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# Maximizing Algebraic Connectivity in Interconnected Networks\*

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Algebraic connectivity, the second eigenvalue of the Laplacian matrix, is a measure of node and link connectivity on networks. When studying interconnected networks it is useful to consider a multiplex model, where the component networks operate together with inter-layer links among them. In order to have a well-connected multilayer structure, it is necessary to optimally design these inter-layer links considering realistic constraints. In this work, we solve the problem of finding an optimal weight distribution for one-to-one inter-layer links under budget constraint. We show that for the special multiplex configurations with identical layers, the uniform weight distribution is always optimal. On the other hand, when the two layers are arbitrary, increasing the budget reveals the existence of two different regimes. Up to a certain threshold budget, the second eigenvalue of the supra-Laplacian is simple, the optimal weight distribution is uniform, and the Fiedler vector is constant on each layer. Increasing the budget past the threshold, the optimal weight distribution can be non-uniform. The interesting consequence of this result is that there is no need to solve the optimization problem when the available budget is less than the threshold, which can be easily found analytically.

Real-world networks are often connected together and therefore influence each other [1]. Robust design of interdependent networks is critical to allow uninterrupted flow of information, power, and goods in spite of possible errors and attacks [2–4]. The second eigenvalue of the Laplacian matrix,  $\lambda_2(L)$ , is a good measure of network robustness [5]. Fiedler shows that algebraic connectivity increases by adding links [6]. Moreover, it is harder to bisect a network with higher algebraic connectivity [7]. The second eigenvalue of the Laplacian matrix is also a measure of the speed of mixing for a Markov process on a network [8]. Boyd et al. maximize the mixing rate by assigning optimum link weights in the setting of a single layer ([9] and [10]).

For multiplex networks (see Fig. 1), a natural question is the following. Given fixed network layers, how should

the weights be assigned to inter-layer links in order to maximize algebraic connectivity?

The behavior of  $\lambda_2$ , in the case of identical weights, i.e., with a fixed coupling weight  $p$  for every inter-layer link, has been studied recently. For instance, Gomez et al. observe that  $\lambda_2(L)$  grows linearly with  $p$  up to a critical  $p^*$ , and then has a non-linear behavior afterwards [11]. Sole-Ribalta et al. analyze the spectrum of multiplex networks with perturbation theory on a decomposed—the intra- and interlayer structure—version of Laplacian matrix [12].

Radicchi and Arenas find bounds for this threshold value  $p^*$  [13]. Sahneh et al. compute the exact value analytically [14].

Martin-Hernandez et al. analyze the algebraic connectivity and Fiedler vector of multiplex structures, with addition of a number of inter-layer links in two configurations; diagonal (one-to-one) and random [15]. They show that for the first case, algebraic connectivity saturates after adding a sufficient number of links. Li et al. adopt a network flow approach to propose a heuristic that improves robustness of large multiplex networks by choosing from a set of inter-layer links with predefined weights [16].

In this letter we remove the constraint of identical interlinking weights and pose the problem of finding the maximum algebraic connectivity for a one-to-one inter-connected structure between different layers in the presence of limited resources. We show that up to the threshold budget  $p^*N$ —where  $p^*$  is the same threshold studied in [11], [13], and [14]—the uniform distribution of identical weights is actually optimal. For larger budgets, the optimal distribution of weights is generally not uniform.

Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  represents a network and by  $\mathcal{V} = \{1, \dots, N\}$  and  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ , we denote the set of nodes and links. For a link  $e$  between nodes  $u$  and  $v$ , i.e.,  $e : \{u, v\} \in \mathcal{E}$ , we define a nonnegative value  $w_{uv}$  as

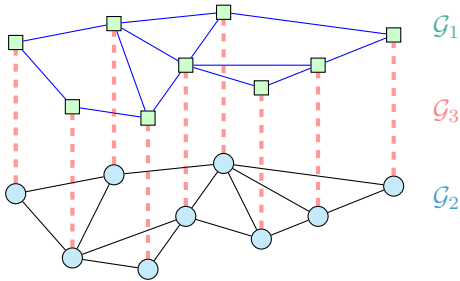


FIG. 1. A schematic of a multiplex network  $\mathcal{G}$  with two layers  $\mathcal{G}_1, \mathcal{G}_2$ , connecting through an inter-layer one-to-one structure  $\mathcal{G}_3$ .

