Switching synchronization in one-dimensional memristive networks

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Switching Synchronization in One-Dimensional Memristive Networks

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We report on an astonishing switching synchronization phenomenon in one-dimensional memristive networks, which occurs when several memristive systems with different switching constants are switched from the high to low resistance state. Our numerical simulations show that such a collective behavior is especially pronounced when the applied voltage slightly exceeds the combined threshold voltage of memristive systems. Moreover, a finite increase in the network switching time is found compared to the average switching time of individual systems. An analytical model is presented to explain our observations. Using this model, we have derived asymptotic expressions for memory resistances at short and long times, which are in excellent agreement with results of our numerical simulations.

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I. INTRODUCTION

Synchronization is the term that is frequently used to describe the coherent dynamics of an ensemble of interconnected dynamical units, namely, dynamical units forming networks. The networks are ubiquitous in nature and technology, and, therefore, it is not surprising that the phenomenon of synchronization has been studied and observed in a wide range of dynamical systems. These systems include, for example, neurons [1, 2], power grids [3, 4], coupled lasers [5], oscillators, social systems [6], etc. A lot of attention has been drawn to the synchronization of chaotic systems [7, 8] – an intriguing emergence of collective dynamics of a number of chaotic units linked with a common signal or signals. Oscillator networks [9] are another example of widely studied systems with synchronization.

In this paper, our attention is focused on memristive (memory resistive) networks. These networks are composed of individual memristive elements [10], which now are of considerable interest for a variety of applications [11]. In these passive resistive electronic devices, the resistance depends on the history of signals applied. There are two types of memristive systems: voltage-controlled and current-controlled [10]. In particular, the voltage-controlled memristive systems are defined by

$$I = R^{-1}(x,V,t)\,V, \qquad (1)$$
$$\dot{x} = f(x,V,t), \qquad (2)$$

where I and V are the current through and voltage across the system, respectively, $R(x,V,t)$ is the memristance (memory resistance), x is an n-component vector of internal state variables and $f(x,V,t)$ is the vector-function. The current-controlled memristive systems are defined in the similar way [10]. Typically, the memristance changes between two limiting values, R_{on} and R_{off}, such that $R_{on} < R_{off}$.

The ability of memristive systems to store and process information on the same physical platform makes them ideal for unconventional computing applications [12, 13]. In fact, boolean logic operations with small memristive networks were experimentally demonstrated few years ago [14]. Moreover, it was theoretically shown that larger memristive networks could solve maze [15] and shortest path optimization [16] problems in a single step compared to multi-step algorithms employed in the conventional computers. Therefore, it's quite important to understand the dynamical properties of memristive networks.

Recently, two of us (VAS and YVP) have found that in one-dimensional memristive networks subjected to adiabatically increasing voltage, the effective switching rates of memristive systems strongly depend on their polarities [17]. It has been demonstrated (on the level of individual memristive elements) that an abrupt (accelerated) switching occurs when the memristance of a given memristive system in the network increases at the given voltage polarity. A slow (decelerated) switching takes

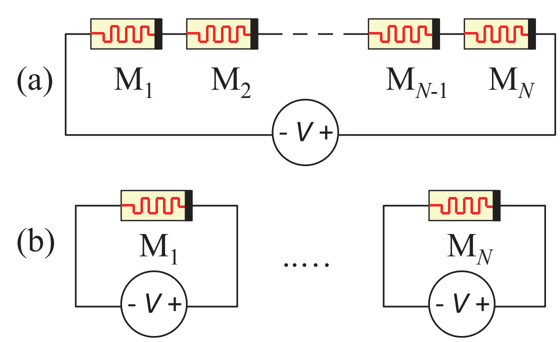

FIG. 1: (Color online) (a) One-dimensional network of N memristive systems M_i connected to a dc voltage source. (b) Memristive systems M_i independently connected to voltage sources.

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place in the opposite case [17]. However, this prior work leaves open the question of the switching behavior beyond the adiabatic limit, namely, when the applied voltage is initially high enough to induce the dynamics of several memristive systems. This is precisely the aim of the present paper, which explores the switching dynamics of one-dimensional memristive networks subjected to sufficiently high voltages. According to our findings, an interesting switching synchronization effect takes place when all memristive systems switch from the high to low resistance state. Our consideration of the switching synchronization effect employs both numerical and analytical techniques.

Fig. 1(a) shows a one-dimensional memristive network connected to a constant voltage source V. The phenomenon of synchronization is exemplified most clearly if the polarities of all memristive systems are the same, and, at the initial moment of time $t = 0$, all memristive systems are in the same high resistance state $R_i(t = 0) = R_{off}$. Our paper focuses precisely on this configuration. Importantly, the same conditions exist in the maze [15] and shortest path problem [16] solving networks.

