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The optical mechanical analogy and nonlinear nonholonomic constraints

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Abstract

In this paper we establish a connection between particle trajectories subject to a nonholonomic constraint and light ray trajectories in a variable index of refraction. In particular we extend the analysis of systems with linear nonholonomic constraints to the dynamics of particles in a potential subject to nonlinear velocity constraints. We contrast the long time behavior of particles subject to a constant kinetic energy constraint (a thermostat) to particles with the constraint of parallel velocities. We show that while in the former case the velocities of each particle equalize in the limit, in the latter case all the kinetic energies of each particle remain the same.

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I. INTRODUCTION

It is well known that there is an analogy between optics and mechanics that inspired much of the classical theory of mechanics and indeed extended to the theory of quantum mechanics. The analogy is based on the fact that trajectories for light rays and particles can be found from variations of functionals that can be put in one to one correspondence: light rays follow the path that minimizes time, particles follow the path that minimizes action. However, there is an important class of systems -nonholonomic mechanical systems- for which the physical paths between two points is not determined by a variational principle. That leaves open the question of how to properly quantize these systems, since the optical-mechanical analogy breaks down. As part of an attempt to bridge this gap, in this paper we develop the optical mechanical analogy for nonholonomic mechanical systems with nonlinear constraints. Nonholonomic systems are not Hamiltonian or indeed variational [1], [2], [3], so this analogy is quite subtle and moreover such systems typically have linear constraints.

A key aspect our analysis is that we are analyzing trajectories and the analogy involves a change of time. Thus we map trajectories to trajectories rather than dynamics to dynamics.

Nonholonomic mechanics is the study of systems subject to nonintegrable constraints on their velocities. The classical study of such systems ([1], [3], [4] and references therein) is concerned with constraints that are linear in their velocities. Nonlinear nonholonomic constraints essentially do not arise in classical mechanics but are however of interest in the study of nonequilibrium or constant temperature dynamics which model the interaction of system with a bath [5], [6], [8], [9], [10]. In this setting the dynamics can be derived using the classical Gauss's principle of least constraint.

In this paper we consider an optical analogy for particle mechanics with nonlinear constraints. This extends our earlier work on nonholonomic systems with linear constraints [11] as well as our earlier work on the thermostat, [12].

We describe firstly the classical optical mechanical analogy where it is shown that the trejectories of a particle subject to a potential are equivalent to those of a light ray with suitable index of refraction. We show that a similar (but more complex) equivalence may be derived for certain systems with nonlinear nonholonomic constraints. We show in particular how to relate the index of refraction of the optical system to the potential of contrained system. This is important because it shows that such systems, which are not variational, can be mapped to systems which are variational – namely certain optical systems. We hope this will be useful for their quantization which depends on having a variational structure. We note that it is possible to quantize certain nonholonomic systems – see for example [13] and the discussion in [14], by taking the limit of radiation field which enforces the constraint or embedding the system in a larger variational system, but there is no universal procedure for quantizting nonholonomic systems, even with linear constraints.

We also contrast the long time behavior of particles subject to a constant kinetic energy constraint to particles with the constraint of parallel velocities. We note also that particularly in the latter case we obtain a kind of "flocking" behavior of the particles (see e.g. [15]) and we hope this work might be useful in future analysis of flocking of biological and other systems.

II. SUMMARY OF CLASSICAL OPTICAL ANALOGY

The optical mechanical analogy stems from the isomorphism between trajectories of a particle of mass m, moving at constant energy E in a potential $V(\mathbf{x})$ (the momentum being $p(\mathbf{x}) = \sqrt{2m(E - V(\mathbf{x}))}$, and that of a light ray that propagates, at constant frequency, in a medium of index of refraction $n(\mathbf{x})$. In each case, if x_i and x_f are the initial and final points, the trajectories are the extrema of their corresponding action functionals:

$$\begin{aligned} S_o &= \int_{x_i}^{x_f} nds \quad \text{(geometric optics)} \\ S_m &= \int_{x_i}^{x_f} pds \quad \text{(mechanics).} \end{aligned} \tag{1}$$

The analogy results from the equivalence of two conservation laws: conservation of momentum in the direction parallel to the surfaces of constant potential (Newton's second law for particles) and conservation of wave vector (or "slowness") in the direction parallel to the surfaces of constant index of refraction (Snel's law for light rays).