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the weight of the link. The Laplacian matrix of  $\mathcal{G}$  can be defined as:

$$L = \sum_{ij \in \mathcal{E}} w_{ij} B_{ij} \quad (1)$$

where  $B_{ij} := (e_i - e_j)(e_i - e_j)^T$  is the incidence matrix for link  $ij$ , and  $e_i$  is a vector with  $i$ th component one and rest of its elements are zero.

For a multiplex network with two layers  $\mathcal{G}_1 = \{\mathcal{V}_1, \mathcal{E}_1\}$  and  $\mathcal{G}_2 = \{\mathcal{V}_2, \mathcal{E}_2\}$  and  $|\mathcal{V}_1| = |\mathcal{V}_2|$ , we consider a bipartite graph  $\mathcal{G}_3 = \{\mathcal{V}, \mathcal{E}_3\}$  with  $\mathcal{E}_3 \subseteq \mathcal{V}_1 \times \mathcal{V}_2$ . The multiplex network  $\mathcal{G}$  is composed from  $\mathcal{G}_1$ ,  $\mathcal{G}_2$ , and  $\mathcal{G}_3$  (Fig. 1). We want to design optimal weights for  $\mathcal{G}_3$  to improve the algebraic connectivity of  $\mathcal{G}$  as much as possible with a limited budget, i.e.,  $\sum w_{ij} = c$ . Using Eq. (1), the Laplacian matrix of  $\mathcal{G}$  (supra-Laplacian matrix), is:

$$L(w) = \sum_{ij \in \mathcal{E}_2 \cup \mathcal{E}_3} B_{ij} + \sum_{ij \in \mathcal{E}_3} w_{ij} B_{ij}, \quad (2)$$

where we use the notation  $L(w)$  to make explicit the dependence of the Laplacian on the interlayer weights  $w$ .

From Eq. (2), the Laplacian,  $L$ , of the combined network takes the form

$$L(w) = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} + \begin{bmatrix} W & -W \\ -W & W \end{bmatrix},$$

where  $L_1$  and  $L_2$  are the Laplacians of the individual layers and  $W = \text{diag}(w)$  with  $w \geq 0$  the inter-layer link weights satisfying the budget constraint  $w^T \mathbf{1} = c$ . We assume the two layers are connected independently, so that  $\lambda_3(L) > 0$ , for all choices of  $c$  and  $w$ .

The second eigenvalue can be characterized as the solution to the optimization problem

$$\lambda_2(L) = \min_{\substack{v \neq 0 \\ v^T \mathbf{1} = 0}} \frac{v^T L v}{\|v\|^2}. \quad (3)$$

The optimal weight problem, then, can be phrased as follows. Given a budget  $c \geq 0$ , solve the problem

$$F(c) := \max_{\substack{w \geq 0 \\ w^T \mathbf{1} = c}} \lambda_2(L(w)). \quad (4)$$

Since  $L$  is an affine function of  $w$ , and  $\lambda_2$  is a concave function of  $L$ , it follows that (4) is a convex optimization problem. In fact, it can be recast as a semi-definite programming problem or SDP (see (17) in the Appendix) and, thus, can be solved efficiently even for large networks using standard numerical methods.

Returning to (3), it is convenient to write  $v$  in component form  $v = (v_1^T, v_2^T)^T$  so that (3) implies

$$\begin{aligned} & v_1^T L_1 v_1 + v_2^T L_2 v_2 + (v_1 - v_2)^T W (v_1 - v_2) \\ & - \lambda_2(L) (\|v_1\|^2 + \|v_2\|^2) \geq 0 \quad \forall v_1^T \mathbf{1} = -v_2^T \mathbf{1}. \end{aligned} \quad (5)$$

