Generalized model of blockage in particulate flow limited by channel carrying capacity.

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We investigate stochastic models of particles entering a channel with a random time distribution. When the number of particles present in the channel exceeds a critical value N, a blockage occurs and the particle flux is definitively interrupted. By introducing an integral representation of the n-particle survival probabilities, we obtain exact expressions for the survival probability, the distribution of the number of particles that pass before failure, the instantaneous flux of exiting particle and their time correlation. We generalize previous results for $N = 2$ to an arbitrary distribution of entry times and obtain new, exact solutions for $N = 3$ for a Poisson distribution and partial results for $N \geq 4$.

I. INTRODUCTION

A stream of particles flowing through a channel may be slowed or blocked if the number of particles present exceeds the carrying capacity of the channel. This phenomenon is widespread and spans a range of lengthscales. Typical examples include vehicular and pedestrian traffic flow, filtration of particulate suspensions and the flow of macromolecules through micro- or nano- channels. A specific, comic book, example of the first category is a bridge that collapses if the combined weight of the vehicular traffic exceeds a threshold. In filtration, experimental data of the fraction of grains retained by a filter mesh can be explained by assuming that clogging may occur when two or more grains are simultaneously present in the same vicinity of a mesh hole, even though isolated grains are small enough to pass through the holes [1]. A biological example is provided by the bidirectional traffic in narrow channels between the nuclear membrane and the cytoplasm [2].

The totally asymmetric simple exclusion effect process (TASEP) provides a theoretical approach to these phenomena. The TASEP is a lattice model with a stochastic dynamics where particles hop randomly from site to site in one direction with the condition that two particles cannot occupy the same site at the same time [3, 4]. At the two extremities of the finite lattice, particles are inserted and removed with two different rates. The model and its extensions provide quantitative descriptions of the circulation of cars and pedestrians [5–10]. In this model blockage due the channel is not possible. The dynamics of the process is controlled by the entering and exiting fluxes and exclusion effects. The so-called bridge models consider two TASEP processes with oppositely directed flows, but allow exchange of particles on the bridge [11–15]. At the microscopic level active motor protein transport on the cytoskeleton has been modeled by a TASEP [16, 17].

Recently, some of the present authors [18, 19] intro-duced a class of continuous time and space stochastic models that are complementary to the TASEP approach in that the blockage is due to the limited carrying capacity of the channel (and not the exclusion effect between particles). In these models particles enter a passage at random times according to a given distribution. In the simplest concurrent model particles move in the same direction and an isolated particle exits after a fixed transit time τ but if $N = 2$ particles are simultaneously present, blockage occurs. If the particle entries follow a homogeneous Poisson process all properties of interest, including the survival probability, mean survival time and the flux and distribution of exiting particles can be obtained analytically. The model has a connection to queuing theory in that it is a generalization of an M/D/1 queue, i.e. one where arrivals occur according to a Poisson process, service times are deterministic and with one server. This queue has many other applications including, for example, trunked mobile radio systems and airline hubs [20–22].

Opposing streams, where blockage is triggered by the simultaneous presence of two particles moving in different directions can be treated within the same framework [18]. Inhomogeneous distributions of entering particles can be treated analytically [23]. It is also possible to obtain exact solutions for when the blockage is of finite duration, rather than permanent [24]. In this case, for a constant flux of incoming particles the system reaches a steady-state with a finite flux of exiting particles that depends on the blockage time τ_b.

The purpose of this article is to explore the properties of the concurrent flow models for any distribution of entry times and when the threshold for blocking is $N > 2$. In addition to the applications described above, this generalized model may also be relevant for internet attacks, in particular denial of service attacks (DoS) and a distributed denial of service attacks (DDoS) where criminals attempt to flood a network to prevent its operation [25–27].

Unfortunately, the method used to solve the models for $N = 2$ [18, 24] applies only to a Poisson distribution and cannot be used even in this case for $N > 2$. In section II A, we develop a new approach providing formal exact expressions of the key quantities describing the kinetics of the model. In section II B, as a first application, we recover the results of the model $N = 2$ that were first obtained by using a differential equation approach [18]. In section II C, we present a complete solution when the entry time distribution is poissonian for $N = 3$. In section II D we consider the case of general N. In section III we investigate the time correlation for $N = 2$ and $N = 3$, and we further explore the model by studying the correlations between the arrival times of the particles. We also explore the connection with the equilibrium properties of the hard rod fluid.

II. CONCURRENT FLOW MODEL

A. Definition and Quantities of Interest

We assume that at $t = 0$ the channel of length L is empty. The first particle enters at a time t_0 that is distributed according to a probability density function $\psi(s)$. The entry of subsequent particles is characterized by the inter-particle time $t_i, i > 0$ between the entry of particle i and $i + 1$. We assume that the t_i are distributed according to $\psi(s)$ and uncorrelated. The total elapsed time is then $t = t_0 + \sum_{i=1}^{n-1} t_i + t'$ when n particles have entered and t' is the time elapsed after the entry of the last particle.

If unimpeded by the presence of another particle, a particle exits after a transit time $\tau > 0$. Blockage occurs when N particles are present in the channel at the same time, which occurs if $t_i + t_{i+1} + \cdots + t_{i+N-2} < \tau$ (see Fig.1 for the case $N = 3$). The behavior of the model depends on the recent history, more specifically the entry times of the particles that are still in the channel. The probability that no particle enters in the interval $[0, t]$ is $1 - \psi_c(t)$ with $\psi_c(t)$ the cumulative distribution $\psi_c(t) = \int_0^t \psi(s)ds$.

The simplest case is a homogeneous Poisson process where the probability density function of particle entry times is $\psi(t) = \lambda e^{-\lambda t}$ where λ is the rate (sometimes called the intensity).

The key quantities describing the process are the probability that the channel is active at time t, namely the survival probability, $p_s(t)$, the average blocking time $\langle t \rangle$ (where the bracket indicates an average over realizations of the process), the number of particles that have exited the channel at time t, $\langle m(t) \rangle$, and the instantaneous particle flux $j(t)$.