Moreover, we assume that Fig. 1(a) network employs voltage-controlled memristive systems with threshold that have been experimentally realized with different materials combinations [11]. In our numerical simulations and analytical calculations presented below, we use a model of voltage-controlled memristive systems with threshold [17] that can be written (for i-th memristive system) in the following form:

$$I_i = R_i^{-1} V_i \quad (3)$$

$$\frac{\mathrm{d}R_i}{\mathrm{d}t} = \begin{cases} \pm\mathrm{sign}(V_i)\beta_i(|V_i| - V_t) & \text{if } |V_i| > V_t \\ 0 & \text{otherwise} \end{cases}, \quad (4)$$

where the memristance R_i serves as an internal state variable [10], β_i is a positive switching constant characterizing the intrinsic rate of memristance change when $|V_i| > V_t$, V_t is the constant positive threshold voltage, and $+$ or $-$ sign is selected according to the device connection polarity. Additionally, it is assumed that the memristance is limited to the interval $[R_{on},\ R_{off}]$. We emphasize that Eqs. (3)-(4) are a particular case of general Eqs. (1)-(2).

II. NUMERICAL RESULTS

In the calculations presented below, we consider the dynamics of a set of N memristive systems characterized by a probabilistic distribution of the parameter β_i. For the sake of simplicity, we consider a flat random distribution of this parameter keeping all other parameters of memristive systems the same. In these calculations, we use the " $-$ " sign in Eq. (4) accounting for the desired device polarity and $R_i(t = 0) = R_{off}$.

Let us, first of all, consider the dynamics of a set of N memristive systems individually subjected to a positive voltage $V \equiv V_i$ (see Fig. 1(b) circuits), $i = 1, .., N$. For this purpose, we perform N independent calculations (for each memristive system from the set). In each calculation, $R_i(t)$ is found as a solution of Eq. (4) with $V_i = V$. Fig. 2(a) shows the result of these calculations for a specific realization of random parameters β_i and $V = 1.05V_t$, which is the average voltage per system in the network considered in the next two paragraphs. It follows from Fig. 2(a) (as well as from Eq. (4)) that the memristances $R_i(t)$ vary linearly with time. It is not surprising that the switchings of individual memristive systems subjected to the same voltage occur at very different rates defined by specific individual values of switching constants β_i.

Next, we consider the collective switching, namely, the switching of memristive systems that form one-dimensional networks as the one sketched in Fig. 1(a). In this case, the network dynamics is found in a single calculation as a solution of N Eqs. (4) coupled through the current I and condition $\sum_{i=1}^{N} V_i = V$. Using Eq. (3), it is not difficult to see that in this case $V_i(t) = (R_i(t)/R(t))V$ with $R(t) = \sum_{i=1}^{N} R_i(t)$.

Fig. 2(b) shows the main result of this paper. This plot demonstrates a surprising switching synchronization effect in which the effective switching rates of unlike systems become the same. This is a truly remarkable behavior that has not been anticipated in the literature. The basic principles of this behavior are related to the phenomenon of the decelerated switching [17] that, unlike our previous investigation [17], takes place simultaneously in every component of the network. Technically speaking, the switching of memristive systems with larger values of β_i can not proceed fast as the decreases of their memristances also suppress the voltage falls across them. At the same time, the voltage falls across memristive systems with smaller values of β_i increase compensating the smallness of their β_i. As a result, the switching of all memristive system occurs coherently with approximately the same effective rate.

Fig. 2(b) also demonstrates that the total switching time for this random realization of the network is longer than the switching time defined by the average value of β_i-s, which is $\langle\beta\rangle$. This typical type of behavior has been seen in the majority of random realizations of the network. Fig. 3 shows some additional data points (extracted from numerical simulations) demonstrating a monotonic increase in the network switching time with distribution width. In fact, according to our analytical theory presented below, the network switching time is actually proportional to $\langle 1/\beta\rangle$ instead of $1/\langle\beta\rangle$ (See Eq. (13)). For a flat random distribution of β_i in the range $[\langle\beta\rangle - \Delta\beta/2,\ \ \langle\beta\rangle + \Delta\beta/2]$ and $N \gg 1$

$$\langle 1/\beta\rangle = \frac{1}{\Delta\beta} \ln \frac{\langle\beta\rangle + \frac{\Delta\beta}{2}}{\langle\beta\rangle - \frac{\Delta\beta}{2}}. \quad (5)$$