Since  $v$  must satisfy  $v_1^T \mathbf{1} = -v_2^T \mathbf{1}$ , we use the following substitution for  $v_1$  and  $v_2$  to separate the  $\mathbf{1}$  subspace and its orthogonal counterpart  $u_i$ :

$$v_1 = \alpha \mathbf{1} + u_1, \quad v_2 = -\alpha \mathbf{1} + u_2, \quad (6)$$

where  $u_i \in \mathbb{R}^N$ , such that  $u_1^T \mathbf{1} = u_2^T \mathbf{1} = 0$ , and  $\alpha$  is some constant. Rewriting the terms in (5), we observe that

$$\begin{aligned} & (v_1 - v_2)^T W (v_1 - v_2) \\ & = (2\alpha \mathbf{1} + u_1 - u_2)^T W (2\alpha \mathbf{1} + u_1 - u_2) \\ & = 4\alpha^2 c + 4\alpha w^T (u_1 - u_2) \\ & \quad + (u_1 - u_2)^T W (u_1 - u_2) \end{aligned}$$

and that

$$\|v_i\|^2 = \|\alpha \mathbf{1}\|^2 + \|u_i\|^2 = \alpha^2 N + \|u_i\|^2 \quad \text{for } i = 1, 2.$$

Thus, Eq. (5) implies that

$$\begin{aligned} & u_1^T L_1 u_1 + u_2^T L_2 u_2 + 4\alpha^2 c + 4\alpha w^T (u_1 - u_2) \\ & \quad + (u_1 - u_2)^T W (u_1 - u_2) - \\ & \quad \lambda_2(L) (2\alpha^2 N + \|u_1\|^2 + \|u_2\|^2) \geq 0 \\ & \quad \forall \alpha, u_1^T \mathbf{1} = u_2^T \mathbf{1} = 0. \end{aligned} \quad (7)$$

In particular, setting  $u_1 = u_2 = 0$  in (7), then, gives the inequality

$$4\alpha^2 c - 2\alpha^2 N \lambda_2(L) \geq 0 \quad \forall \alpha$$

which can only be true if  $\lambda_2(L) \leq \frac{2c}{N}$ . Thus for the two-layer problem described above, we have the bound

$$F(c) \leq \frac{2c}{N}. \quad (8)$$

Now we turn our attention to the question of attainability of (8). This question is answered by the following theorem.

**Theorem 1.** *The inequality in (8) can only be satisfied as equality if  $w = \frac{c}{N} \mathbf{1}$ .*

*Proof.* Suppose the weights  $w$  are chosen such that the Laplacian  $L$  satisfies  $\lambda_2(L) = \frac{2c}{N}$ . Then (7) simplifies to

$$\begin{aligned} & u_1^T L_1 u_1 + u_2^T L_2 u_2 + 4\alpha w^T (u_1 - u_2) \\ & \quad + (u_1 - u_2)^T W (u_1 - u_2) \\ & \quad - \frac{2c}{N} (\|u_1\|^2 + \|u_2\|^2) \geq 0 \quad \forall \alpha, u_1^T \mathbf{1} = u_2^T \mathbf{1} = 0. \end{aligned}$$

This can only be true if the linear coefficient in  $\alpha$ ,  $4w^T (u_1 - u_2)$ , vanishes for every choice of  $u_1, u_2$  satisfying  $u_1^T \mathbf{1} = u_2^T \mathbf{1} = 0$ . This implies that  $w$  is parallel to  $\mathbf{1}$  and, since  $w^T \mathbf{1} = c$ , the theorem follows.  $\square$

The previous theorem shows that when the bound (8) is attained, it can only be attained by the uniform choice of weights  $w = \frac{c}{N} \mathbf{1}$ . The next theorem characterizes exactly the budgets for which the bound is attained.

**Theorem 2.** For a given two-layer network, define the constant

$$c^* := N \min_{\substack{u_1^T \mathbf{1} = u_2^T \mathbf{1} = 0 \\ u_1 + u_2 \neq 0}} \frac{u_1^T L_1 u_1 + u_2^T L_2 u_2}{\|u_1 + u_2\|^2} \quad (9)$$

Then, for all budgets  $c \geq 0$ ,  $F(c) = \frac{2c}{N}$  if and only if  $c \leq c^*$ .