The analogy implies that the physical trajectories between x_i and x_f can be computed either for a light ray or for a particle, provided one has the equivalence

$$p(\mathbf{x}) = \sqrt{2m[E - V(\mathbf{x})]} = n(\mathbf{x}). \tag{2}$$

Notice that p and n have different units, but this is irrelevant in determining the geometry of the trajectories since the respective units amount to multiplicative constants in their actions. The optical mechanical analogy gained further prominence with the advent of quantum mechanics, and the early search of a wave mechanics for particles. The natural question is: if geometric optics is the small wave length limit of wave optics, what plays the role of a wave length λ for particles, in such a way that Newtonian mechanics is recovered in the limit of small λ? The optical mechanical analogy provides the natural correspondence:

$$p(\mathbf{x}) \propto n(\mathbf{x}) \propto \frac{1}{\lambda(\mathbf{x})}. \tag{3}$$

Since p and λ have different units there must be a constant of proportionality between them: $p = h/\lambda$, the celebrated De Broglie's relation, with the proportionality constant (Planck's universal constant) determined experimentally.

We remark that the optical analogy may also be rephrased as follows (see e.g. [16] and [17]). We simply replace the classical mechanical action with

$$\int_a^b [\mathbf{k} \cdot \dot{\mathbf{x}} - \omega(\mathbf{x}, \mathbf{k}, t)]dt\,. \tag{4}$$

Then the usual Hamilton's equations with $\mathbf{k}$ playing the role of momentum gives rise to the Hamilton-Jacobi equation for the light ray (although not the Eikonal equation). The variational principle in this case is analogous to Hamilton's principle in that it is an unconstrained variational principle with fixed time at the edpoints but with energy not fixed. This is in constrast to the Maupertuis principle where the energy E is fixed and which is the focus of this paper. Further dicusssion on optics and mechanics may be found for example in [18], [19], [20].

III. LIGHT RAY EQUATION

We now analyze the general dynamics of light rays in a medium with isotropic index of refraction. We start with the Lagrangian (which represents the optical length) in such a medium:

$$\begin{aligned} \mathcal{L} &= \frac{ds}{dt} n \\ &= \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, n(\mathbf{x}). \end{aligned} \tag{5}$$

To compute the dynamics we observe:

$$\begin{aligned}
\frac{d\mathcal{L}}{d\dot{\mathbf{x}}} &= \hat{\mathbf{t}} n(\mathbf{x}). \\
\frac{d}{dt}\frac{d\mathcal{L}}{d\dot{\mathbf{x}}} &= \frac{d\hat{\mathbf{t}}}{ds}\frac{ds}{dt} n + \hat{\mathbf{t}}\left(\nabla n \cdot \dot{\mathbf{x}}\right) \\
&= \frac{d\hat{\mathbf{t}}}{ds} + \hat{\mathbf{t}}\left(\frac{\nabla n}{n} \cdot \hat{\mathbf{t}}\right) \qquad (6)
\end{aligned}$$

where $\hat{\mathbf{t}}$ is the unit tangent vector to the ray. We also have

$$\begin{aligned}
\nabla\mathcal{L} &= \frac{ds}{dt}\nabla n \\
&= \frac{\nabla n}{n} \qquad (7)
\end{aligned}$$

Combining the above equations we get

$$\frac{d\hat{\mathbf{t}}}{ds} = (\hat{\mathbf{t}} \times \nabla \ln n) \times \hat{\mathbf{t}}\,. \qquad (8)$$

We can understand this in terms of wave fronts by writing the refraction law that relates the curvature of the light ray with the index of refraction. We recall the following:

Consider the trajectory of a light ray propagating in an arbitrary two–dimensional index of refraction $n(\mathbf{x})$ (the argument easily extends to three dimensions).

The problem of the curvature of a light ray in an arbitrary index of refraction was treated by Born and Wolf [18] in their classic "Principles of Optics". Here we re-derive the same result using a slightly different approach for completeness, as in [11]. We discretize the problem into lines of constant n, as in Figure (1).