The dashed curve in Fig. 3 shows the perfect agreement

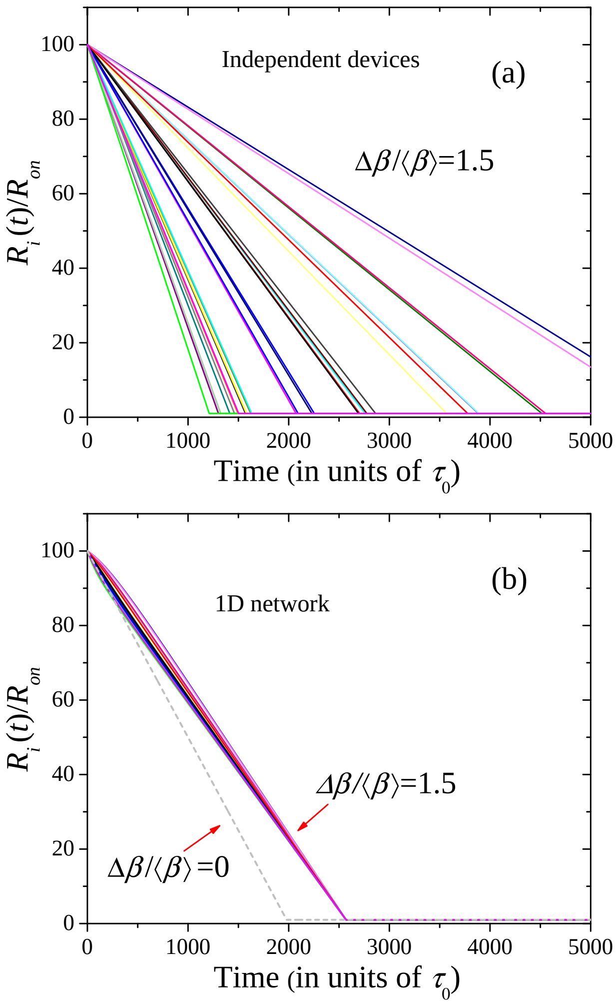

FIG. 2: (Color online) The memristance $R_i(t)$ of $N = 30$ memristive systems with $R_{off}/R_{on} = 100$ (a) individually subjected to the same voltage $V = V_i = 1.05V_t$ (see Fig. 1(b) circuits), and (b) forming 1D network (as in Fig. 1(a)) subjected to $V = 1.05NV_t$. These plots have been obtained with a random flat distribution of parameters β_i in the interval $[\langle\beta\rangle - \Delta\beta/2,\ \langle\beta\rangle + \Delta\beta/2]$. The time is measured in units of $\tau_0 = R_{on}/(\langle\beta\rangle V_t)$.

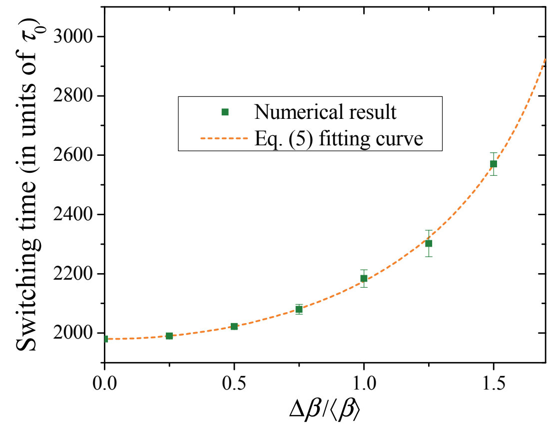

FIG. 3: (Color online) The network switching time as a function of the distribution width $\Delta\beta$. The numerical points have been found for a network of $N = 1000$ memristive systems with a flat random distribution of switching constants assuming $V = 1.05NV_t$. An averaging over 11 random realizations of networks has been performed. The error bars show the standard deviation of each point. The dashed curve is plotted assuming that the switching time is proportional to $\langle 1/\beta\rangle$ given by Eq. (5).

of Eq. (5) with our numerical results.

Even if the distribution of β_i-s is not flat then one can show that for *any* distribution of β_i the difference $\langle 1/\beta\rangle - 1/\langle\beta\rangle \geq 0$. Indeed, in accordance with the Cauchy-Schwarz inequality we have $\langle 1/\beta\rangle\langle\beta\rangle = \left(\int \mathrm{d}\beta f(\beta)/\beta\right)\left(\int \mathrm{d}\beta f(\beta)\beta\right) \geq \left(\int \mathrm{d}\beta f(\beta)\right)^2 = 1$, where $f(\beta)$ is the distribution function of positive switching rates β_i.