*Proof.* By Theorem 1, the upper-bound  $\frac{2c}{N}$  for  $F(c)$  can be attained only in the case of uniform weights  $w = \frac{c}{N} \mathbf{1}$ . In this case we write  $L = L(c)$ . For all  $c \geq 0$ , one can check that  $\frac{2c}{N}$  is always an eigenvalue of  $L(c)$ , with eigenvector  $(\mathbf{1}^T, -\mathbf{1}^T)^T$ . Since  $L(c)$  is positive semi-definite and  $\lambda_1(L(c)) = 0$ , it follows that  $\lambda_2(L(c)) \leq \frac{2c}{N}$ . Thus, we have  $F(c) = \frac{2c}{N}$  if and only if  $\lambda_2(L(c)) \geq \frac{2c}{N}$ .

Recalling the variational characterization of  $\lambda_2(c)$  in (3), we observe that  $\lambda_2(L(c)) \geq \frac{2c}{N}$  if and only if the following inequality holds for every choice of  $v \neq 0$ , with  $v^T \mathbf{1} = 0$  or, equivalently, for every choice of  $\alpha$ ,  $u_1$  and  $u_2$  according to the substitution (6):

$$\begin{aligned} 0 &\leq v^T L v - \frac{2c}{N} \|v\|^2 \\ &= v_1^T L_1 v_1 + v_2^T L_2 v_2 + \frac{c}{N} \|v_1 - v_2\|^2 - \frac{2c}{N} (\|v_1\|^2 + \|v_2\|^2) \\ &= u_1^T L_1 u_1 + u_2^T L_2 u_2 - \frac{c}{N} \|u_1 + u_2\|^2. \end{aligned}$$

This inequality holds for all  $u_1^T \mathbf{1} = u_2^T \mathbf{1} = 0$  if and only if  $c \leq c^*$  as defined in (9), completing the proof.  $\square$

The threshold obtained by Eq. (9) is exactly equivalent to the threshold found in [14] (see Theorem 3 in the Appendix):

$$c^* = N \lambda_2 \left( \left( L_1^\dagger + L_2^\dagger \right)^\dagger \right), \quad (10)$$

where  $L^\dagger$  represents the Moore-Penrose pseudoinverse of  $L$ . At the threshold a rough lower-bound for  $\lambda_2(L)$  is

$$\lambda_2(L) = \frac{2}{N} c^* \geq \min\{\lambda_2(L_1), \lambda_2(L_2)\}. \quad (11)$$

One way to see this is to observe that:

$$\frac{u_1^T L_1 u_1 + u_2^T L_2 u_2}{\|u_1 + u_2\|^2} \geq \frac{\|u_1\|^2 + \|u_2\|^2}{\|u_1 + u_2\|^2} \min\{\lambda_2(L_1), \lambda_2(L_2)\}.$$

Inequality (11) then follows from the parallelogram law [17]. An upper bound for  $\lambda_2(L)$  is given in [11]

$$\lambda_2(L) \leq \frac{1}{2} \lambda_2(L_1 + L_2). \quad (12)$$

In the special case of identical layers ( $L_1 = L_2$ ) with corresponding nodes connected, the bound in (12) is attained with uniform weights at the threshold budget  $c^*$  [13]. This can be seen by combining (11) and (12). Therefore, in this case, uniform weights are optimal for budgets  $c \leq c^*$ , and increasing the budget beyond  $c^*$  cannot increase the algebraic connectivity, regardless of the weight allocation.

For general structures, it is possible to substantially improve the algebraic connectivity by increasing the budget beyond  $c^*$  using an optimal weight distribution. Figs. 2a and 2b compare the optimal value of  $\lambda_2(L)$  to the one obtained by the uniform distribution as the budget  $c$  varies for two different network structures. In both cases, the optimal distribution gives a higher algebraic connectivity after the threshold.

