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We present an analytical and numerical study of the parking lot model (PLM) of granular relaxation and make a connection to the aging dynamics of dense colloids. As we argue, the PLM is a Kinetically Constrained Model which features astronomically large equilibration times and displays a characteristic aging behavior on all observable time scales. The density of parked cars displays quasi-equilibrium Gaussian fluctuations interspersed by increasingly rare intermittent events, quakes, which can lead to an increase of the density to new record values.

Defining active clusters as the shortest domains of parked cars which must be re-arranged to allow further insertions, we find that their typical length grows logarithmically with time for low enough temperatures and show how the number of active clusters on average gradually decreases as the system approaches equilibrium. We further characterize the aging process in terms of the statistics of the record sized fluctuations in the interstitial free volume which lead to quakes and show that quakes are uncorrelated and that they can be approximately described as a Poisson process in logarithmic time.

I. INTRODUCTION

The Parking Lot Model (PLM) is an off-lattice model where identical cars are placed on a one-dimensional parking strip with no marked bays. Its origin can be traced back to the one dimensional random packing problem first considered by Renyi [1] decades ago, where identical objects, i.e. 'cars', are inserted in random positions until no interstitial space remains which is large enough to accommodate yet another car. Its (more recent) physical applications allow both insertion and removal and include molecule ad- and de-sorption within a crowded surface area [2], and the compactification of granular materials [3]. Certain glassy features of the PLM dynamics were discussed in Ref. [4] but, in spite of intense theoretical focus on Kinetically Constrained Model (KCM) for their connections to glassy dynamics [5], it has largely gone unnoticed that the PLM qualifies as a KCM.

The asymptotic properties of the PLM average observables have been explored previously [6]. Here we investigate in more detail its spatial and temporal complexity, showing in particular that a key property of glassy dynamics, dynamical heterogeneity in time and space, is present in this model and is related to record sized fluctuations and their statistics, as also seen in other glassy systems [14–18]. Furthermore, the 'thermal' model version presently investigated furnishes a prime example of decelerating aging dynamics controlled by kinematic constraints. Our analysis clarifies a key model assumption made in a recent description of particle motion in dense colloidal suspensions [7, 8]. Specifically, the PLM features reversible fluctuations similar to in-cage rattlings of dense colloids together with irreversible releases of free volume. The latter are associated with escapes in a free-energy landscape [9, 10] and are connected to a cooperative and increasingly rare restructuring of the spatial domains present in the system.

The basic mechanism behind the model's decelerating dynamics is that the kinetic constraint provided by car impenetrability becomes harder to overcome as the density increases. A minute and increasingly rare $O(1/N)$-change to the car density lowers the free energy, but concomitantly raises the free energy barrier which must be overcome to further increase the density. The non-trivial spatial structure associated to increasing free-energy barriers [11] is indeed responsible for the dynamical behaviour of the PLM: let an *active cluster* or *active domain* be a group as small as possible of adjacent cars which must be rearranged in order to create an interstitial space sufficient to accommodate an additional car. As we shall see, the size of active clusters grows logarithmically with the system's age, and the characteristic time for their rearrangement by means of random moves grows exponentially with their size, similarly to what is observed in both recent experiments [12, 13] and in numerical model simulations [7, 8] of aging in dense colloids.

The aging dynamics of the PLM is induced by record fluctuations [14–18], in this case free-energy fluctuations able to bring the system across a series of ever increasing free energy barriers. Such fluctuations trigger non-equilibrium *quake*, in our case the rearrangement of an active cluster followed by the insertion of an extra car. Specifically, PLM quakes are induced by the appearance of an interstitial volume wide enough to accommodate one additional car. These fluctuations are rare in dense systems, occur at a decelerating rate on a linear time scale but at a nearly constant rate when viewed on a logarithmic time-scale. Correspondingly, the model's dynamics transverses metastable states of growing duration, each characterized by reversible fluctuations around a fixed average number of parked cars and each modified by an irreversible quake.

The paper is organized as follows: In the next Section, we introduce the PLM; in Sec. III, we analyze its hierarchical configuration space structure and, using heuristic arguments, sketch the dynamical consequences this structure generically brings; in Sec. IV we introduce a rejection-less Monte Carlo algorithm able to relax the system to equilibrium by transversing the required number of metastable states; in Sec. V we make connections with the phenomenology of kinetically constraint models, draw some conclusions and offer an outlook on future work. In the Appendix, we detail the algorithm used to obtain our numerical results.

II. THE PARKING LOT MODEL

A brief description of the PLM is given here to fix the notation. Our parking lot is a strip delimited by two rigid walls to avoid center-of-mass drifts, has no marked bays, and can at most accommodate L cars of unit width [1]. At a given time, $N \leq L$ cars are present, all parked perpendicularly to the strip's longitudinal axis. A configuration is equally well specified by a list of $N+1$ interstitial spaces I_i, which, for $0 < i < N$, separate parked cars i and $i+1$, with I_0 separating the left wall from the first car and I_N separating the last car from the right wall. We gloss over the distinction between an interstitial space and its size, or length.