Additionally, the difference between $\langle 1/\beta\rangle$ and $1/\langle\beta\rangle$ normally grows with the distribution width. For example, if all odd central momenta are negative or equal to zero (as in the case of the Gaussian distribution) then this difference cannot be less than $(\langle\beta^2\rangle - \langle\beta\rangle^2)/\langle\beta\rangle^3$. Thus, our observation of the switching time increase is valid on average for any distribution of β_i.

III. ANALYTICAL MODEL

If the initial memristances are the same and the applied voltage exceeds the combined threshold voltage NV_t, then one can realize that the voltage fall across any memristive system exceeds its threshold voltage at any time. Moreover, in the case of a distribution of initial memristances, the same is true either from $t = 0$ or after an initial equilibration period. Therefore, in the region of parameters of interest, Eq. (4) can be generally written as

$$\dot{R}_i(t) = -\beta_i \left[V_i(t) - V_t\right], \qquad (6)$$

where $i = 1, ..., N$, $V_i(t) = VR_i(t)/R(t)$, and $R(t) = \sum_{i=1}^{N} R_i(t)$ is the total memristance.

Let us search for the solution of Eq. (6) in the form

$$R_i(t) = C_i(t)e^{-\beta_i \int\limits_0^t \frac{V}{R(\tau)}\mathrm{d}\tau}, \qquad (7)$$

where $C_i(t)$ is a time-dependent function and the integral in the exponent is actually the charge $\left(\int_0^t \left[V/R^{-1}(\tau)\right] \mathrm{d}\tau = q(t)\right)$ flown through the network by the time t. This form of the solution is natural, if we formally solve Eq. (6) as a linear equation with respect to $R_i(t)$. Substituting Eq. (7) into Eq. (6), one can find

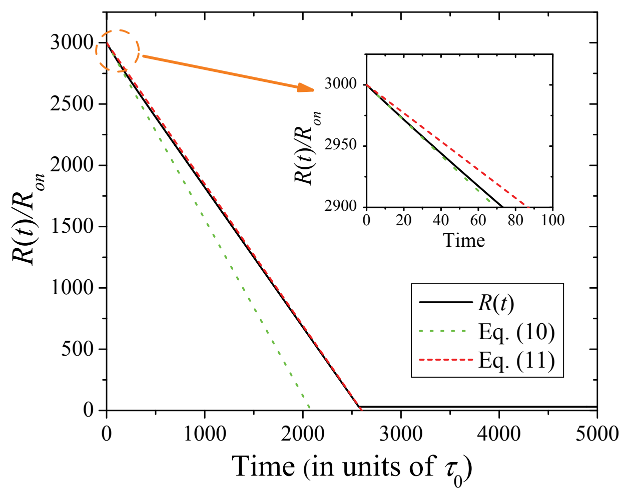

FIG. 4: (Color online) Comparison of the asymptotic expressions (Eqs. (10) and (11)) for the time-dependence of the total memristance $R(t)$ and numerical (exact) solution. The numerical solution (solid black curve) has been obtained for the same realization of memristive systems, model and simulation parameters as in Fig. 2. Eq. (10) curve is plotted in the linear approximation.

the following expression for $R_i(t)$:

$$R_i(t) = R_i(0)e^{-\beta_i q(t)} + \beta_i V_t e^{-\beta_i q(t)} \int_0^t e^{\beta_i q(t')} \mathrm{d}t'. \quad (8)$$

Taking into account that $R_i(0) = R_{off}$, the sum of Eqs. (8) yields

$$R(t) = \sum_{i=1}^{N} e^{-\beta_i q(t)} \left[R_{off} + \beta_i V_t \int_0^t e^{\beta_i q(t')} \mathrm{d}t' \right]. \quad (9)$$

As $q(t)$ can be expressed through $R(t)$ (see the definition of $q(t)$ below Eq. (7)), Eq. (9) can be considered as the nonlinear integral equation for $R(t)$.

While it is difficult to find the exact solution $R(t)$ from Eq. (9), this equation can be effectively used to derive the asymptotic behavior of $R(t)$ in the most important limiting cases. In particular, in the short time limit, one can expand $R(t) = R_0(1-at+bt^2+O(t^3))$ using unknown constants a and b, and get $q(t) = V(t+at^2/2+O(t^3))/R_0$. Using Eq. (9) one can find

$$R(t) = NR_{off} - \langle\beta\rangle\delta V t + \frac{D\beta V \delta V t^2}{2NR_{off}} + O(t^3), \ t \to 0, \quad (10)$$

where $\delta V = V - NV_t$ is the voltage excess above the combined threshold voltage NV_t, and $D\beta = \langle\beta^2\rangle - \langle\beta\rangle^2$ is the dispersion of switching constants β_i. While the first and second terms in Eq. (10) are related to the evolution of individual memristive systems, the third term, being proportional to the dispersion, is always positive and describes the collective evolution of the network. Note that the expression (10) is valid only when the second and third terms are small compared to the first one.