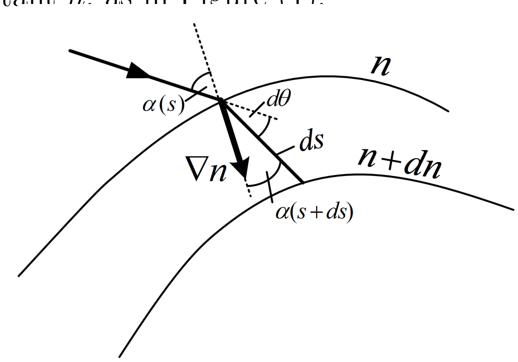

FIG. 1: Discretization of the trajectory of a light ray in a spatially dependent index of refraction n

Snel's law for a ray refracting on one of this lines is

$$\begin{aligned}
n(s)\sin\alpha(s) &= n(s+ds)\sin\alpha(s+ds) \qquad (9) \\
&= n(s+ds)\sin\left(\alpha - d\theta\right), \qquad (10)
\end{aligned}$$

where $\alpha(s)$ is the angle the light ray makes with the normal to the surface of constant n, s is the arc length and $d\theta$ is the change of the angle of the tangent to the curve [See Figure (1)].

Now expand the right hand side of Equation (10) to obtain

$$\frac{d\theta}{ds} = \frac{n'(s)}{n(s)} \tan\alpha(s). \tag{11}$$

Since α is the angle of the tangent to the ray with the normal to the light ray,

$$\frac{dn(s)}{ds}\frac{1}{\cos\alpha(s)} = |\nabla n|, \tag{12}$$

and from this equation we obtain the general expression for the curvature of the light ray

$$\frac{d\theta}{ds} \equiv \kappa(s) = \frac{|\nabla n|}{n(s)} \sin\alpha(s), \tag{13}$$

Thus

$$\kappa = |\nabla \ln(n)| \sin\alpha \tag{14}$$

$$|\kappa| = |\nabla \ln(n) \times \hat{\mathbf{t}}| . \tag{15}$$

Also, we have

$$\frac{d\hat{\mathbf{t}}}{ds} = \kappa\hat{\mathbf{n}}, \tag{16}$$

with $\hat{\mathbf{n}}$ the unit vector normal to the ray. Since the normal to the ray is perpendicular to $\hat{\mathbf{t}}$ and is in the plane spanned by $\hat{\mathbf{t}}$ and ∇n, we have the following equation for the light ray:

$$\frac{d\hat{\mathbf{t}}}{ds} = \hat{\mathbf{t}} \times \left(\nabla \ln(n) \times \hat{\mathbf{t}}\right) . \tag{17}$$

IV. SYSTEMS WITH NONLINEAR NONHOLONOMIC CONSTRAINTS

We now turn to nonholonomic mechanics – mechanics for systems with nontrivial velocity constraints – contraints which cannot be writen as constraints on positions, or as holonoimic constraints. The standard setting for nonholonomic systems (see e.g. [1]) is the following: one has a mechanical systems defined a configuration space Q, which we take to be a smooth manifold, and locally one has n coordinates $q_i(t)$ and m (linear in the) velocity-dependent constraints of the form

$$\sum_{i=1}^{n} a_i^{(j)}(\mathbf{q})\dot{q}_i = 0, \quad j = 1, \cdots, m. \tag{18}$$

The constraints are assumed to be *nonintegrable*, i.e. they are not equivalent to a set of position constraints. They constrain motion only in velocity space but not in position space and the entire position space is accessible to the system. Equivalently one says that the constraints define a nonintegrable *distribution* on the tangent bundle of the configuration space – at each point the velocities are retricted to a subspace of the velocity space.

Let $L(q_i, \dot{q}_i)$ be the system Lagrangian. and suppose the m velocity constraints are represented by the equation $A(q)\dot{q} = 0$ where $A(q)$ is an $m \times n$ matrix and $\dot{q}$ is a column vector. Let λ be a row vector of Lagrange multipliers which are used to define the virtual forces which are necessary to impose the constraints. The equations we obtain are thus

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \lambda A(q), \quad A(q)\dot{q} = 0. \tag{19}$$

Note that such systems are not variational and the dynamics may not be derived by appending the constraint to the Lagrangian by Lagrange multipliers, i.e. forming the augmented Lagrangian. If one simply appends the contraint(s) to the Lagrangian one arrive at so called vakonomic mechanics which is not equilvalent to the correct Newtonian dynamics – there are extra terms in the equations make the dynamics variational. Details on this issue may be found in [1] and work cited therein.