The survival probability can be expressed as the sum over all n-particle survival probabilities $q(n,t)$, i.e. the joint probability of surviving up to t and that n particles
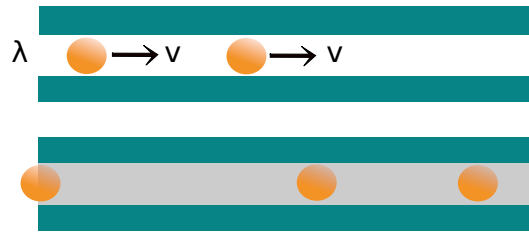

FIG. 1. (Color online) Concurrent flow model $N = 3$. Particles enter the left hand side of a channel of length L randomly. Top: two particles cross and exit the channel in a time τ. Bottom: If a third particle enters while the two previous particles are still in the channel, a blockage occurs instantaneously.

have entered the passage during this time,

$$p_s(t) = \sum_{n=0}^{\infty} q(n,t). \tag{1}$$

For general N and $n > N - 1$, $q(n,t)$ can be expressed as:

$$q(n,t) = \int_0^{\infty} \cdots \int_0^{\infty} \left[\prod_{i=0}^{n-1} dt_i \psi(t_i) \right] \int_0^{\infty} dt'(1 - \psi_c(t'))$$
$$\left[\prod_{j=1}^{n-N+1} \theta \left(\sum_{m=0}^{N-2} t_{j+m} - \tau \right) \right] \delta \left(t - \sum_{i=0}^{n-1} t_i - t' \right), \tag{2}$$

where $\theta(x)$ the Heaviside step function. The first n integrals correspond to the arrival of n particles in the channel, with time intervals t_i, the integral over t' imposes that no particle enters after particle n. The Heaviside functions account for the constraint that no consecutive sequence of N particles can be simultaneously in the channel, i.e. in a time interval smaller than τ and the δ function imposes that the observation time t is equal to the sum of the time intervals t_i plus t_0 and t'.

For $0 \leq n \leq N - 1$ there is no constraint on the particle time interval so the probability $q(n,t)$ is expressed as the joint probability of n independent and identically distributed events,

$$q(n,t) = \int_0^{\infty} \left[\prod_{i=0}^{n-1} dt_i \psi(t_i) \right] \int_0^{\infty} dt'(1 - \psi_c(t'))$$
$$\delta \left(t - \sum_{i=0}^{n-1} t_i - t' \right) \tag{3}$$

and

$$q(0,t) = 1 - \psi_c(t). \tag{4}$$

Once the $q(n,t)$, and hence $p_s(t)$, are known we can obtain several useful quantities. The probability density

function of the blocking time, $f(t)$ is simply related to $p_s(t)$ [28]

$$f(t) = -\frac{dp_s(t)}{dt}. \tag{5}$$

Defining the Laplace transform as $\tilde{f}(u) = \int_0^\infty dt e^{-ut} f(t)$, one infers

$$\tilde{f}(u) = 1 - u\tilde{p}_s(u). \tag{6}$$

The mean blocking time is given by

$$\langle t \rangle = \int_0^\infty dt\, t f(t) = \tilde{p}_s(0). \tag{7}$$

The instantaneous flux of particles exiting the channel can be obtained by noting that, if a particle exits the channel at time t, at most $N-1$ particles can enter the channel between t and $t-\tau$ if no blockage is to occur. Since blockage is irreversible the flux tends to 0 when the time increases, $j(\infty) = 0$ for all value of N. The total flux is given by the sum

$$j(t) = \sum_{n=1}^\infty j(n,t), \tag{8}$$

where $j(n,t)$ is the partial flux where a particle exits the channel at time t such that the channel is still open and n particles have already entered, for $n \geq N$

$$j(n,t) = \int_0^\infty dt'(1 - \psi_c(t')) \int_0^\infty \left[\prod_{i=0}^{n-1} dt_i \psi(t_i)\right]$$
$$\left[\prod_{j=1}^{n-N+1} \theta\left(\sum_{m=0}^{N-2} t_{j+m} - \tau\right)\right] \delta\left(t - \sum_{i=0}^{n-1} t_i - t'\right)$$
$$[\delta(t' - \tau) + \sum_{k=1}^{N-2}(\delta(t' + \sum_{w=1}^k t_{n-w} - \tau))],\ n \geq N. \tag{9}$$

The condition that a particle exits at time t is expressed in terms of δ functions. The last line of the above equation corresponds to different situations. The isolated delta function corresponds to a single particle in the channel, while the sum corresponds to the presence of $N-1$ particles with the first one exiting (which was the first to enter). For $n < N$ blocking is not possible, so Eq.(9) is replaced by one without the Heaviside functions.

Finally, the number of particles that have exited at time t can be obtained by integrating over the particle flux

$$\langle m(t) \rangle = \int_0^t dt' j(t'). \tag{10}$$

We can also obtain the distribution of particles exiting the channel. Let $h(m,t)$ denote the probability that

blockage occurs in the interval $(0,t)$ and that m particles have exited during this time. Its time evolution is given by

$$\frac{dh(m,t)}{dt} = \int_0^\infty \prod_{i=0}^{m+N-1} dt_i \psi(t_i) \prod_{j=1}^m \theta\left(\sum_{p=0}^{N-2} t_{j+p} - \tau\right)$$
$$\theta(\tau - \sum_{p=1}^{N-1} t_{m+p})\delta(t - \sum_{i=0}^{m+N-1} t_i),\ m \geq 1. \tag{11}$$

The upper part of the right hand side corresponds to the event where $m + N$ particles have entered at time t, and there was no blockage involving the first $m + N - 1$ particles. The second Heaviside function corresponds to the constraint that the last N particles are blocked in the channel, with the $N + m^{th}$ particle entering at time t.

One can check that

$$\langle m(t) \rangle = \sum_{m=0}^\infty m h(m,t) = \int_0^t j(t')dt'. \tag{12}$$

We now consider the specific cases $N = 2$ and $N = 3$.