In Fig. 2c, we plot the first five eigenvalues of  $L$  (omitting the zero eigenvalue) for a multiplex with identical weights on the inter-layer links. Because  $\frac{2c}{N}$  is always an eigenvalue and  $\lambda_3(L) > \frac{2c}{N}$  for  $c \rightarrow 0$ , increasing  $c$ ,  $\lambda_2(L)$  and  $\lambda_3(L)$  cross. For the same multiplex with optimal distribution of inter-layer weights, we plot the eigenvalues in Fig. 2d. When increasing the budget beyond the threshold; we observe that, in this example, the second and third eigenvalues coalesce and are less than  $\frac{2c}{N}$ . Since (4) is a convex optimization problem, we know the optimal  $w_i$ 's vary continuously with  $c$ , and smoothly away from the finite set of budgets where eigenvalue multiplicities change.

When  $c \leq c^*$ , the Fiedler vector is  $v = \frac{1}{\sqrt{2N}}[\mathbf{1}, -\mathbf{1}]$  and the Fiedler cut distinguishes the layers [13–15]. For  $c > c^*$ , due to the multiplicity of  $\lambda_2(L)$ , there is a corresponding Fiedler eigenspace. Therefore, the two layers are not as easily recognizable as before.

In Fig. 2, we also observe that for  $c > c^*$ ,  $\lambda_2$  increases more slowly. Moreover, as Fig. 3 shows, for a multiplex of two scale free network layers (more results in Fig. 4 in the Appendix), we can have very non-uniform weights in this case. These optimal weights represent the importance of each link in improving the algebraic connectivity of the whole network.

In summary, we have shown that before a threshold budget, the largest possible algebraic connectivity is a linear function of the budget and can only be attained by the uniform weight distribution. Since the threshold budget is always strictly positive, for low enough budgets it is not necessary to solve (4). On the other hand, for larger budgets, (4) can be solved with efficient semi-definite programming solvers to find the optimal weights. In particular, heuristic methods based solely on the information of each layer are too blunt to notice this threshold phenomenon.