In the current setting we are interested in a nonlinear constraints of the form $\phi^i(q, \dot{q}) = 0$ These again may be implemented using Lagrange multipliers, by differentiating the constraint and enforcing the system to lie on the resultant hypersurface defined by this constraint. This is equivalent to Gauss's principle of least constraint (see e.g. [6],[1]).

In the linear setting (see [1]), the system energy is preserved. This is not true in the nonlinear setting.

Another feature of nonholonomic systems is that volume may not be preserved in the phase space even in the absence of external friction ([1] [13]). In the systems below volume is also not preserved in general.

We begin by considering a Newtonian system subject to an external force. We impose the constraint by a Lagrange multiplier in accordance with Gauss's principle of Least Constraint.

A. One constraint

Consider an N dimensional vector $\mathbf{V} = (\dot{x}_1, \cdots, \dot{x}_N)$ and an N dimensional force $\mathbf{F} = (f_1, \cdots, f_N)$. The constraint is imposed by a "time dependent viscosity" $\eta(t)$.

For the velocity dependent constraint

$$G(\mathbf{v}) = 0$$

$$\dot{\mathbf{v}} \;=\; \mathbf{F} - \eta(t)\nabla G$$

and

$$\dot{\mathbf{v}} \;=\; \mathbf{F} - \frac{\nabla G \cdot \mathbf{F}}{\nabla G \cdot \nabla G}\nabla G$$

guarantees that the constraint is satisfied $dG/dt = 0$.

1. *Constant velocity constraint*

We now show that the constant velocity constraint gives us a nice relation with the optical mechanical analogy, using the formulation of Section III.

The constant velocity (or constant kinetic energy) constraint corresponds to the following G:

$$G(\mathbf{v}) = v^2 - v_0^2. \tag{20}$$

Hence

$$\begin{aligned}\dot{\mathbf{v}} \;&=\; \mathbf{F} - \frac{\mathbf{F}\cdot\mathbf{v}}{v_0^2}\mathbf{v} \\ &= \frac{\mathbf{F}(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{F}\cdot\mathbf{v})\mathbf{v}}{v_0^2} \\ &= \frac{\mathbf{v}\times(\mathbf{F}\times\mathbf{v})}{v_0^2}.\end{aligned} \tag{21}$$

Using the constancy of the speed we have $\mathbf{t} = \mathbf{v}/v_0$, and

$$\begin{aligned}\dot{\mathbf{v}} \;&=\; \frac{d\mathbf{v}}{ds}v_0 \\ &= \frac{d\mathbf{t}}{ds}v_0^2,\end{aligned} \tag{22}$$

which, combined with (21) gives

$$\frac{d\mathbf{t}}{ds} = \mathbf{t} \times \left(\frac{\mathbf{F}}{v_0^2} \times \mathbf{t}\right). \tag{23}$$

Given (21) and (17) we have the equivalence

$$\mathbf{F} = v_0^2 \nabla \ln(n), \tag{24}$$

In other words, for the constant velocity constraint, the optical mechanical analogy is expressed by the equation

$$\frac{U(\mathbf{x})}{v_0^2} = -\ln n(\mathbf{x}) + \text{Constant}. \tag{25}$$

Thus the trajectory (but not the dynamics) of a particle in a constant velocity constraint with potential $U(\mathbf{x})$ is the same as that of a light ray moving in an index refraction given by

$$n(\mathbf{x}) = Ae^{-U(\mathbf{x})/v_0^2}, \tag{26}$$

with A a constant. We note that the potential is related to the natural log of the index of refraction up to the addition of a constant, but this constant becomes multiplicative for the index of refraction and therefore becomes irrelevant for the geometry of the light-ray trajectory.

Example–constant gravity

We now consider a particular example where we have a constant gravitational field.

$$\mathbf{F} = g\hat{\mathbf{j}}.$$

Thus we have

$$\dot{v}_y = g - g\frac{v_y^2}{v_0^2} \tag{27}$$

$$\dot{v}_x = -g\frac{v_y v_x}{v_0^2} \tag{28}$$

Since the speed is constant, we write

$$\mathbf{v} = v_0(\sin\theta, \cos\theta) \tag{29}$$

and rewrite (28) as

$$\dot{v}_x = -g\sin\theta\cos\theta. \tag{30}$$

Also,

$$\dot{v}_x = v_0 \frac{d\sin\theta}{dy}\frac{dy}{dt} = v_0^2 \frac{d\sin\theta}{dy}\cos\theta,$$