B. Explicit Solution for $N = 2$

Since each Heaviside function in Eq.(2) depends on only one variable, the multiple integrals can always be calculated. Taking the Laplace transforms of Eq.(2) and Eq.(3), one obtains

$$\tilde{q}(n,u) = \tilde{\psi}(u)(\frac{1}{u} - \tilde{\psi}_c(u)) \left[\int_\tau^\infty dt\, e^{-ut}\psi(t)\right]^{n-1}. \tag{13}$$

Using Eq.(1) and $\tilde{\psi}_c(u) = \frac{\psi\tilde{(}u)}{u}$, we obtain the Laplace transform of the survival probability.

$$\tilde{p}_s(u) = \sum_{n=0}^\infty \tilde{q}(n,u)$$
$$= \frac{1 - \tilde{\psi}(u)}{u}\left(1 + \frac{\tilde{\psi}(u)}{1 - \int_\tau^\infty e^{-ut}\psi(t)dt}\right). \tag{14}$$

Therefore, the mean time of blockage is

$$\langle t \rangle = \hat{t}\left[1 + \frac{1}{\int_0^\tau \psi(t)dt}\right], \tag{15}$$

where $\hat{t} = \tilde{\psi}'(0) = \int_0^\infty dt\, t\psi(t)$ is mean inter particle time. To interpret Eq.(15) we note that $\psi_c(\tau) = \int_0^\tau \psi(t)dt$ gives the probability that two consecutive particles are separated by a time smaller than τ.

The integral representation of the n-particle survival probabilities approach presented in section II A allows us
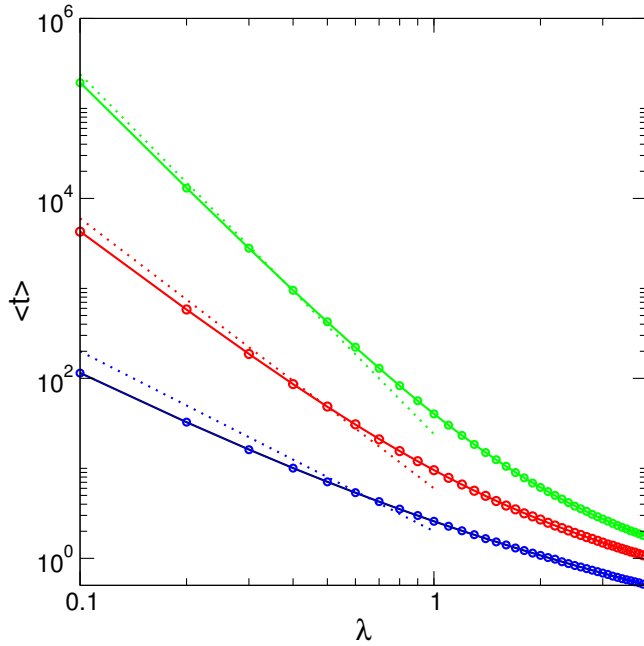
FIG. 2. (Color online) Mean time of blocking $\langle t \rangle$ as a function of the intensity, λ, for a Gamma distribution for $\alpha = 2, 3, 4$ (from bottom to top), from numerical simulation (circles) and Eq.(15) (full lines). Dotted lines correspond to the asymptotic behavior, Eq.(17).

to consider distributions other than the poissonian. In particular, we study the Gamma distribution,

$$\psi(t) = \frac{\lambda^\alpha t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} \qquad (16)$$

where α is a shape parameter. For $\alpha = 1$ this reduces to the poissonian case. For $\alpha > 1$ the maximum of the distribution occurs at a finite time rather than $t = 0$. This may be more appropriate for the description of traffic flow where there is a gap between successively arriving particles. The mean time of blocking is equal to

$$\langle t \rangle = \frac{\alpha}{\lambda}\left(1 + \frac{\Gamma(\alpha)}{\Gamma(\alpha) - \gamma(\tau, \alpha)}\right), \qquad (17)$$

where $\Gamma(\alpha)$ and $\gamma(\alpha, x)$ are the Gamma and incomplete Gamma functions, respectively. When $\lambda \tau < 1$, one obtains

$$\langle t \rangle = \frac{1}{\lambda}\frac{\alpha!}{(\lambda\tau)^\alpha}. \qquad (18)$$

Figure 2 shows $\langle t \rangle$ versus $\lambda \tau$ for $\alpha = 2, 3, 4$. One observes an excellent agreement between simulation data (circles) and the exact formula, Eq.(17). As expected, the mean time $\langle t \rangle$ diverges when λ goes to zero. The asymptotic behavior, Eq.(18) provides a good approximation of simulation data when $\lambda \tau < 1$.

By taking $\alpha = 1$ in the Gamma distribution, which corresponds to a homogeneous Poisson process, the mean time of blockage is given by

$$\langle t \rangle = \frac{2 - e^{-\lambda\tau}}{\lambda(1 - e^{-\lambda\tau})}, \qquad (19)$$

a result previously obtained by using a master equation for the time evolution of the $q(n, t)$ [18].

The mean flux $j(t)$ can be obtained by using Eqs.(8,9)

$$j(t) = \sum_{n=1}^\infty \int_0^\infty dt'(1 - \psi_c(t')) \int_0^\infty \left[\prod_{i=0}^{n-1} dt_i \psi(t_i)\right]$$
$$\left[\prod_{j=1}^{n-1} \theta(t_j - \tau)\right] \delta\left(t - \sum_{i=0}^{n-1} t_i - t'\right)[\delta(t' - \tau)]. \qquad (20)$$

The multiple integral can be factorized and the flux is given by :

$$j(t) = \sum_{n=1}^\infty \int_0^\infty dt_0 \psi(t_0) \int_0^\infty dt'(1 - \psi_c(t')) \left[\int_\tau^\infty dt\psi(t)\right]^{n-1}$$
$$\delta\left(t - \sum_{i=0}^{n-1} t_i\right)[\delta(t' - \tau)]. \qquad (21)$$

In Laplace space, the summation over n can be performed and $\tilde{j}(u)$ is given by

$$\tilde{j}(u) = \frac{(1 - \psi_c(\tau))e^{-u\tau}\tilde{\psi}(u)}{1 - \int_\tau^\infty e^{-ut}\psi(t)dt}. \qquad (22)$$

With a Poisson distribution $\psi(t) = \lambda e^{-\lambda t}$, we have

$$\tilde{j}(u) = \frac{\lambda e^{-(u+\lambda)\tau}}{u + \lambda(1 - e^{-(u+\lambda)\tau})}. \qquad (23)$$

By taking the inverse Laplace transform, the mean flux $j(t)$ can be expressed as a series

$$j(t) = \lambda e^{-\lambda t} \sum_{n=1}^\infty \left[\frac{1}{n!}(\lambda(t - (n+1)\tau))^n \theta(\lambda(t - (n+1)\tau))\right], \qquad (24)$$

as obtained previously by using a master equation approach [19].