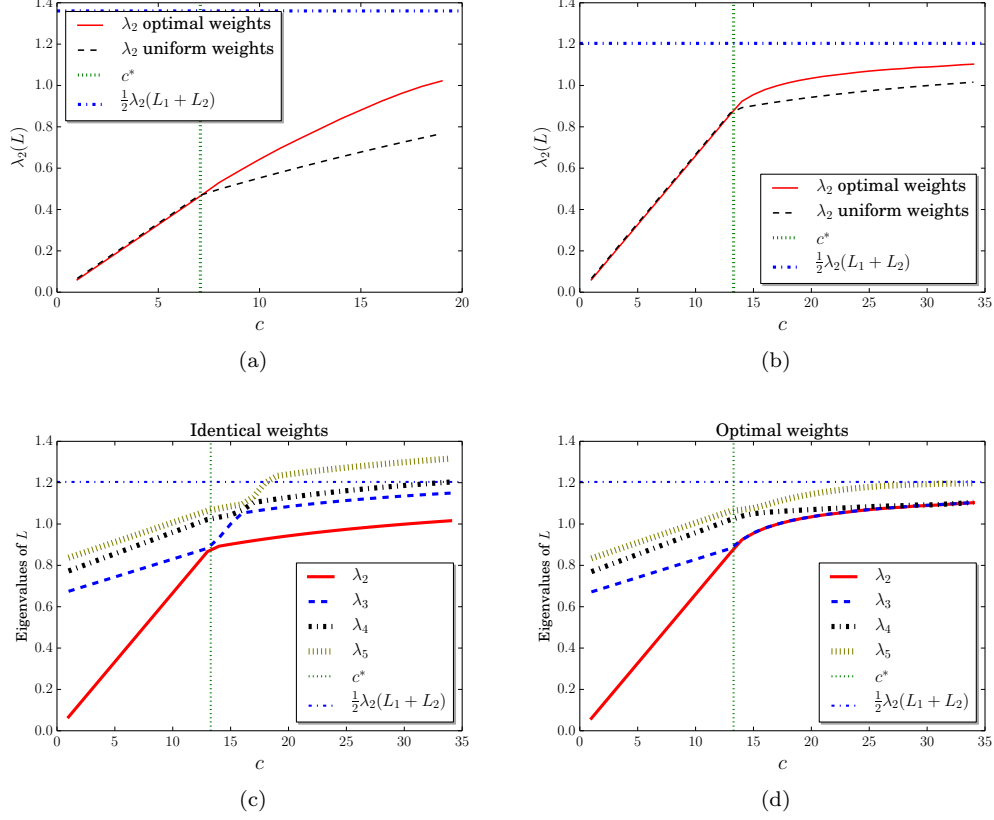


FIG. 2. (a) and (b) Plots of  $\lambda_2(L)$  with different amount of available budget. The solid (red) line is for the optimal weights and the dashed (black) line is for uniform weights. The threshold budget and upper-bound is shown with vertical (green) dotted and horizontal (blue) dot-dashed lines respectively. The upper-bound is from Eq. (12) and the threshold is from Eq. (10). (a) A structure of two Erdős-Rényi networks each with 30 nodes and (b) a structure of two scale-free networks each with 30 nodes. (c) First five eigenvalues of Laplacian matrix of  $\mathcal{G}$  considering a uniform distribution of weights for the multiplex in (b). (d) First five eigenvalues of Laplacian matrix of  $\mathcal{G}$  considering an optimal distribution of weights for the multiplex in (b).

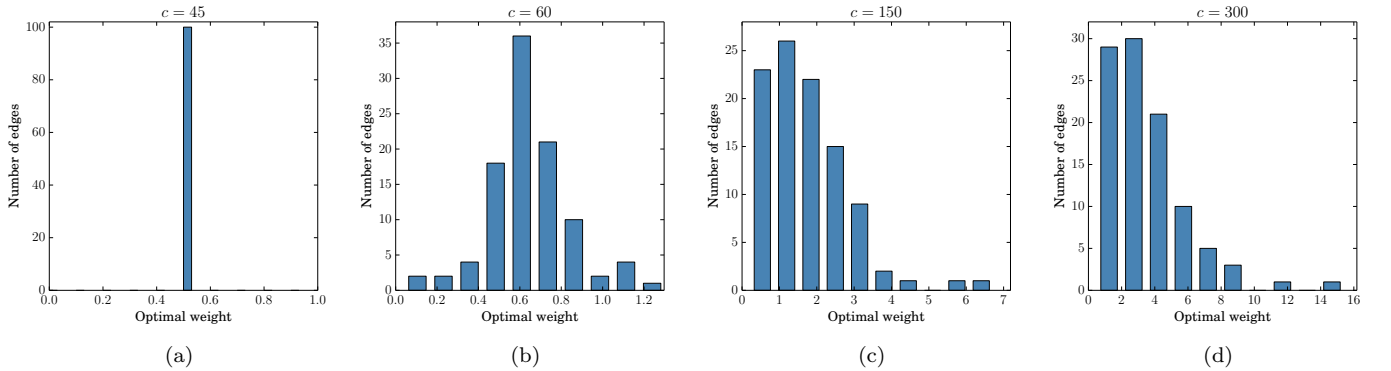


FIG. 3. Optimal weight distribution for different amount of budgets. The structure of a multiplex with two scale free network layers, with  $N = 100$  nodes and  $|\mathcal{E}_1| = 196$  and  $|\mathcal{E}_2| = 291$ . In (a) budget is lower than threshold and uniform distribution is optimal. In this example, the threshold budget  $c^*$  is 51.4.

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## APPENDIX

We have defined the threshold budget  $c^*$  as

$$c^* = N \min_{\substack{u_1^T \mathbf{1} = u_2^T \mathbf{1} = 0 \\ u_1 + u_2 \neq 0}} \frac{u_1^T L_1 u_1 + u_2^T L_2 u_2}{\|u_1 + u_2\|^2}, \quad (13)$$

where  $L_1$  and  $L_2$  are the Laplacian matrices for the two individual layers.