which, combined with (30) gives

$$\frac{d\sin\theta}{dy} = -\frac{g}{v_0^2}\sin\theta \tag{31}$$

or

$$\sin\theta = Ce^{-\alpha y},$$

with $\alpha = g/v_0^2$. Now, using Snel's law

$$n(y)\sin\theta = \text{Const}$$

we get in fact that

$$n(y) \propto e^{\alpha y}.$$

In general, using (31)

$$n\frac{d(1/n)}{dy} = \frac{1}{v_0^2}\frac{dV}{dy}, \tag{32}$$

or

$$-\frac{d\ln(n)}{dy} = \frac{1}{v_0^2}\frac{dV}{dy}, \tag{33}$$

and

$$\ln n(y) = -\frac{V(y)}{v_0^2} + \text{Constant} \tag{34}$$

We note that in the analysis above both the potential and index depend only on y. This relects the nature of the analogy between the optical and mechanical problems where the change in the potential in a given direction is determined by the change of optical index in that direction.

B. Dynamics of many particles with a single velocity constraint

We recall the work of [12]. We assume we have N particles with equals mass.

Consider the case of N particles in one dimension subject to a constant gravitational force $f = mg$ and and with constant kinetic energy. In the absence of the constraint the particles move independently and the kinetic energy fluctuates. We can show that the constraint induces correlations and that the long time behavior corresponds to *all* particles moving with the same velocity, regardless of the initial conditions.

As above the equation of motion of the n-th particle is

$$\dot{v}_n = g - \frac{\sum_{m=1}^{N} g v_m}{\mathbf{V}^2} v_n. \tag{35}$$

Of course $\mathbf{V}^2 = \sum v_n(t)^2$ is preserved by the dynamics. Define

$$u_q = \frac{1}{N} \sum_n v_n e^{iqn}, \tag{36}$$

with $q = \frac{2\pi}{N} k,\ \ k = 0, 1, \cdots, (N-1)$. Also define a (constant) mean quadratic velocity as $v_M^2 = \frac{\mathbf{V}^2}{N}$.

Substitute these two transformations in (35) to obtain

$$\dot{u}_q(t) = g\delta_{q,0} - \frac{g u_0(t)}{v_M^2} u_q(t). \tag{37}$$

From (37) the equation of motion for u_0 is

$$\dot{u}_0 = g \left(1 - \frac{u_0^2}{v_M^2} \right). \tag{38}$$

with solution (and long time limit) given by:

$$u_0(t) = v_M \tanh(gt/v_M) \to v_M.$$

The solution for $u_q(t)$ for $q > 0$ is given by

$$u_q(t) = \frac{u_q(0)}{\cosh(gt/v_m)}.$$

In the long time limit, $u_q(t) \to 0$. Substituting in (36) we see that the long time solution is

$$v_n(t \to \infty) = v_M.$$

This means that in this particular example, at long times, the constraint enforces all particles to move *with the same velocity* v_M. In the absence of the constraint, the velocities are of course independent, and the total energy is conserved.

In the constrained case the long time behavior for each particle position is a linear increase, meaning that, although the kinetic energy is constant, the potential energy is linearly decreasing: $\dot{U}_n = -mgv_M$.

The extension to non-equal masses is essentially immediate. The main result is that the long time behavior remains the same: regardless of the mass differences, the asymptotic velocities are all the same. This means of course that in that case equipartition does not occur. One can also apply the analysis to the case of equal mass particles with different charges in an electric field.

We remark that this one dimensional analysis is useful for understanding the multi-particle case. To obtain an optical analogy we consider below higher dimensions.

C. Many particles–More than one constraint

We now consider many particles with multiple constraints.

$$G_k(\mathbf{v}) = 0, \quad k = 1, \cdots, m \tag{39}$$

$$\dot{\mathbf{v}} = \mathbf{F} - \sum_{k=1}^{m} \eta_k \nabla G_k, \tag{40}$$

where $\mathbf{v}$ has dimensions $D \times N$, with N the number of particles and D the spatial dimensionality.