No particle exits the channel between 0 and τ; indeed, the flux is obviously equal to 0 in this interval and rises instantaneously to a maximum, $j_{\max} = \lambda e^{-\lambda\tau}$ which itself is maximum when $\lambda = \frac{1}{\tau}$, and then decreases to 0.

For a Gamma distribution with an integer value of α, the Laplace transform of the flux can be obtained explicitly, but increasing α it rapidly leads to lengthy expressions.

Figure 3 displays the time evolution of the mean flux $j(t)$ for different values of λ and $\alpha = 3$. In all cases, the flux becomes nonzero for $t > \tau$, corresponding to the exit of a first particle. For $\lambda\tau \geq 1$, $j(t)$ displays a strong maximum at a time t_m slightly larger than τ and decays
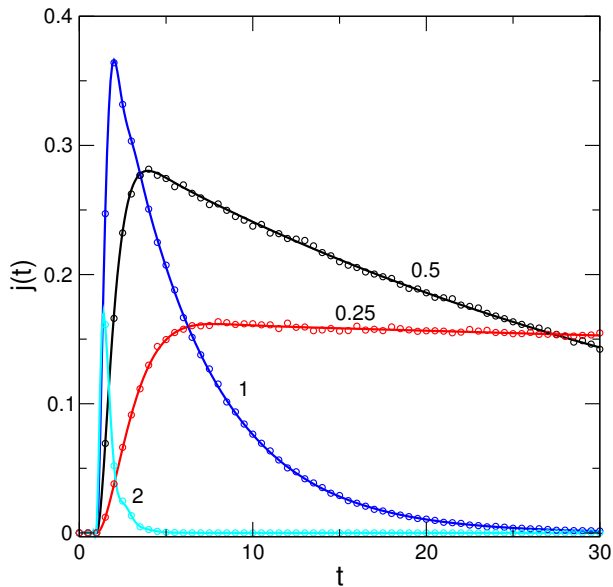
FIG. 3. (Color online) Mean flux $j(t)$ as a function of time for a Gamma distribution with $\alpha = 3$ and for different values of λ ($\tau = 1$). The solid lines show the exact expression, Eq.(22) and the circles show simulation results.

to 0. For $\lambda\tau = 0.5$, the maximum of the flux is shifted to a time $t_m \simeq 3\tau$ and the typical decay time is around 100τ. For $\lambda\tau = 0.25$, $j(t)$ increases up to a quasi-plateau and the typical decay time is larger than 1000τ, which corresponds to a physical situation where a large number of particles exit the channel before the definitive clogging. Note that for a given value of λ the flux is much larger than for a Poisson distribution. However, it approaches zero for sufficiently long times with a characteristic time equal to the mean blocking time.

We also consider the probability, $h(m, t)$, that blockage occurs in the interval $(0, t)$ and that during this time m particles exit the channel. The time evolution of this function is given by

$$\frac{dh(m,t)}{dt} = \int_0^\infty \prod_{i=0}^{m+1} dt_i \psi(t_i) \prod_{j=1}^{m} \theta(t_j - \tau)$$
$$\theta(\tau - t_{m+1})\delta\left(t - \sum_{i=0}^{m+1} t_i\right). \quad (25)$$

Two particles have to be in the channel for the system to block, so the interval between two consecutive particles has to be less than τ (the θ function). The previously entering particles exited the channel without blockage.

Taking the Laplace transform we obtain for $m \geq 0$

$$\tilde{h}(m, u) = \frac{\tilde{\psi}(u)}{u} \int_0^\tau \psi(t')e^{-ut'} dt' \left[\int_\tau^\infty dt\psi(t)e^{-ut}\right]^m. \quad (26)$$

The probability that the channel is blocked can be expressed as the sum over partial probabilities $h(n, t)$, namely $h(t) = \sum_{m=0}^\infty h(m, t)$. By using Eq.(26), one infers $\lim_{t\to\infty} h(t) = \lim_{u\to 0} uh(u) = 1$, as because blockage is certain to occur, a result valid for any distribution $\psi(t)$. Finally, we note the following sum rule, $\sum_{n\geq 0}(q_s(n, t) + h(n, t)) = 1$ - all configurations of the process are either blocked or unblocked.

For the Poisson process, an explicit expression can be obtained

$$\tilde{h}(m, u) = \frac{\lambda^{m+2}}{u(\lambda + u)^{m+2}} \left[1 - e^{-(\lambda+u)\tau}\right] e^{-(\lambda+u)m\tau}. \quad (27)$$

Performing the Laplace inversion we obtain $h(m, t)$ as obtained previously [19]. As expected, $h(m, t)$ is equal to zero for $t < m\tau$ corresponding to the minimum time necessary for m particles to exit the channel. For the Gamma distribution with $\alpha = 2$ we obtain

$$\tilde{h}(m, u) = \frac{1}{u(u+\lambda)^{2(m+2)}} (e^{-(u+\lambda)\tau m} \lambda^{2(m+2)}$$
$$(1 + (u+\lambda)\tau)^m (1 - e^{-\tau(u+\lambda)}(1 + (u+\lambda)\tau))). \quad (28)$$

For the Gamma distribution, we plot in Fig. 4 the time evolution of $h(m, t)$ as a function of time with $m = 0, 1, 2$ for $\alpha = 2$ and $\lambda = 2$. As expected, $h(m, t) = 0$ for $t < m\tau$, which can be explained by the fact that the minimum time for having a configuration where m particles exit the channel must be at least larger than $m\tau$. Similarly, the transient time associated with $h(m, t)$ increases with m, and corresponds to rare events when m increases.