**Theorem 3.** *The threshold budget  $c^*$  defined in (13) satisfies*

$$\frac{c^*}{N} = \lambda_2 \left( (L_1^\dagger + L_2^\dagger)^\dagger \right) \quad (14)$$

*Proof.* We begin by rewriting the minimization in (13):

$$\begin{aligned} \min_{\substack{u_1^T \mathbf{1} = u_2^T \mathbf{1} = 0 \\ u_1 + u_2 \neq 0}} \frac{u_1^T L_1 u_1 + u_2^T L_2 u_2}{\|u_1 + u_2\|^2} &= \min_{\substack{u^T \mathbf{1} = 0 \\ u \neq 0}} \min_{\substack{u_1^T \mathbf{1} = u_2^T \mathbf{1} = 0 \\ u_1 + u_2 = u}} \frac{u_1^T L_1 u_1 + u_2^T L_2 u_2}{\|u\|^2} \\ &= \min_{\substack{u^T \mathbf{1} = 0 \\ u \neq 0}} \frac{1}{\|u\|^2} \min_{\substack{u_1^T \mathbf{1} = u_2^T \mathbf{1} = 0 \\ u_1 + u_2 = u}} (u_1^T L_1 u_1 + u_2^T L_2 u_2). \end{aligned} \quad (15)$$

To solve the inner minimization problems, we introduce Lagrange multipliers to find that the minimizing  $u_1$  and  $u_2$  satisfy

$$L_1 u_1 = \nu \mathbf{1} + \mu, \quad L_2 u_2 = \eta \mathbf{1} + \mu.$$

Taking an inner product of each of these with the  $\mathbf{1}$  vector shows that

$$\nu = \eta = -\frac{\mu^T \mathbf{1}}{N},$$

so that

$$u_1 = L_1^\dagger \left( \mu - \frac{\mu^T \mathbf{1}}{N} \mathbf{1} \right), \quad u_2 = L_2^\dagger \left( \mu - \frac{\mu^T \mathbf{1}}{N} \mathbf{1} \right).$$

Thus, without loss of generality,  $\mu$  can be taken to be orthogonal to  $\mathbf{1}$ . With this form,  $u_1$  and  $u_2$  are already orthogonal to  $\mathbf{1}$  as well. In order to satisfy the constraint  $u_1 + u_2 = u$ , we must have

$$\left(L_1^\dagger + L_2^\dagger\right)\mu = u, \quad \text{i.e.,} \quad \mu = \left(L_1^\dagger + L_2^\dagger\right)^\dagger u.$$

From this, we see that the minimizing  $u_1$  and  $u_2$  of the inner minimization problem in (15) satisfy

$$u_1 = L_1^\dagger \left(L_1^\dagger + L_2^\dagger\right)^\dagger u, \quad u_2 = L_2^\dagger \left(L_1^\dagger + L_2^\dagger\right)^\dagger u,$$

giving a minimum value of

$$\begin{aligned} u_1^T L_1 u_1 + u_2^T L_2 u_2 &= u^T \left(L_1^\dagger + L_2^\dagger\right)^\dagger L_1^\dagger L_1 L_1^\dagger \left(L_1^\dagger + L_2^\dagger\right)^\dagger u + \\ &\quad u^T \left(L_1^\dagger + L_2^\dagger\right)^\dagger L_2^\dagger L_2 L_2^\dagger \left(L_1^\dagger + L_2^\dagger\right)^\dagger u \\ &= u^T \left(L_1^\dagger + L_2^\dagger\right)^\dagger L_1^\dagger \left(L_1^\dagger + L_2^\dagger\right)^\dagger u + \\ &\quad u^T \left(L_1^\dagger + L_2^\dagger\right)^\dagger L_2^\dagger \left(L_1^\dagger + L_2^\dagger\right)^\dagger u \\ &= u^T \left(L_1^\dagger + L_2^\dagger\right)^\dagger \left(L_1^\dagger + L_2^\dagger\right) \left(L_1^\dagger + L_2^\dagger\right)^\dagger u \\ &= u^T \left(L_1^\dagger + L_2^\dagger\right)^\dagger u. \end{aligned} \tag{16}$$

Here, we have used the identity  $A^\dagger A A^\dagger = A^\dagger$ .