Define

$$A_j = \nabla G_j \cdot \mathbf{F} \tag{41}$$

$$B_{ij} = \nabla G_i \cdot \nabla G_j \tag{42}$$

then

$$\dot{\mathbf{v}} = \mathbf{F} - \sum_{k=1}^{m} \left(B^{-1}A\right)_k \nabla G_k \tag{43}$$

D. Parallel velocity constraint in $2D$

We now consider the case of N particles with $N-1$ constraints that enforce parallel motion (for related work see [7])

$$G_i = \dot{x}_i \dot{y}_{i+1} - \dot{y}_i \dot{x}_{i+1}, \quad i = 1, \cdots, N-1.$$

The equations of motion are

$$\ddot{x}_1 = f_{1x} - \eta_1 \dot{y}_2 \tag{44}$$

$$\ddot{y}_1 = f_{1y} + \eta_1 \dot{x}_2 \tag{45}$$

$$\ddot{x}_k = f_{kx} - \eta_k \dot{y}_{k+1} + \eta_{k-1} \dot{y}_{k-1} \quad (k = 2, \cdots N-1) \tag{46}$$

$$\ddot{y}_k = f_{ky} + \eta_k \dot{x}_{k+1} - \eta_{k-1} \dot{x}_{k-1} \quad (k = 2, \cdots N-1) \tag{47}$$

$$\ddot{x}_N = f_{Nx} + \eta_{N-1} \dot{y}_{N-1} \tag{48}$$

$$\ddot{y}_N = f_{Ny} - \eta_{N-1} \dot{x}_{N-1} \tag{49}$$

where (f_{ix}, f_{iy}) denote the (x, y) compoments of the force acting on the ith particle.

For conservative forces, the above equations guarantee that the individual energies for each particle are conserved. Also, for general forces, it guarantees that the *direction* θ_k for each of the particles is the same.

In terms of

$$\begin{aligned} z_j &= \dot{x}_j + i\dot{y}_j \\ &= v_j e^{i\theta_j} \\ f_j &= f_{jx} + i f_{jy} && (50) \\ &= F_j e^{i\alpha_j}, && (51) \end{aligned}$$

where F_j denotes the magnitude of the force on the jth particle, the equations of motion are:

$$\dot{z}_1 = f_1 + i\eta_1 z_2 \tag{52}$$

$$\dot{z}_k = f_k + i\eta_k z_{k+1} - i\eta_{k-1} z_{k-1} \quad (k = 2, \cdots N-1) \tag{53}$$

$$\dot{z}_N = f_N - i\eta_{N-1} z_{N-1}. \tag{54}$$

The above implies, using the telescoping property of the sum,

$$\sum_{j=1}^{N} z_j \dot{z}_j = \sum_{j=1}^{N} f_j z_j \tag{55}$$

We now rewrite the equation explicitly in terms of the polar form of z_j.

We note that $\dot{z}_j = \dot{v}_j e^{i\theta_j} + i v_j \dot{\theta}_j e^{i\theta_j}$. Hence, we have:

$$\sum_j \dot{v}_j v_j e^{i2\theta_j} + i v_j^2 \dot{\theta}_j e^{i2\theta_j} = \sum_j F_j v_j e^{i(\alpha_j+\theta_j)} = \sum_j F_j v_j e^{i(\alpha_j-\theta_j)} e^{i2\theta_j}. \tag{56}$$

We now implement the constraint of parallelism between velocities, which implies that θ_j equal a common value, θ. Then, dividing by the common exponential of $i\theta$, the imaginary part of the above equation becomes

$$\sum_{j=1}^{N} v_j^2 \dot{\theta} \ = \ \sum_{j=1}^{N} F_j v_j \sin(\alpha_j - \theta). \tag{57}$$

Thus we have

$$\dot{\theta} = \frac{\sum_{j=1}^{N} F_j v_j \sin(\alpha_j - \theta)}{\sum_{j=1}^{N} v_j^2}\,. \tag{58}$$

From this equation we can extract an optical mechanical analogy for each particle. First we use the fact the curvature for the trajectory for each particle is given by

$$\kappa_j = \frac{d\theta}{ds_j} = \frac{\dot{\theta}}{v_j}, \tag{59}$$

and we have conservation of energy

$$v_j(\mathbf{x}) = \sqrt{2[E_j - V(\mathbf{x})]}. \tag{60}$$

Now consider the following two cases:

1. Constant gravity g

For the gravitational case $F_j = g$, $\alpha_j = -\pi/2$ and we obtain

$$\dot{\theta} = -g\cos\theta \frac{\sum_{j=1}^{N} v_j}{\sum_{j=1}^{N} v_j^2} \tag{61}$$

Also, since the force is constant (independent of position) and the angles θ_j are equal, we have

$$\dot{v}_i = \dot{v}_j,$$

the relative velocities are constants of motion.