C. Explicit Solution for $N = 3$

For the first three partial probabilities, there is no constraint and one easily obtains that $q(0, t) = 1 - \psi_c(t)$, and for $i = 1, 2$ the probabilities are given in terms of the Laplace transforms $q(i, u) = \left(\frac{1-\tilde{\psi}(t)}{u}\right)\tilde{\psi}(u)^i$. For a Poisson process, one recovers that $q(0, t) = e^{-\lambda t}$, $q(1, t) = \lambda te^{-\lambda t}$ and $q(2, t) = \frac{(\lambda t)^2}{2}e^{-\lambda t}$. For $n > 2$, Eq.(2) becomes

$$q(n, t) = \int_0^\infty dt'(1 - \psi_c(t')) \int_0^\infty \prod_{i=0}^{n-1} dt_i \psi(t_i)$$
$$\times \prod_{j=1}^{n-2} \theta(t_j + t_{j+1} - \tau)\delta\left(t - \sum_{i=0}^{n-1} t_i - t'\right). \quad (29)$$

The constraint, imposed by the θ function, requires that the sum of two consecutive time intervals be less than τ.
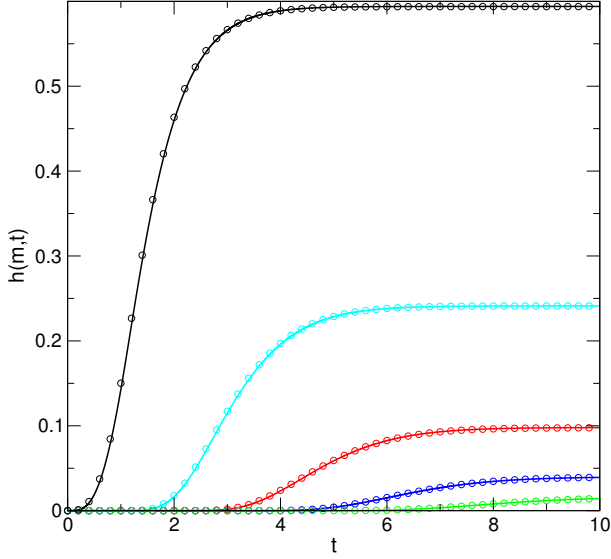
FIG. 4. (Color online) The probability $h(m,t)$ as a function of time for a Gamma distribution with $\alpha = 2$, $\lambda = 2$ and $m = 0, 1, 2, 3, 4$ (from top to bottom). The full curves show the exact expression, Eq. (28) and circles show simulation results.

Taking the Laplace transform of Eq.(29), one obtains

$$\tilde{q}(n,u) = \frac{\tilde{\psi}(u)(1 - \tilde{\psi}(u))}{u} \int_0^\infty dt \psi(t) e^{-ut} r(n-1, t, u),$$
(30)

where the auxiliary function $r(n-1, t, u)$ is given by

$$r(n-1, t, u) = \int_0^\infty \prod_{i=1}^{n-2} dt_i \psi(t_i) e^{-ut_i} \prod_{j=1}^{n-2} \theta(t_j + t_{j+1} - \tau),$$
(31)

where $t_{n-1} = t$. A recurrence relation can be written for $r(n,t,u)$

$$r(n,t,u) = \int_{\max(\tau - t, 0)}^\infty dt' \psi(t') e^{-ut'} r(n-1, t', u),$$
(32)

with $r(1, t, u) = 1$.

Let us introduce the generating function $G_r(z, t, u)$ defined as

$$G_r(z, t, u) = \sum_{n=1}^\infty z^{n-1} r(n, t, u).$$
(33)

The explicit form of which is given in Appendix A.

For $n = 0, 1, 2$ the partial probabilities $q(n, t)$ correspond to those of a Poisson process. For $n > 2$, by using the generating function $G_r(z, t, u)$, the Laplace transform of $q(n, t)$ is given by

$$\tilde{q}(n, u) = \frac{\lambda}{(\lambda + u)^2} \int_0^\infty dt \lambda e^{-(\lambda + u)t} \left. \frac{\partial^{n-2} G_r(z, t, u)}{\partial z^{n-2}} \right|_{z=0}.$$
(34)

After some calculation, one obtains

$$\begin{aligned} q(3, t) &= \theta(t - \tau) \lambda^3 e^{-\lambda t} \left[\tfrac{1}{2} \tau (t - \tau)^2 + \tfrac{1}{6} (t - \tau)^3 \right] \\ q(4, t) &= \lambda^4 e^{-\lambda t} \left[\theta(t - \tau) \frac{(t - \tau)^4}{12} - \theta(t - 2\tau) \frac{(t - 2\tau)^4}{24} \right]. \end{aligned}$$
(35)

By using Eqs.(30) and (33), the Laplace transform of the survival probability $p_s(t)$ is

$$\begin{aligned} \tilde{p}_s(u) =& \tilde{q}(0, u) + \tilde{q}(1, u) + \\ & \frac{\psi(u)(1 - \tilde{\psi}(u))}{u} \int_0^\infty dt \psi(t) e^{-ut} G_r(1, t, u). \end{aligned}$$
(36)

By inserting the solution of Eq.(A2), the Laplace transform of the survival probability is given by

$$\begin{aligned} \tilde{p}_s(u) =& \frac{\lambda}{(\lambda + u)^2} \left[1 + \frac{u}{\lambda} + A(1, u) \left[1 + \frac{\lambda}{s_2} (1 - e^{-s_2 \tau}) \right] \right. \\ & \left. + B(1, u) \left[1 + \frac{\lambda}{s_1} (1 - e^{-s_1 \tau}) \right] \right], \end{aligned}$$
(37)

where

$$\begin{aligned} A(1, u) &= \frac{\lambda e^{s_2 \tau} (s_2 - \lambda)(s_1 + s_2)}{\Delta} \\ B(1, u) &= \frac{\lambda e^{s_1 \tau} (s_1 - \lambda)(s_1 + s_2)}{\Delta}, \end{aligned}$$
(38)

with

$$\Delta = e^{(s_1 + s_2)\tau} s_1 s_2 (s_1 - s_2) + \lambda (s_2^2 e^{s_2 \tau} - s_1^2 e^{s_1 \tau}).$$
(39)