An empty lot has a single interstitial space $I_0 = L$, and the first insertion generates two interstitial spaces $I_0 = q(I_0 - 1)$ and $I_1 = (1-q)(I_0 - 1)$, where q is a random number drawn from the uniform distribution in the unit interval. In general, a new insertion into an existing interstitial $I_i > 1$ splits the latter into two parts. First the indices of the interstitials from $i+1$ and onwards are incremented by one, and then the i^{th} and $i+1^{\text{st}}$ values are recalculated as $I_i \leftarrow q(I_i - 1)$ and $I_{i+1} \leftarrow (1-q)\,(I_i - 1)$. For a car removal, we set $I_i \leftarrow I_i + I_{i+1} + 1$, and decrement by one the indices of the interstitials from the rightmost one and down to I_{i+2}.

Starting from an empty lot, random insertions succeed as long as interstitial spaces larger than unity exist. When this no longer applies, a 'random loosely packed' configuration is reached which can only be changed by two-car processes [4]: either a 'bad parker' is removed leaving sufficient space for two 'good parkers', or the opposite process occurs. Such metastable situation is here dubbed *stage zero* because it turns out to be the first in a hierarchy of metastable states. The average parked car density at stage zero was analytically shown by Renyi [1] to approach, for $L \to \infty$, $\rho_0 = 0.7475\dots$, a value also close to numerical results obtained for a two dimensional version of the same problem [19].

In the 'thermal' version of the model discussed later in more detail the basic energy scale is defined by assigning zero energy to parked car and unit energy to free cars. Hence low temperatures correspond to values $T \ll 1$, and the 'greedy' random packing algorithm just mentioned corresponds to $T = 0$ dynamics, where only insertion attempts are possible. In general, the temperature T can be so low and the chemical potential so high that any leaving car is immediately replaced by another, the latter inserted in the same slot but with a slightly shifted position. A series of such removal/insertion processes thus amounts to small positional changes of already parked cars, which is similar to in-cage rattlings of a dense colloid [20].

In each panel of Fig. 1 the line represents the average over 20 independent simulations of the difference between the equilibrium car density and the density obtained in simulations starting with an empty lot. Simulations were conducted using a 'naive' version of the Waiting Time Method (WTM) [21], a rejection-less algorithm which inserts and removes cars at times calculated from the likelihood that these moves would succeed in a standard Metropolis algorithm. See also Sec. V A where a coarser and more efficient version of the WTM is described. The left-hand panel shows that at temperature $T = 0.80$ the system equilibrates very quickly. In the right-hand panel, the temperature is $T = 0.42$, and the asymptotic value reached by the ordinate is clearly different from zero, indicating the presence of the 'zeroth stage' metastable state mentioned above. The horizontal line segment has ordinate $1 - \rho_0$, and marks the point at which the $T = 0$ greedy dynamics on average grinds to a halt. As expected, the thermal algorithm gets closer to the equilibrium density than the quench does. Note the difference in the time scales of the relaxation processes occurring at $T = 0.80$ and $T = 0.42$.

III. HIERARCHICAL STRUCTURE AND SPATIAL AND TEMPORAL HETEROGENEITY

In this section the hierarchical structure of PLM configurations and its relation to spatial and temporal heterogeneity is discussed with no mention of a specific dynamical rule. We do assume, however, that the rule in question is 'blind', in the sense that a large number of failed random attempts are needed to successfully rearrange d cars in a preassigned way. Our conclusions, which are qualitative but general, are confirmed in the next section, where a numerical approach is considered.

Defining $I^0_{\text{max}} \geq 1$ as the largest interstitial in a system which has not yet reached metastability, we identify a stage zero metastable state, $M_0(1)$, as the configuration reached when the condition $I^0_{\text{max}} < 1$ becomes fulfilled for the first time. Such state corresponds to a loosely packed configuration, where no additional cars can be inserted without previous removals. Using k independent simulations, each starting from an empty lot, $M_0(k), \quad k = 1, 2, \dots k_{\text{max}}$ loosely packed configurations with the same statistical properties can be generated, which possibly contain slightly different numbers of cars.

