{22}}\left[|\mathbf{m}|(\dot{x}_1 + g_1(x_1',x_2'))^2 + (m_{22}(\dot{x}_2 + g_2(x_1',x_2')) + m_{12}(\dot{x}_1 + g_1(x_1',x_2')))^2\right]}, \end{aligned} \tag{B.6}$$

By taking the ratio between the forward and backward probability, we find

$$\begin{aligned} &\frac{P_F(\mathbf{x} \to \mathbf{x}'|\mathbf{x}, t)}{P_B(\mathbf{x}' \to \mathbf{x}|\mathbf{x}', t+\Delta t)} = \\ &= \mathrm{e}^{-\frac{\Delta t(k_1 T x_1 \dot{x}_1 + k_2 x_2((T+\Delta T)\dot{x}_2 - \Delta T \epsilon \dot{x}_1))}{T(T+\Delta T)}} \\ &= \mathrm{e}^{\Delta t\left(\frac{\dot{Q}_{11}}{T+\Delta T} + \frac{\dot{Q}_{22}}{T} - \frac{\Delta T}{T(T+\Delta T)}\dot{Q}_{12}\right)} \end{aligned} \tag{B.7}$$

where Q_{ij} is given by eq. (20).

By iterating the above procedure, and considering a trajectory from $\mathbf{x}$ to $\mathbf{x}'$ in a finite time interval $[t, t, +\tau]$, from eq. (B.7), we find

$$\begin{aligned} &\frac{P_F(\mathbf{x} \to \mathbf{x}'|\mathbf{x}, t)}{P_B(\mathbf{x}' \to \mathbf{x}|\mathbf{x}', t+\tau)} = && \text{(B.8)} \\ &= \mathrm{e}^{-\int_t^{t+\tau} \mathrm{d}t' \left(\frac{k_1 x_1 \dot{x}_1}{T+\Delta T} + \frac{k_2 x_2 \dot{x}_2}{T} - \frac{\Delta T}{T(T+\Delta T)}\epsilon k_2 x_2 \dot{x}_1\right)} && \text{(B.9)} \\ &= \mathrm{e}^{-\frac{k_1({x_1'}^2 - x_1^2)}{2(T+\Delta T)} - \frac{k_2}{2T}({x_2'}^2 - x_2^2) - Q_{12}\frac{\Delta T}{T(T+\Delta T)}} && \text{(B.10)} \\ &= \mathrm{e}^{-\frac{k_1({x_1'}^2 - x_1^2)}{2(T+\Delta T)} - \frac{k_2}{2T}({x_2'}^2 - x_2^2) + \frac{k_2 \Delta T(Q_{21}/k_1 + \epsilon(x_1' x_2' - x_1 x_2))}{T(T+\Delta T)}}, && \text{(B.11)} \end{aligned}$$

where we have used $Q_{12} = -k_2(Q_{21}/k_1 + \epsilon(x_1'x_2' - x_1x_2))$. Eq. (B.9) corresponds to eq. (32) in the main text. The last set of equations hold at the level of single trajectories, and for any time interval τ.

C. STEADY STATE FT

Here we derive the FTs introduced in section IV A.

Equation (B.9) gives the long time FT for the quantity Q_{12}. Indeed by noticing that the quantity Q_{12} scales linearly with the time on average, as discussed above (see eq. (29) in the main text), while the differences ${x_i'}^2 - x_i^2$ are time independent, we can write the following approximate equality

$$\frac{P_F(\mathbf{x} \to \mathbf{x}'|\mathbf{x}, t)}{P_B(\mathbf{x}' \to \mathbf{x}|\mathbf{x}', t+\tau)} \underset{\tau\to\infty}{\simeq} \mathrm{e}^{-Q_{12}\frac{\Delta T}{T(T+\Delta T)}} \tag{C.1}$$

which is the FT for Q_{12} for large τ. This corresponds to neglecting the boundary terms in in eq. (B.10), i.e. energy difference stored in the potentials $k_i x_i^2/2$. By noticing that if the quantity Q_{12} is associated to a given forward trajectory, then $-Q_{12}$ is associated to the corresponding backward trajectory, and by integrating the lhs of eq. (C.1) over all those trajectories with a fixed value of Q_{12}, and neglecting the contribution of any initial distribution of the variable $\mathbf{x}$ for the forward trajectories and of $\mathbf{x}'$ for the backward trajectories, we obtain the long-time integrated FT for Q_{12} which reads

$$\frac{P(Q_{12}, \tau)}{P(-Q_{12}, \tau)} \underset{\tau\to\infty}{\simeq} \mathrm{e}^{-Q_{12}\frac{\Delta T}{T(T+\Delta T)}}. \tag{C.2}$$