From the generating function, $G_r(z, t, u)$, one can also obtain global quantities, like the mean blocking time $\langle t \rangle = \tilde{p}_s(0)$

Let $g = \sqrt{|1 - 4e^{-\lambda \tau}|}$ and $\nu = \frac{\lambda \tau}{2}$ then, after some calculation, one obtains for $\lambda \tau > 2 \ln(2)$

$$\lambda \langle t \rangle = \frac{2 e^\nu \sinh(\nu) + g e^{\lambda \tau}}{-g - 2 \sinh(\nu) e^{-\nu} + e^\nu (\sinh(\nu) + g \cosh(\nu))} + 1.$$
(40)

and for $\lambda \tau < 2 \ln(2)$

$$\lambda \langle t \rangle = \frac{2 e^\nu \sin(\nu) + g e^{\lambda \tau}}{-g - 2 \sin(\nu) e^{-\nu} + e^\nu (\sin(\nu) + g \cos(\nu))} + 1.$$
(41)

Fig. 5 shows the mean blocking time $\langle t \rangle$ of the models with $N = 2, 3, 4, 5$ for a Poisson distribution obtained by simulation and for $N = 2, 3$ by using the analytic expressions. We observe a perfect agreement between simulation data and exact expressions for $N = 2$ Eq.(19) and $N = 3$ Eq.(40,41). More generally, one observes a divergence of the mean blocking time as $\lambda \tau$ goes to 0 and indeed performing a first-order expansion of Eq.(41) in $\lambda \tau$ gives

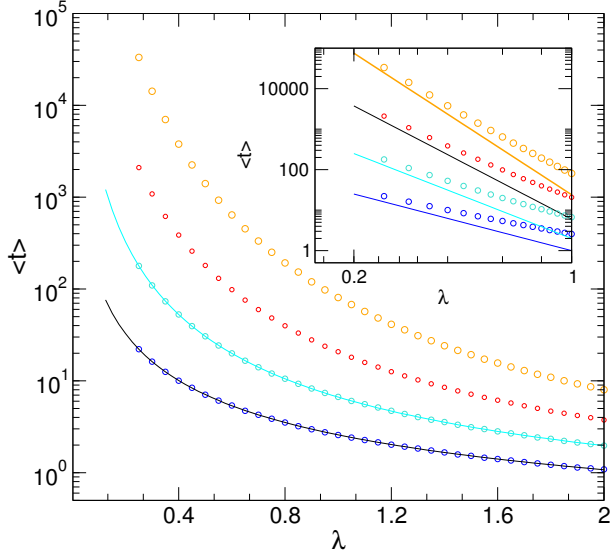$$\langle t \rangle \simeq \frac{2\tau}{(\lambda \tau)^3}.$$
(42)

FIG. 5. (Color online) Mean time of blocking as a function of the intensity $\lambda\tau$ for $N = 5, 4, 3, 2$, top-to-bottom, from numerical simulation (circles) and Eq.(19) $N = 2$ and Eqs.(40,41) $N = 3$ (full curves) for a Poisson distribution. The inset compares the asymptotic formula, Eq.(54), with simulation results.
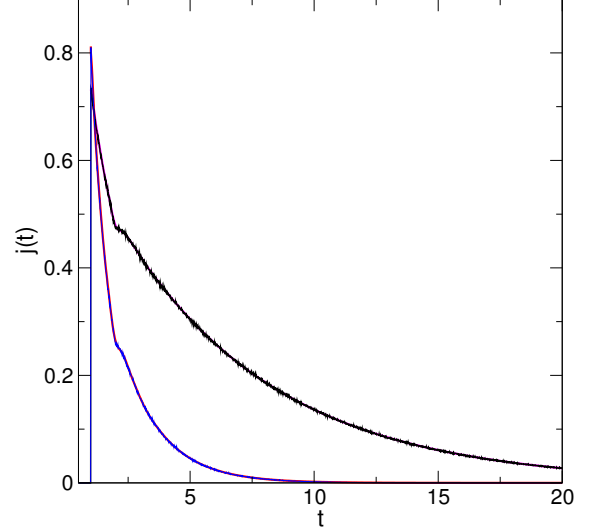

FIG. 6. (Color online) Mean flux $j(t)$ as a function of time for a Poisson distribution for $\lambda = 1, 2$ ($\tau = 1$). The solid lines show the exact expression, (inverse Laplace transform of Eq.(45)) accurately matching simulation results (noisy lines).

The mean flux $j(t)$ can be also obtained by using Eq.(9) and the auxiliary functions $r(n, t, u)$ and it comes for the Laplace transform $\tilde{j}(n, u)$ (for $n \geq 1$)

$$\tilde{j}(n, u) = e^{-u\tau}\tilde{\psi}(u)\left((1 - \psi_c(\tau))\int_0^\infty dt e^{-ut}\psi(t)r(n - 1, t, u)\right.$$
$$\left. + \int_0^\tau dt\psi(t)(1 - \psi_c(\tau - t))r(n - 1, t, u)\right).$$
(43)

By summing over n (accounting for the boundary terms $\tilde{j}(1, u)$ and $\tilde{j}(2, u)$, the Laplace transform $\tilde{j}(u)$ is expressed as

$$\tilde{j}(u) = e^{-u\tau}\tilde{\psi}(u)(1 - \psi_c(\tau))\int_0^\infty dt e^{-ut}\psi(t)G_r(1, t, u)$$
$$+ e^{-u\tau}\tilde{\psi}(u)\int_0^\tau dt\psi(t)(1 - \psi_c(\tau - t))G_r(1, t, u)$$
$$+ \tilde{j}(1, u).$$
(44)

By using Eq.(A3) and the expression of the generating function $G_r(1, t, u)$, the Laplace transform of the flux can be expressed as

$$\tilde{j}(u) = \frac{\lambda e^{-(u+\lambda)\tau}}{\lambda + u}\left[A(1, u)\left(e^{s_1\tau}\left(1 + \frac{\lambda}{s_1}\right) - \frac{\lambda}{s_1}\right)\right.$$
$$\left. + B(1, u)\left(e^{s_2\tau}\left(1 + \frac{\lambda}{s_2}\right) - \frac{\lambda}{s_2}\right)\right],$$
(45)

where $A(1, u)$ and $B(1, u)$ are given by Eq.(38).