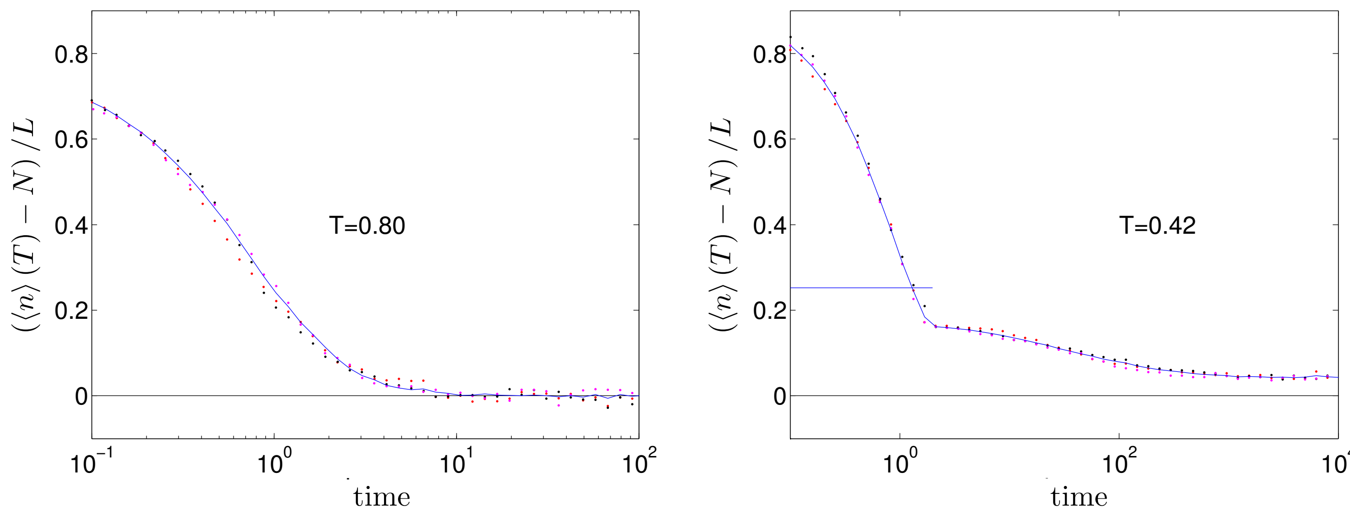

Figure 1: In each panel, the blue line shows an average over 20 independent simulations of the deviation of the car density from its equilibrium value $(\langle n \rangle (T) - N)/L$. All simulations start from an empty lot of length $L = 1000$. Three different trajectories are shown as data points to illustrate the fluctuations around the average deviation. Left-hand panel: $T = 0.80$. The ordinate converges rapidly to zero, signaling that equilibrium is reached. Right-hand panel: $T = 0.42$. The horizontal line segment with ordinate $1 - \rho_0$ marks the value of the at which the $T = 0$ greedy dynamics grinds to a halt. We can see that our 'naive' thermal algorithm comes close to equilibrium, and then remains trapped in a long-lasting metastable state.

In the limit of large $k_{\max}$ and L, we finally obtain

$$\rho_0 = \left\langle \frac{M_0(k)}{L} \right\rangle_k = 0.7475\ldots \quad (1)$$

for the averaged car density in stage zero, consistent with Renyi's analytical result [1].

Even though no additional insertions into any configuration $M_0(k)$ are possible, removing the l^{th} car will produce enough space for the insertion of two cars wherever the condition $I_l^1 \stackrel{\text{def}}{=} I_{l-1}^0 + I_l^0 > 1$ is satisfied. Starting now from a state $M_0(k)$ and repeating whenever possible and as long as possible the random removal of one car followed by the insertion of two cars in the empty slot thus generated, a stage-one metastable state $M_1(k, 1)$ is eventually reached. In such state removing one car never allows the insertion of two cars, because $I_{\max}^1 \stackrel{\text{def}}{=} \max_l \{I_l^1\} < 1$. Repeating the above procedure m times, with $M_0(k)$ as starting point, and stopping as soon as the condition $I_{\max}^1 < 1$ is satisfied, generates a series of stage one metastable states $M_1(k, m)$. Each of these can only be modified by randomly searching for two adjacent cars whose simultaneous removal makes room for three cars. This step can be iterated until all possibilities are exhausted. Proceeding along this line, we can now define a hierarchy

$$M_r(k, m, n \ldots) \subset M_{r-1}(k, m, n, \ldots) \ldots \subset M_0(k), \quad (2)$$

where in a configuration $M_r(k, m, n \ldots)$ the largest of the sums of all possible sets of $r + 1$ adjacent interstitial spaces obeys $I_{\max}^r < 1$. The symbol $\subset$ used in, say, $M_1(3, 0) \subset M_0(0)$ indicates that state $M_1(3, 0)$ is generated dynamically starting from state $M_0(0)$, but does not indicate a static set inclusion relationship. The car density at level r is

$$\rho_r = \left\langle \frac{M_r(k, m, n \ldots)}{L} \right\rangle_{k,m,n,\ldots}, \quad (3)$$

where the average is taken over all the available indices. The critical car densities separating each of the first five levels of the hierarchy from its successor were obtained numerically for $L = 1000$ and are given, with $\pm 1\sigma$ error bars, by: $\rho_0 = 0.7476 \pm 4 \cdot 10^{-4}$; $\rho_1 = 0.8587 \pm 3 \cdot 10^{-4}$; $\rho_2 = 0.8992 \pm 3 \cdot 10^{-4}$; $\rho_3 = 0.9205 \pm 2 \cdot 10^{-4}$; and $\rho_4 = 0.9343 \pm 2 \cdot 10^{-4}$. Renyi's result corresponds to the first value listed.