Similarly eq. (B.11) gives the FT for the quantity Q_{21}, indeed by neglecting the boundary terms, one obtains

$$\frac{P_F(\mathbf{x} \to \mathbf{x}'|\mathbf{x}, t)}{P_B(\mathbf{x}' \to \mathbf{x}|\mathbf{x}', t+\tau)} \underset{\tau\to\infty}{\simeq} \mathrm{e}^{\frac{k_2}{k_1}Q_{21}\frac{\Delta T}{T(T+\Delta T)}}, \tag{C.3}$$

while the long-time integrated FT for Q_{21} reads

$$\frac{P(Q_{21}, \tau)}{P(-Q_{21}, \tau)} \underset{\tau\to\infty}{\simeq} \mathrm{e}^{\frac{k_2}{k_1}Q_{21}\frac{\Delta T}{T(T+\Delta T)}}. \tag{C.4}$$

D. THE TRANSIENT FT

We prepare our system such that at $t \to -\infty$ the temperature difference is vanishing $\Delta T = 0$, and then at $t = 0$ we "turn on" the temperature difference ΔT, and start measuring the heat currents for $t \geq 0$. Thus the initial PDF for our system reads

$$P(x_1, x_2, t=0) = P_{ss}(x_1, x_2, \Delta T = 0), \tag{D.1}$$

where $P_{ss}(x_1, x_2, \Delta T = 0)$ is given by eq. (A.3). We prepare the system in the same way along the backward trajectories: at $t' = \tau - t \to -\infty$ we take $\Delta T = 0$, and then at $t' = 0$ we "turn on" the temperature difference ΔT and start measuring the heat currents along the backward trajectories. Thus we have

$$\begin{aligned}\frac{P(x_1, x_2, t=0)}{P(x_1', x_2', t'=0)} &= \frac{P_{ss}(x_1, x_2, \Delta T = 0)}{P_{ss}(x_1', x_2', \Delta T = 0)} = \\ &= \mathrm{e}^{\frac{k_1}{2T}({x_1'}^2 - x_1^2) + \frac{k_2}{2T}({x_2'}^2 - x_2^2)}.\end{aligned} \tag{D.2}$$

Thus, combining the last equation with eq. (B.9), we obtain the following expression

$$\begin{aligned}&\frac{P(x_1, x_2, t=0)P_F(\mathbf{x} \to \mathbf{x}'|\mathbf{x}, 0)}{P(x_1', x_2', t'=0)P_B(\mathbf{x}' \to \mathbf{x}|\mathbf{x}', \tau)} = \\ &= \exp\left[\frac{\Delta T}{T(T+\Delta T)}\int_0^\tau \mathrm{d}t'\, k_1 x_1 \dot{x}_1 + \epsilon k_2 x_2 \dot{x}_1\right] \\ &= \exp\left[-\frac{\Delta T}{T(T+\Delta T)}Q_1\right],\end{aligned} \tag{D.3}$$

which is a FT that holds for any $\tau > 0$ and for the specific protocol where the system is prepared with $\Delta T = 0$ at the beginning of both the forward and the backward trajectories. The last equation corresponds to eq. (35) in the main text.

We now derive the integrated FTs introduced in section IV B. Upon integration over all the microscopic trajectories with a fixed value of Q_1, eq. (D.3) gives the integrated FT

$$\frac{P(Q_1, \tau)}{P(-Q1, \tau)} = \exp\left[-\frac{\Delta T}{T(T+\Delta T)}Q_1\right], \tag{D.4}$$

which also holds for any $\tau > 0$.