Substituting back into (15), we have

$$\frac{c^*}{N} = \min_{\substack{u^T \mathbf{1} = 0 \\ u \neq 0}} \frac{u^T \left(L_1^\dagger + L_2^\dagger\right)^\dagger u}{\|u\|^2}.$$

Since  $L_1$  and  $L_2$  are positive semidefinite, so are  $L_1^\dagger$  and  $L_2^\dagger$  and, consequently, so are  $L_1^\dagger + L_2^\dagger$  and  $\left(L_1^\dagger + L_2^\dagger\right)^\dagger$ . Since the component networks are assumed connected, the nullspace of  $\left(L_1^\dagger + L_2^\dagger\right)^\dagger$  is spanned by the vector  $\mathbf{1}$ . The Rayleigh quotient in (16) is therefore minimized over the orthogonal complement of the eigenspace associated with the first eigenvalue of  $\left(L_1^\dagger + L_2^\dagger\right)^\dagger$  and the theorem follows.  $\square$

#### SDP formulation of (4)

We can pose the problem of assigning weights to inter-layer edges with a limited budget  $c$  (similarly to the construction in [10]):

$$\begin{aligned} &\underset{w_{ij}}{\text{maximize}} && \lambda \\ &\text{subject to} && \sum_{ij \in \mathcal{E}_3} w_{ij} B_{ij} + L_0 + \mu e e^T - \lambda I_n \succeq 0 \\ & && \sum_{ij \in \mathcal{E}_3} w_{ij} \leq c \\ & && w_{ij} \geq 0 \end{aligned} \tag{17}$$

where  $L_0 = \sum_{i,j \in \mathcal{E}_1 \cup \mathcal{E}_2} B_{ij}$ . We know  $L \succeq 0$  and  $\lambda_1 = 0$ . Due to this redundancy in Laplacian matrix, parameter  $\mu$  is employed to avoid the zero eigenvalue.

Problem (17) is a convex SDP [18] and can be efficiently solved for arbitrary large networks with applying sub-gradient methods. Solution of (17) gives us the optimal algebraic connectivity, and optimal weights for interconnection links. Investigating these results, reveals striking features for designing networks links. We consider the case of a two-layer network with one-to-one interlayer links.

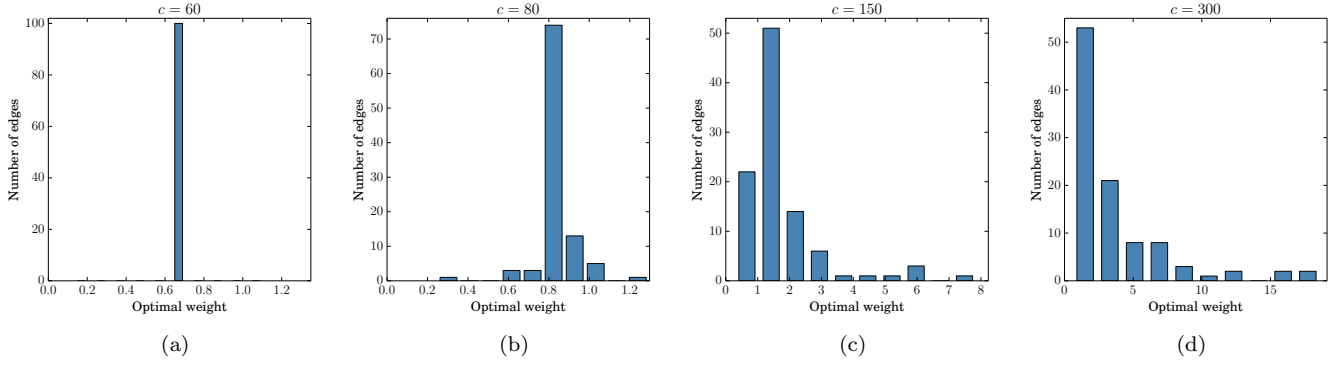


FIG. 4. Optimal weight distribution for different amount of budgets. The structure of a multiplex with two scale free network layers, with  $N = 100$  nodes and  $|\mathcal{E}_1| = 358$  and  $|\mathcal{E}_2| = 362$ . In (a) budget is lower than threshold and uniform distribution is optimal. In this example, the threshold budget  $c^*$  is 64.

#### More results for optimal inter-layer weight distribution

In Figure 4, we plot the optimal weight distribution for a multiplex of two Erdős-Renyi network layers.