To obtain a physical process with a proper timescale, we follow Ben-Naim et al. [6] in assuming that cars move independently, only constrained by the free volume of interstitial spaces left to them by their neighbors. Independent car motion translates in turn into interstitial spaces I_l^r, $1 \le l \le N$, with a marginal distribution that is uniform but collectively constrained in their total length, $\sum_{l=1}^N I_l^r = L - N$. In that case, one finds [4] that their effective distribution is asymptotically exponential,

$$Q(I) \sim \frac{N}{L-N}\, e^{-\frac{NI}{L-N}} = \frac{1}{\langle I\rangle} e^{-\frac{I}{\langle I\rangle}}, \qquad L, N \to \infty. \quad (4)$$

The result is not too surprising, representing merely an exponential distribution with the mean interstitial length, $\langle I\rangle = \frac{L-N}{N}$, as its cut-off. Then, the probability that an interstitial opens up to fit in the $N+1^{\rm st}$ car of unit size, is given by

$$p_{N+1|N}(d_r) = \int_1^{L-N} dI\, Q(I) \sim e^{-\frac{N}{L-N}} \stackrel{\rm def}{=} e^{-d_r}, \quad (5)$$

where the relation $\langle I\rangle\, d_r = 1$ is used to define the typical size d_r of the domain in which r cars need to collectively moved to provide an opening of unit size.

In a large system, at each level r of the hierarchy many domains $d_r \sim r$ may coexist. Those are the "soft spots" where further insertions are most likely to occur, separating areas that are *minutely* more resistant to insertions at this level. Dynamical activity will wander from one domain to the next until all successful insertions at level r have taken place. At this point, the domains characteristic of the next level will start to play their role. The spatially localized dynamical activity, which as we just argued is typical of the PLM, is also the manifestation of dynamic spatial heterogeneity [22].

Turning to temporal heterogeneity, or intermittency, we note that $p_{N+1}(r_d)$ defines the rate at which the rare fluctuations occur which trigger a quake, i.e., the demise of a domain of size d_r and the corresponding insertion of an extra car. Quakes occurring at level r determine the time Δt_r it takes to go from the $r^{\rm th}$ to the $r+1^{\rm st}$ level of the hierarchy. This time grows as $t \sim \tau\, e^{d_r}$, where τ is a constant. Conversely, we can say that the size of such domains grows logarithmically in time,

$$d_r(t) \sim \ln\frac{t}{\tau}. \quad (6)$$

The logarithmic growth of the size of such *active domains* is a key property of the cluster model discussed in Ref. [8] and also represents a key prediction of the record dynamics description of colloidal aging [7].

To further connect our record dynamics picture with previous work [6] on the PLM, we also note that

$$d_r = \frac{1}{\langle I\rangle} = \frac{N}{L-N} = \frac{\rho_r}{1-\rho_r}, \quad (7)$$

where $\rho_r = N/L$ is the car density when domains have size d_r. Then, for times t such that $d_r \gg 1$ and $T \to 0$ the density approaches $\rho_\infty = 1$ as

$$\rho_r(t) = \frac{d_r}{1+d_r} \sim \rho_\infty - \frac{1}{\ln\frac{t}{\tau}},$$

as expected for the PLM [6].

IV. THERMAL DYNAMICS

To check the dynamical behavior just described in qualitative terms, we turn to the numerical analysis of a 'thermal' version of the PLM, where parking a car changes its energy from $\epsilon = 1$ to $\epsilon = 0$. Cars are in contact with a thermal energy reservoir at temperature T and, in the lack of interactions, the mean energy per car and the mean number of parked cars are given by

$$\langle\epsilon(T)\rangle = \frac{\exp(-1/T)}{1+\exp(-1/T)}; \quad \langle n(T)\rangle = \frac{L}{1+\exp(-1/T)} \quad (8)$$

in thermal equilibrium. In the above, both temperature and energy are dimensionless and the Boltzmann constant is set to one. The kinematic constraint forbidding the spatial overlap of parked cars has no effect on thermal equilibrium properties but has a strong effect on the time scale needed to achieve thermalization.