We consider now the explicit expression of Q_1 and Q_2, and using eq. (17), we express these quantities quantities as

$$\begin{aligned}Q_1 &= \frac{k_2}{2}({x_1'}^2 - x_1^2) + \epsilon k_2 \int_t^{t+\tau} \mathrm{d}t' x_2 \dot{x}_1, \\ Q_2 &= \frac{k_2}{2}({x_2'}^2 - x_2^2) + \epsilon k_1 \int_t^{t+\tau} \mathrm{d}t' x_1 \dot{x}_2 \\ &= \frac{k_2}{2}({x_2'}^2 - x_2^2) \\ &\quad + \epsilon k_1 \left[(x_1' x_2' - x_1 x_2) - \int_t^{t+\tau} \mathrm{d}t' x_2 \dot{x}_1\right],\end{aligned}$$

We can thus recast the expression of Q_1 in terms of Q_2, and find

$$Q_1 = -\frac{k_2}{k_1}Q_2 + \frac{k_1}{2}({x_1'}^2 - x_1^2) + \frac{k_2^2}{2k_1}({x_2'}^2 - x_2^2) + \epsilon k_2 (x_1' x_2' - x_1 x_2).$$

By substituting the last equation into eq. (D.3) and neglecting the boundary terms in the long time limit, we find

$$\frac{P(x_1, x_2, t=0)P_F(\mathbf{x} \to \mathbf{x}'|\mathbf{x}, 0)}{P(x_1', x_2', t'=0)P_B(\mathbf{x}' \to \mathbf{x}|\mathbf{x}', \tau)} \underset{\tau\to\infty}{\simeq} \mathrm{e}^{\frac{k_2 \Delta T}{k_1 T(T+\Delta T)}Q_2},$$

and upon integration over all the microscopic trajectories with a fixed value of Q_2 we find the integrated FT

$$\frac{P(Q_2, \tau)}{P(-Q_2, \tau)} \underset{\tau\to\infty}{\simeq} \exp\left[\frac{k_2 \Delta T}{k_1 T(T+\Delta T)}Q_2\right]. \tag{D.5}$$

[1] Bérut A, Imparato A, Petrosyan A and Ciliberto S 2016 *Phys. Rev. Lett.* **116** 068301 URL http://link.aps.org/doi/10.1103/PhysRevLett.116.068301

[2] Bodineau T and Derrida B 2004 *Phys. Rev. Lett.* **92**(18) 180601 URL http://link.aps.org/doi/10.1103/PhysRevLett.92.180601

[3] Jarzynski C and Wójcik D K 2004 *Phys. Rev. Lett.* **92**(23) 230602 URL http://link.aps.org/doi/10.1103/PhysRevLett.92.230602

[4] Van den Broeck C, Kawai R and Meurs P 2004 *Phys. Rev. Lett.* **93**(9) 090601 URL http://link.aps.org/doi/10.1103/PhysRevLett.93.090601

[5] Visco P 2006 *Journal of Statistical Mechanics: Theory and Experiment* **2006** P06006 URL http://stacks.iop.org/1742-5468/2006/i=06/a=P06006

[6] Evans D J, Searles D J and Williams S R 2010 *The Journal of Chemical Physics* **132** 024501 URL http://scitation.aip.org/content/aip/journal/jcp/132/2/10.1063/1.3279124

[7] Crisanti A, Puglisi A and Villamaina D 2012 *Phys. Rev. E* **85**(6) 061127 URL http://link.aps.org/doi/10.1103/PhysRevE.85.061127

[8] Zamponi F, Bonetto F, Cugliandolo L F and Kurchan J 2005 *Journal of Statistical Mechanics: Theory and Experiment* **2005** P09013 URL http://stacks.iop.org/1742-5468/2005/i=09/a=P09013

[9] Sarracino A, Villamaina D, Gradenigo G and Puglisi A 2010 *EPL (Europhysics Letters)* **92** 34001 URL http://stacks.iop.org/0295-5075/92/i=3/a=34001

[10] Dotsenko V, Maciołek A, Vasilyev O and Oshanin G 2013 *Phys. Rev. E* **87**(6) 062130 URL http://link.aps.org/doi/10.1103/PhysRevE.87.062130

[11] Fogedby H C and Imparato A 2011 *Journal of Statistical Mechanics: Theory and Experiment* **2011** P05015 URL http://stacks.iop.org/1742-5468/2011/i=05/a=P05015

[12] Fogedby H C and Imparato A 2012 *Journal of Statistical Mechanics: Theory and Experiment* **2012** P04005 URL http://stacks.iop.org/1742-5468/2012/i=04/a=P04005