Consider particle j: for given initial conditions for each particle velocities and positions v_{0k} and y_{0k} we have that all the velocities can be written in terms of y_j the y coordinates of particle j:

$$v_k = v_k(y_j).$$

The curvature of the particle is the following function of position:

$$\kappa_j = -\cos\theta \frac{g}{v_j} \frac{\sum_{j=1}^N v_j}{\sum_{j=1}^N v_j^2}$$

and, using (14), with $\sin\alpha = \cos\theta$, we find the optical mechanical analogy for constant gravity

$$\frac{d}{dy_j} \ln n(y_j) = -\frac{g}{v_j} \frac{\sum_{j=1}^N v_j}{\sum_{j=1}^N v_j^2}\,. \tag{62}$$

This thus gives an expression for the equivalent index of refraction that gives light ray dynamics similar to -but not identical to- particles moving a constant gravitational potential subject to parallel velocity contraints. Notice that, in our multi-particle case, there is an index of refraction for each particle (or light ray), determined by the value of each one particle constant of motion v_j [and E_j through Eq. (60)] . In addition, each light ray has a complex dynamics since the motion is influenced by the instantaneous position of the rest of the particles. An interesting special case is as follows, where the individual v_j are constants of motion.

2. Constant magnetic field

Consider now the case of a uniform field but particles with different mass, so that $F_j = \omega_j v_j$. Also, since the force is perpendicular to the velocity $\sin(\alpha_j - \theta) = 1$, and we obtain

$$\dot{\theta} = \omega = \frac{\sum_{j=1}^N \omega_j v_j^2}{\sum_{j=1}^N v_j^2}, \tag{63}$$

the particles rotate proportionately to a weighted average of their bare individual angular velocities. The optical mechanical analogy is given as above by (62).

Remark: We make the following remark about velocity type constraints. It is possible to construct an interesting Hamiltonian that conserves kinetic energy, that for a so-called ultrarelativistic particle. However, this Hamiltonian is singular, that is, not hyperregular (see [21]) . (A Hamiltonian (or Lagrangian) is said to be hyperegular if the corresponding Legendre transform or equivalently its inverse is a diffeomorphism. This is essentially equivalent to the Hessian of the kinetic energy being an invertible matrix. Singular Lagrangians are usually handled in general by the Dirac's method (see[22]).)

Consider the following Hamiltonian:

$$H(\mathbf{x},\mathbf{p}) = c|\mathbf{p}| + V(\mathbf{x}) \tag{64}$$

where c is constant (the velocity of light in a vacuum). One can compute the Hamiltonian equations of motion and one finds that $\dot{\mathbf{x}} = \mathbf{p}/|\mathbf{p}|$. Hence the magnitude of the velocity is conserved and one can check that the dynamics gives that of the system with constant velocity constraint discussed above. We note that this may be viewed as a ultrarelativistic (singular) limit of the relativistic Hamiltonian $H = c\sqrt{\mathbf{p}^2 + m^2c^2} + V(\mathbf{x})$ with corresponding Lagrangian $L = -mc\sqrt{c^2 - \dot{\mathbf{x}}^2} - V(\mathbf{x})$. This is limit one obtains as the magnitude of the velocity approaches c. Since

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{mc}{\sqrt{c^2 - \dot{\mathbf{x}}^2}}\dot{\mathbf{x}} \tag{65}$$

we see that as the velocity approaches c the momentum becomes large leading to the above ultrarelativistic limit of the Hamiltonian.

V. CONCLUSION

In conclusion we have demonstrated, as far as we know for the first time, that there is a natural optical analogy for the motion of particles subject to nonlinear nonholonomic constraints. This extends the classical analogy and our earlier work on the case of holonomic constraints.

This analogy, as in the classical setting, only maps trajectories to trajectories. However, this is important because it shows that such systems, which are not variational, can be mapped to systems which are variational – namely certain optical systems. We hope this will be useful for their quantization which depends on having a variational structure.

Further, our work leads to an interesting contrasting description of the motion of particles subject to a thermostat constraint and particles with enforced parallel directional velocities. We intend in future work to extend these ideas to the analysis of dynamical and controlled flocking.

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