To see how the effect comes about, we note that Eq. (8) for $\langle n(T)\rangle = \rho_r L$ defines a series of characteristic temperatures

$$T_r = \frac{1}{\ln\left(\frac{\rho_r}{1-\rho_r}\right)} = \frac{1}{\ln(d_r)}, \qquad T_r < T_{r-1} \ldots < T_0, \quad (9)$$

each corresponding to the equilibrium density at the 'edge' between metastable states r and $r+1$. For $T > T_r$, the equilibrium car density satisfies $\langle n(T)\rangle/L < \rho_r$, and, consequently, a dynamical process starting from an empty lot reaches equilibrium before reaching a metastable state of stage $r+1$ or higher. In particular, for $T > T_0 \approx 0.921$ the equilibrium car density is too low for the kinematic constraint to play any role.

The equilibrium thermal density at $T = 0.8$, $\langle\rho\rangle = .777$, is only slightly above the Renyi density and, as shown in the left panel of Fig 1, the calculated discrepancy $\langle\rho\rangle - N/L$ quickly equilibrates and vanishes. The right panel shows corresponding data for $T = 0.42$, where the equilibrium thermal density, $\langle\rho\rangle = .966$ is way above the Renyi density and where no equilibrium is reached within the time scales considered.

In order to equilibrate starting from an empty lot, a system quenched to a low temperature must surmount a number of growing free energy barriers. This forms the basis of the PLM aging dynamics since, as shown below, the equilibration time $t_{{\rm eq},r}$ can easily outlast the patience of any observer. To estimate $t_{{\rm eq},r}$ at temperature T_r, we use Eqs. (6) and (9), finding, for an unspecified numerical constant C,

$$t_{{\rm eq},r} = \exp(C\exp(1/T_r)), \quad (10)$$

a quantity which becomes astronomically large when $\rho_r \to 1$ and $T_r \to 0$.

To explore the thermal aging dynamics of the PLM, the Metropolis algorithm is woefully inadequate, since it

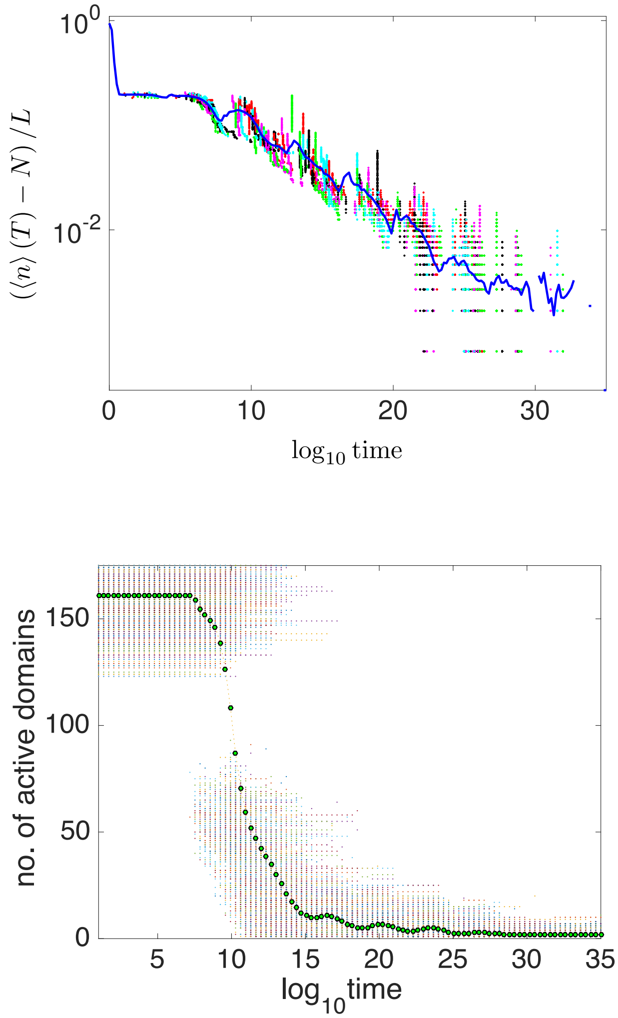

Figure 2: Upper panel: The blue line depicts the average of 2^{10} independent trajectories, each consisting of the difference between the equilibrium average of the car density at $T = 0.35$, and the time dependent car density when starting from an empty parking lot of length $L = 1000$. The ordinate goes through a fast initial decrease, followed by a considerably slower relaxation toward zero. To convey an idea of the size of the fluctuations, 40 trajectories are depicted as dots. Negative data values are present in the late stages of the relaxations and are omitted. Lower panel: the dots show the number of active domains present is the system versus the logarithm of time for 2^{10} independent simulations. The green circles depict averages over the data points..

would spend most computer resources to generate and reject configurations. We use instead an adaptation of the Waiting Time Method (WTM) [21], a rejection-less algorithm whose application to the PLM is sketched in the Appendix. The key points are: *i)* the algorithm generates for each system state a list of possible moves, each associated with a time at which the move would happen in a sequence of random attempts. The move with the shortest waiting time is carried out. *ii)* The algorithm use the temperature value needed to equilibrate the system at that temperature, see e.g. the upper panel of Fig. 2.