[13] Fogedby H C and Imparato A 2014 *Journal of Statistical Mechanics: Theory and Experiment* **2014** P11011 URL http://stacks.iop.org/1742-5468/2014/i=11/a=P11011

[14] Ciliberto S, Imparato A, Naert A and Tanase M 2013 *Phys. Rev. Lett.* **110**(18) 180601 URL http://link.aps.org/doi/10.1103/PhysRevLett.110.180601

[15] Ciliberto S, Imparato A, Naert A and Tanase M 2013 *Journal of Statistical Mechanics: Theory and Experiment* **2013** P12014 URL http://stacks.iop.org/1742-5468/2013/i=12/a=P12014

[16] Koski J, Sagawa T, Saira O, Yoon Y, Kutvonen A, Solinas P, Möttönen M, Ala-Nissila T and Pekola J 2013 *Nature Physics* **9** 644–648 URL http://www.nature.com/nphys/journal/v9/n10/full/nphys2711.html

[17] Kotar J, Leoni M, Bassetti B, Lagomarsino M C and Cicuta P 2010 *Proceedings of the National Academy of Sciences* **107** 7669–7673 (*Preprint* http://www.pnas.org/content/107/17/7669.full.pdf) URL http://www.pnas.org/content/107/17/7669.abstract

[18] Sassa Y, Shibata S, Iwashita Y and Kimura Y 2012 *Phys. Rev. E* **85**(6) 061402 URL http://link.aps.org/doi/10.1103/PhysRevE.85.061402

[19] Curran A, Lee M P, Padgett M J, Cooper J M and Di Leonardo R 2012 *Phys. Rev. Lett.* **108**(24) 240601 URL http://link.aps.org/doi/10.1103/PhysRevLett.108.240601

[20] Ziehl A, Bammert J, Holzer L, Wagner C and Zimmermann W 2009 *Phys. Rev. Lett.* **103**(23) 230602 URL http://link.aps.org/doi/10.1103/PhysRevLett.103.230602

[21] Bulnes Cuetara G, Esposito M and Imparato A 2014 *Phys. Rev. E* **89**(5) 052119 URL http://link.aps.org/doi/10.1103/PhysRevE.89.052119

[22] Bérut A, Petrosyan A and Ciliberto S 2014 *EPL (Europhysics Letters)* **107** 60004 URL http://stacks.iop.org/0295-5075/107/i=6/a=60004

[23] Meiners J C and Quake S R 1999 *Phys. Rev. Lett.* **82**(10) 2211–2214 URL http://link.aps.org/doi/10.1103/PhysRevLett.82.2211

[24] Bartlett P, Henderson S I and Mitchell S J 2001 *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* **359** 883–895 ISSN 1364-503X URL http://rsta.royalsocietypublishing.org/content/359/1782/883

[25] Hough L A and Ou-Yang H D 2002 *Phys. Rev. E* **65**(2) 021906 URL http://link.aps.org/doi/10.1103/PhysRevE.65.021906

[26] Doi M and Edwards S F 1988 *The Theory of Polymer Dynamics* International series of monographs on physics (Clarendon Press) ISBN 9780198520337 URL http://books.google.fr/books?id=dMzGyWs3GKcC

[27] Herrera-Velarde S, Euán-Díaz E C, Córdoba-Valdés F and Castañeda-Priego R 2013 *Journal of Physics: Condensed Matter* **25** 325102 URL http://stacks.iop.org/0953-8984/25/i=32/a=325102

[28] Sekimoto K 2010 *Stochastic Energetics* Lecture Notes in Physics (Springer) ISBN 9783642054112

[29] Seifert U 2005 *Phys. Rev. Lett.* **95** 180601

[30] Bérut A, Imparato A, Petrosyan A and Ciliberto S 2016 *Journal of Statistical Mechanics: Theory and Experiment* **2016** to be published

[31] Zwanzig R 2001 *Nonequilibrium Statistical Mechanics* (Oxford University Press)

[32] Imparato A, Peliti L, Pesce G, Rusciano G and Sasso A 2007 *Phys. Rev. E* **76**(5) 050101 URL http://link.aps.org/doi/10.1103/PhysRevE.76.050101

[33] Imparato A, Jop P, Petrosyan A and Ciliberto S 2008 *Journal of Statistical Mechanics: Theory and Experiment* **2008** P10017 URL http://stacks.iop.org/1742-5468/2008/i=10/a=P10017