Because the right-hand-side of Eq.(45) can be factorized by $e^{-u\tau}$, it implies that $j(t) = 0$ for $t < \tau$, which corresponds to the minimum time for a particle to exit the channel.

The mean flux $j(t)$ is plotted as as function of time for $\lambda = 1, 2$ with a Poisson distribution for $\lambda = 1, 2$ (Fig. 6). A discontinuity appears at $t = \tau$ where the flux is maximum $j(\tau) = \lambda$. At $t = \tau$, the flux is given by

$$j(\tau) = \lambda(1 + \lambda\tau)e^{-\lambda\tau} \quad (46)$$

which corresponds to events where a particle exits between t and $t + dt$ such that 0 or 1 particle is still in the channel. The flux decay exhibits a visible cusp at $t = 2\tau$ which corresponds to the non-analytical structure of the solution. At long times, the flux decays to 0, with a typical time which becomes larger when λ decreases.

The joint probability $h(m, t)$ can also be obtained with the function $r(n, t, u)$. For $m \geq 1$ its time evolution is given by

$$\frac{dh(m, t)}{dt} = \int_0^\infty \prod_{i=0}^{m+2} dt_i\psi(t_i)\prod_{j=1}^m \theta\left(t_j + t_{j+1} - \tau\right)$$
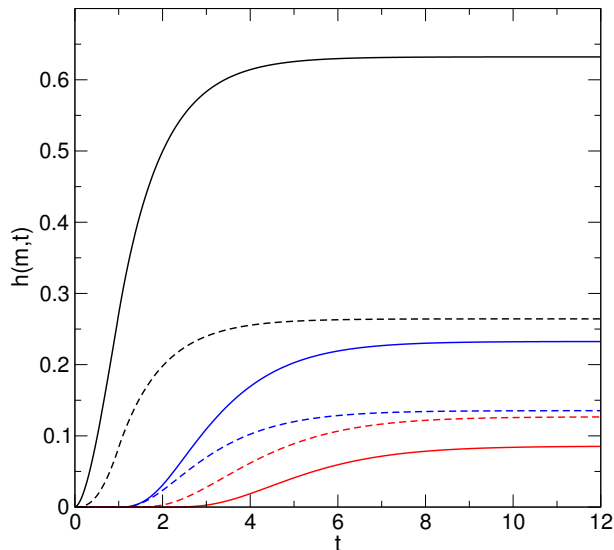$$\theta(\tau - t_{m+1} - t_{m+2})\delta(t - \sum_{i=0}^{m+2} t_i). \quad (47)$$

FIG. 7. (Color online) Probability distributions $h(m,t)$ versus time t for a Poisson distribution, for $m = 0, 1, 2$ (from top to bottom) and $\lambda = 1$. The solid lines correspond to the model with $N = 2$ and dashed lines correspond to the model with $N = 3$.

Taking the Laplace transform gives

$$\tilde{h}(m,u) = \frac{\tilde{\psi}(u)}{u} \int_0^\infty \prod_{i=1}^{m+2} dt_i \psi(t_i) e^{-ut_i}$$
$$\prod_{j=1}^{m} \theta(t_j + t_{j+1} - \tau)\theta(\tau - t_{m+1} - t_{m+2}) \quad (48)$$

that can be expressed using the function $r(n,t,u)$ as

$$\tilde{h}(m,u) = \frac{\tilde{\psi}(u)}{u} \int_0^\tau dt \psi(t) e^{-ut}$$
$$\int_0^{\tau-t} dt' \psi(t') e^{-ut'} r(m+1, t', u). \quad (49)$$

Figure 7 shows the time evolution of the probability distributions $h(m,t)$ for a Poisson distribution with $\lambda = 1$. The probability that zero or one particle ($m = 0, 1$) particle exits is smaller for $N = 2$ than for $N = 3$. For $m \geq 2$ the order reverses (e.g. for $m = 2$, case shown). This is because, for a given value of $\lambda\tau$, more particles exit before blockage as N increases.

D. Partial Solution for $N \geq 4$

We have seen that for $N = 3$ the product of Heaviside functions in Eq.(2) leads to a simple recurrence relation Eq.(32). For $N \geq 4$ the task is much more difficult because one needs to introduce auxiliary functions that depend on $N - 2$ time variables. These functions are related by an integral equation that cannot be converted

to an ordinary differential equation. We therefore propose an approximate treatment of the dynamics. For the model where the blockage occurs when N particles enter the channel between $t - \tau$ and t, the first $N - 2$ partial probabilities $q(i,t)$ obey differential equations identical to those of a Poisson process

$$\frac{dq(0,t)}{dt} = -\lambda q(0,t) \quad (50)$$

and

$$\frac{dq(n,t)}{dt} = -\lambda q(n,t) + \lambda q(n-1,t), \; 1 \leq n \leq N-1. \quad (51)$$

For $n > N - 1$, there is a constraint on the interval between particles, but for $n = N$, the time evolution is simply given by

$$\frac{dq(N,t)}{dt} = -\lambda q(N,t) + \lambda \sum_{s=0}^{N-2} \frac{(\lambda\tau)^s}{s!} e^{-\lambda\tau} q(N-1-s, t-\tau).$$
$$(52)$$

The gain term reflects the fact that blockage only occurs with N particles, $N - 1$ terms correspond to the cases where there may be from 0 to $N - 1$ particles in the channel.