At the zero'th stage, car insertions do not require prior removals, and the algorithm inserts and removes single cars with the frequencies required to approach equilibrium. At low temperatures equilibrium densities are high, and the equilibration process will reach a stage where at least r contiguous cars need to be rearranged to increase the density. Such minimal cluster of cars correspond to the active domains d_r defined in Sec. III. When $r > 0$, removing a single car is in most cases followed by a re-insertion in the same slot at a slightly different position. As argued, this amounts to quasi-equilibrium fluctuations within the metastable state. These fluctuations are bypassed in our numerical algorithm using two steps: first, the time needed to remove r contiguous car using blind attempts is drawn from an exponential distribution whose average is proportional to the Arrhenius time $\exp(r/T)$. An active domain is randomly chosen among those available, and its cars are all removed. This creates a void and pushes the system back into its zero' th dynamical stage. In the second step, zero' th stage dynamics is utilized, until stage r' with $1 \leq r' \leq r + 1$ is reached. If $r' = r + 1$ an extra car has been inserted the event is registered as a quake, while if $r' < r$ the system simply enters a lower active stage. In both cases, the steps just described are repeated *ad libitum*. Note that, once the system is near thermal equilibrium, the insertion of additional cars through the zero'th stage dynamical step becomes unlikely and the dynamics enters a fluctuation regime where active clusters of size near the equilibrium cluster size are continuously removed and recreated.

The data shown in the upper panel of Fig. 2 are based on the differences between the equilibrium car density $\langle n(T) \rangle / L = 0.9656$ at $T = 0.35$ and the calculated car density N/L at the same temperature for 2^{10} independent simulations, all starting with an empty lot. The continuous blue line shows the average value of the differences, and the dots show appr. 40 of our data sets to give an idea of the fluctuations while keeping the figure uncluttered. The negative fluctuations present in the final stages of the relaxation are omitted in order to be able to use a vertical logarithmic scale. The initial phase of the relaxation ends when the density reaches the Renyi value $\rho_0 = 0.7475\ldots$, i.e., the value which delimits the lower boundary of the first metastable state. What then follows is, on average, a slow decay of the ordinate toward its equilibrium value, i.e., zero. The equilibration process can also be followed by monitoring the number of active domains. In the lower panel of the same figure, the number of active domains present at a given time is extracted from the same set of simulations and plotted as dots versus the logarithm of time. The circles represent the average number of domains at a given time.

Figure 3 illustrates how record dynamics predictions fit the low T dynamics of the PLM, based on estimates obtained from our 2^{10} independent runs. Let t_k denote the time at which the k'th quake occurs, and define the 'logarithmic waiting times' as the differences $\delta_k = \log t_k - \log t_{k-1}$. In a Poisson process with average $\mu_q \propto \ln t$, the logarithmic waiting times are independent and exponentially distributed stochastic variables. The insert in the left panel of the figure shows that different δ_k have correlation $C_\delta(k) = \delta_{k0}$, indicating the required statistical independence. The main figure shows that the PDF of the log-waiting times has an exponential trend with a superimposed structure not imputable to statistical fluctuations. The average number of quakes (not shown) grows logarithmically in time with a small superimposed oscillation. In summary, the quake process is structurally somewhat richer than a log-Poisson process, but the latter provides a reasonable simplified statistical description of the salient events of the dynamics.

The length of the active domains marks the dynamical stage reached by the system and is plotted in the right panel of the figure vs. the logarithm of time. Active domains of many different sizes replace each other in rapid succession and their seeming co-existence at the same time is only due to insufficient graphical resolution. Longer and longer active domains are seen to develop as the system ages and the first time they appear is marked by the circle at the leftmost edge of every plateau. The same circle also marks an increase in the level of metastability or dynamical stage of the system. The red dotted line is a fit of the position of such events vs. the logarithm of time. The clear logarithmic trend is in excellent agreement with Eq. (6).

V. CONCLUSIONS AND OUTLOOK

Kinetically constrained models have simple equilibrium statistical mechanical and thermodynamical property. However, an equilibrium or steady-state state can be hard to reach since many dynamical paths in their configuration spaces are blocked by kinematic constraints. The PLM is a *bona fide* kinetically constrained model, a fact not prominently featured in its origin and history. The constraint, no overlaps allowed in the parking lot, becomes increasingly hard to overcome as the density of parked cars increases. This leads to a rich aging dynamics, which coexists with a completely trivial thermodynamics. The PLM's equilibration time grows superexponentially as a function of the inverse temperature, a non-Arrhenius relaxation behavior which matches the cooperative nature of the moves required to relax the system. Equilibration is only achievable using a rejectionless algorithm which can access to the required time scales by coarse-graining away all quasi-equilibrium fluctuations. What remains is a series of heterogeneous and intermittent non-equilibrium events, connected to the rearrangements of active domains of contiguous cars needed to insert of an additional car. These events require climbing free energy barriers of growing height and can be approximately described as a Poisson process with average proportional to $\ln t$, a property which, as discussed in Ref. [23] is tantamount to pure aging behavior.