A different asymptotic expression can be found in the long time limit, namely, when $\beta_i q(t) \gg 1$. This limit also implies the optimal synchronization condition $\delta V \ll V$ as demonstrated below. When $\beta_i q(t) \gg 1$, the main contribution to the right-hand side of Eq. (9) comes from the upper limit of the integral with respect to t'. Using this observation one can derive the following main term of the long time asymptotic

$$R(t) = (NR_{off} - \beta_H \delta V t)(1 + O(\delta V/V)), \ \delta V \to +0, \quad (11)$$

where $\beta_H = \langle 1/\beta\rangle^{-1}$.

To specify the applicability conditions of Eq. (11), one can calculate $q(t)$ using Eq. (11). Then the condition $\beta_i q(t) \gg 1$ can be presented as

$$\frac{\beta_i V}{\beta_H \delta V} \ln\left(\frac{NR_{off}}{R(t)}\right) \gg 1. \quad (12)$$

Eq. (12) can be sub-divided into the optimal synchronization condition $\delta V \ll V$ (also observed in our numerical studies) and the condition of long times such that the total resistance $R(t)$ is much less than its initial value NR_{off}. The total switching time T for the network can be easily computed substituting $R(T) = NR_{on}$ in Eq. (11). This gives

$$T = \frac{N(R_{off} - R_{on})}{V - NV_t} \left\langle \frac{1}{\beta} \right\rangle, \ \delta V \ll V. \quad (13)$$

Thus the switching time T is the time it takes for all memristive systems to change their resistances from R_{off} to R_{on}. In other words, for memristive systems initially in R_{off}, it is the shortest time such that $R_i(t) = R_{on}$ for any i. We see that the switching time T of the network is inversely proportional to the voltage excess above the combined threshold voltage NV_t and proportional to the total change of the network resistance. Furthermore, in the typical situations when $R_{off} \gg R_{on}$, R_{on} in the nominator of Eq. (13) can be omitted.

Fig. 4 shows a comparison of the numerically obtained solution with the asymptotic expressions given by Eqs. (10) and (11). Clearly, the asymptotic expressions are in the excellent agreement with the numerical solution for $R(t)$.

It is interesting to note that Eq. (11) also delivers a good approximation for all times when $\delta V \ll V$. This allows to find the approximate expression for individual resistances $R_i(t)$ for all moments of time, which reproduces exactly the asymptotic behavior (11) for long times and the first two terms of (10) for short times for the total resistance $R(t)$. Thus, an approximated expression for $R_i(t)$ can be obtained substituting Eq. (11) into Eq. (8). Assuming that $\delta V \ll V$, which is the optimal synchronization condition, one can obtain

$$R_i(t) = R_{off} - \frac{\beta_H \delta V}{N} t - R_{off} \left(1 - \frac{\beta_H}{\beta_i}\right) \frac{\delta V}{V_t N} + R_{off} \left(1 - \frac{\beta_H}{\beta_i}\right) \frac{\delta V}{V_t N} e^{-\frac{\beta_i V}{R_{off} N} t}. \quad (14)$$

We emphasize that the exponential (last) term in Eq. (14) decays on a short time scale. Clearly, the ratio of this short time scale to the total switching time T, Eq. (13), is $\delta V/V \ll 1$.

Moreover, it is easy to notice that the first two terms in the right-hand side of Eq. (14) are dominant at long times. These terms do not depend on the system's index i and thus are the same for all memristive systems. Consequently, the long-time memristances are nearly the same. This observation confirms our numerical results (see, Fig. 2(b)).

IV. CONCLUSION

In conclusion, we have discovered an interesting synchronization pehnomenon taking place in one-dimensional memristive networks with elements characterized by a distribution of switching constants. When the switching occurs from the high to low resistance state, the systems with larger switching constants slow down their switching as the voltage falls across these systems decrease faster compared to the voltages across the systems with smaller switching constants. As a result, the switching of all memristive systems occurs coherently with nearly the same effective rate regardless the specific switching constants of individual systems. This simple picture explains the mechanism of the synchronization effect that is most pronounced when the applied voltage slightly exceeds the combined threshold voltage of memristive systems. We have also demonstrated that the network switching time is independent on the number of memristive systems (for an appropriately scaled applied voltage) and is defined by the harmonic mean of switching constants.

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