Let us connect our present findings to dense colloidal suspensions as described in Refs. [7, 8], where key experimental properties are reproduced by a 'cluster model' based on the idea that colloidal particles belong to clusters whose collapse controls all irreversible movements in the systems. To describe an aging colloid, the probability density $P(h)$ that a cluster of size h collapse must decrease very quickly, e.g. exponentially, with h. The origin of the clusters and of the form of $P(h)$ is however left unexplained in Refs. [7, 8]. Irreversible particle motion was analyzed experimentally by Yunker et al. [12] who defined irreversible events as those which disrupt at least three nearest neighbour relationships. These authors find that, as the system ages, irreversible changes require the correlated motion of increasingly large clusters. The similarity with the active domains of the PLM is clear, since in order to introduce an extra car we need the cooperative motion of an increasingly large domain. The probability that a PLM domain be re-arranged hence decreases exponentially with its size, a property which is shared by the cluster model and which was already used by Adam and Gibbs [24] to describe the approach to the glass transition. Identifying a PLM domain with a cluster of correlated particles in a colloid points to a statistical mechanism shaping the form of $P(h)$. Stability increases with cluster size as irreversible events are connected to a correspondingly decreasing free volume or, equivalently, to a local increase in density. The repulsive short range interactions between colloidal particles prevent such irreversible events from happening unless a spontaneous collective fluctuation of h particles provides the free space needed. Such fluctuation becomes exponentially unlikely.

Our analysis indicates that the typical length scale of the domains which have to be rearranged in order to approach equilibrium grows logarithmically with time. Once equilibrium is reached, the typical size of domains will be larger the longer the equilibration process. Hence the average domain size will rapidly increase with decreasing temperature. The intimate relation between the two properties is clear in the context of the PLM, but could possibly be more generally valid when approaching the glass transition. The issue can be investigated experimentally by studying the persistency of neighborhood relations in particle clusters and its resolution would shed light on the nature of the glass transition.

Finally, it seems reasonable to speculate that a similar analysis would apply to other familiar models of slow relaxation and jamming, such as the East model [25] or

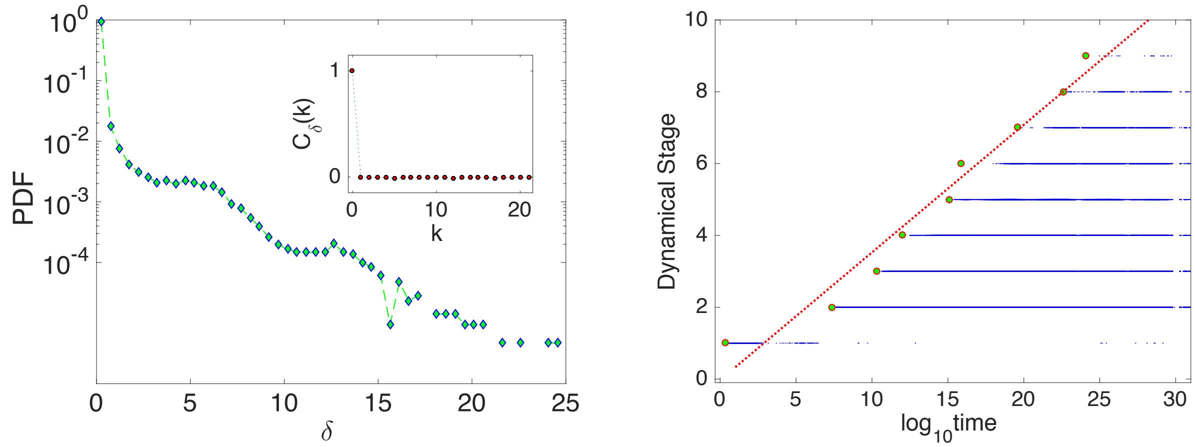

Figure 3: Left: The main figure shows the PDF of the differences $\delta_k = \log t_k - \log t_{k-1}$, where t_k marks the occurrence of the k'th quake. The data are extracted from 1024 independent trajectories run at $T = 0.35$, all starting from an empty parking lot of length $L = 1000$. The insert shows that the correlation function of the series $\delta_1, \delta_2 \dots$ is a Kroneker delta indicating that successive quakes are independent events. Right: The length of the active domains present in the systems defines the dynamical stage of the system and is plotted vs. the logarithm of time using blue points. In any small time interval, domains of different sizes are generated in rapid succession, giving the false impression that domains of different length can coexist. The circles mark the shortest time at which an active domain of a certain length first appears, and the red line is linear fit of the data vs. the logarithm of time.

the Backgammon model [11]. In the East model, an entire domain of unfrustrated spins has to collectively activate to dislodge and move a single frustrated spin on its boundary, merely to be able to expand by a minute increment. Similarly, in the Backgammon model, N uncoupled particles are spread over n domains, where the energy of the system is proportional to n. Particles hop randomly between domains until, by some chance fluctuation of size $\sim N/n$, a domain empties out and becomes inaccessible, leaving $n - 1$ domains, each of minutely larger occupation on average. Thus, these models share the same phenomenology of clusters of variables requiring ever new records in the size of collective activations that are exponentially unlikely in their size to progress towards a marginally improved state.

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Appendix:

A. The Waiting Time Method

The gist of the WTM is to determine the possible moves in a given situation, draw for each of these a waiting time from an exponential distribution with a suitable average, and carry out the move with the shortest waiting time. The WTM satisfies detailed balance and eventually generates the Boltzmann equilibrium distribution but, at low temperatures, does so much faster than the Metropolis algorithm. The algorithm is particularly simple to apply to the PLM, whose degrees of freedom have no mutual interactions.

Our version of the algorithm generates a stochastic series $t_0 < t_1 < t_2 < \dots$ recording the times at which the system configuration undergoes a change. Depending on the severity of the constraints, the algorithm goes through several incarnations, or 'stages'. At stage zero, interstitials are available which can accommodate a car, while in the k'th stage, $k = 1, 2, \dots L - 2$, a domain consisting of k contiguous cars must be rearranged, in order to create the space for an additional car. To reach thermal equilibrium for $T > T_0$, only the zeroth stage of the algorithm is needed while for temperatures in the range $T_{k-1} > T > T_k, \quad k > 1$, k stages are required.

Initially, $t_0 = 0$, the lot is empty, and the zeroth stage of the algorithm is applied: Each car is assigned a waiting time $\tau_i^{\rm free}, \quad i = 0, 1 \dots (L - 1)$, drawn from the exponential distribution with unit average. The car with the lowest waiting time, say $\tau_0^{\rm free}$, is selected for a change of status to 'parked', the global time is updated to $t_1 = t_0 + \tau_0^{\rm free}$ and the waiting times of the cars which remain parked are synchronized to t_1, i.e., $\tau_i^{\rm free} \leftarrow (\tau_i^{\rm free} - \tau_0^{\rm free}), \quad i = 1, 2 \dots (L - 1)$. To complete the first update, the newly parked car is assigned a waiting time $\tau_0^{\rm parked}$, drawn from the exponential distribution with average $e^{1/T}$.

Subsequent updates follow the same pattern as above: time t_n is obtained from t_{n-1} by adding the shortest available waiting time; all other waiting times are synchronized to t_n, and a new waiting time for the last car moved is drawn from an exponential distribution whose average is either 1 or $e^{1/T}$. The first choice applies if the last move was a car removal, and the second if it was a car placement.

As mentioned, for $T_1 < T < T_0$, the dynamics thermalizes in a metastable state of type M_0, where insertions are by definition impossible without previous removals.

With the previous scheme, a car removal would with high probability be followed by a re-insertion in the same slot, since this is the only possible sequence unless the sum of the two interstitials adjacent to the car removed is larger than one. Removal/re-insertion sequences constitute the bulk of the pseudo-equilibrium fluctuations in the metastable state but do not change the number N of parked cars, and do not further the equilibration process. Rather than waiting for a car removal which allows the placement of two cars to happen by chance, stage one makes the move and draws the waiting time associated to it from an exponential distribution, whose average is calculated as follows: Let n_0 denote the number of pairs of adjacent interstitials with total length larger than one (note that $n_0 > 0$ in a metastable state of type M_0) and define the above average as

$$\mu_0(n_0) = \frac{N}{n_0} e^{1/T}. \tag{11}$$

The first term on the right hand side of the equation is the average number of random removals needed to select a car surrounded by one out of n_0 interstitial pairs. The second is the Arrhenius factor associated with its removal. Once the move is carried out and the global time is updated, the algorithm returns to the stage zero update, which continues until a new metastable state of type M_0 is identified.

Stage k dynamics entails reshuffling a domain of k adjacent cars. This is done by first removing the respective cars, and by then returning to stage zero to fill up the opening thus created. The waiting time for removing k adjacent cars is drawn from an exponential distribution, whose average is taken to be

$$\mu_0(n_k) = \frac{1}{n_k} \binom{N}{k} e^{k/T}, \tag{12}$$

where n_k is the number of domains of length r_k present in the system. The initial factor accounts for the number of choices for the placement of the domain, the binomial coefficient is the average number of attempts needed to place k cars out of N in contiguous positions, and the exponential is the Arrhenius factor corresponding to the removal of a group of k cars. As was the case for $k = 1$, the algorithm returns to stage zero to fill in the empty space left by the removal. We note that removing k cars does not guarantee, for $k > 1$, that k cars can be successfully re-inserted in the vacant space.

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