For $n > N$, the dynamics of $q(n,t)$ for $n > N$ can be approximated as follows

$$\frac{dq(n,t)}{dt} = -\lambda q(n,t) + \lambda q(n-1, t-\tau) e^{-\lambda\tau}$$
$$+ \lambda \sum_{s=1}^{N-2} \int_0^\tau dt_1 K_s(t_1) e^{-\lambda\tau} q_s(n-1-s, t-\tau-t_1).$$
$$(53)$$

where we have introduced a kernel $K_s(t)$. We then consider two physical situations. In the first, the last s particles are assumed to have entered the channel in an infinitesimal time interval and the kernel is given by $K_s(t) = \frac{(\lambda\tau)^s}{s!}\delta(t)$. This choice overestimates the survival probability. $N-2$ particles can be in the channel (so can enter between $t - \tau$ and t) when a new particle enters. The other particles enter between time 0 and $t - \tau$. This fails to take into account some blocking. In the second case we take $K_s(t) = \lambda\frac{(\lambda t)^{s-1}}{(s-1)!}e^{-\lambda t}$ which is proportional to the probability that $s-1$ particles enter in $(0,t)$. This choice underestimates the survival probability. When a particle enters at time t there may be a maximum of $N-2$ particles in the channel to avoid blocking. If a particle arrives at a time t_1 between $t - \tau$ and t, there may be a maximum of $N - 3$ particles between t and $t - t_1$ and no particle between $t - t_1$ and $t - t_1 - \tau$.

Taking the Laplace transform of Eq.(53), we calculate two different generating functions corresponding to the two kernels, and the corresponding mean survival times. These bracket the exact value and for $\lambda\tau \ll 1$ the two solutions approach the same limit:

$$\langle t \rangle = \frac{(N-1)!}{(\lambda\tau)^N}. \quad (54)$$

To obtain exact results for $N \geq 4$ is a challenging problem. We therefore finish this section by presenting some numerical results that illustrate the general trends. The inset of Fig. 5 compares the asymptotic behavior for mean blocking time, Eq.(54) with simulation results for $N = 2$ to $N = 5$. We observe that the scaling law provides provides a good description of the process for $\lambda\tau \leq 0.5$.

In Fig. 8 we present numerical results for the mean flux of exiting particles as a function of time. This quantity acquires a non-zero, maximum, value at $t = \tau$ given by

$$j(\tau) = \lambda \sum_{i=1}^{N-2} \frac{(\lambda\tau)^i}{i!} e^{-\lambda\tau}. \tag{55}$$

This expression corresponds to events where a particle exits between t and $t + dt$ such that $0, 1.., N - 2$ particles are still in the channel. For $t > \tau$, we observe a drastic increase of the characteristic decay time as N increases (see the lower figure of Fig. 8). For $N = 2$, $j(t)$ is very small for $t > 3\tau$, while for $N = 7$, the flux is almost constant during two decades. More generally, the decay occurs on a timescale comparable to the mean time to blockage, Eq. (54), that diverges rapidly as N increases.

III. CORRELATIONS

A. General Results

Here we investigate how the entrance of new particles is constrained by the previously entering particles. To that end, the relevant quantity is the time correlation function $C(t)$ that represents the density function that any two particles have a time separation t. $C(t)$ can be expressed as the sum of partial correlation functions $c(n, t)$ that correspond to the probability density that the first and last particles of sequence of $n + 1$ particles are separated by t.

$$C(t) = \sum_{n=1}^{\infty} c(n, t). \tag{56}$$

The partial correlation function $c(n, t)$, the joint probability of having a particle at $t = 0$ and the nth particle at time t, can be written as

$$c(n, t) = \int_0^\infty \prod_{i=1}^{N-2+n} dt_i c^{(N-2)}(t_1, ..., t_{N-2}) \prod_{j=N-1}^{N-2+n} \psi(t_j)$$
$$\times \left[\prod_{j=1}^{n} \theta \left(\sum_{m=0}^{N-2} t_{j+m} - \tau \right) \right] \delta \left(t - \sum_{i=N-1}^{N+n-2} t_i \right), \tag{57}$$

where $c^{(N-2)}(t_1, ..., t_{N-2})$ is the joint probability of having $N - 1$ particles such that the first and the second particles are separated by a duration of t_1, the second
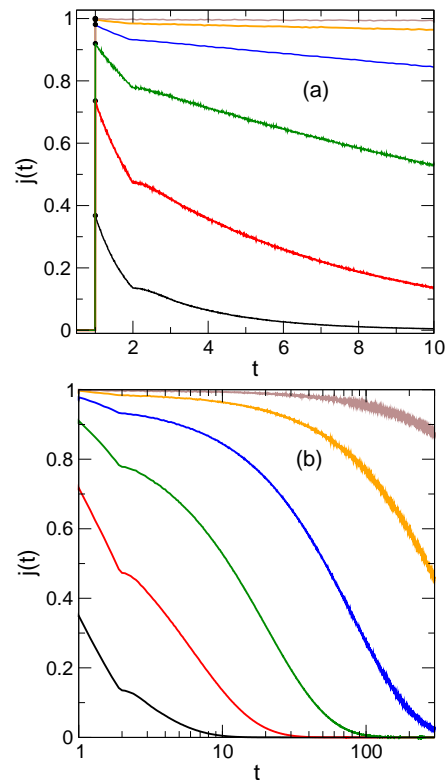

FIG. 8. (Color online) Mean flux $j(t)$ versus time t for $N = 2, 3, 4, 5, 6, 7$ (from bottom to top) for a Poisson distribution with $\lambda = 1$. Top: short time behavior. At $t = 1$, circles correspond to the exact values of the mean flux, Eq.(55). Bottom: linear-log plot showing the long-time behavior. ($\tau = 1$)

and the third particles by a duration of $t_2, ..$ and the $N - 2$ and $N - 1$ particles by t_{N-2}. We can write this probability as

$$c^{(N-2)}(t_1, ..., t_{N-2}) = \int dt_0 c^{(N-2)}(t_0, ..., t_{N-3})$$
$$\times \psi(t_{N-2})\theta(\sum_{j=1}^{N-2} t_j - \tau). \tag{58}$$

This definition of the correlation function considers all trajectories, including those that end before a given time t. As a result, the correlation function approaches zero at long times. It seems more interesting to keep only trajectories which survive until time t.

To generate an infinite sequence of particles corresponding to a trajectory of the model, let us consider the following rejection-free algorithm: Accounting for the constraints of the model (less than N particles must enter the channel in the time interval τ), we introduce the

discrete stochastic equation

$$t_n = \max(\tau - \sum_{j=1}^{N-2} t_{n-j}, 0) + \eta, \qquad (59)$$

where η is a random number generated from the ψ distribution and $t_{n-j}, j = 1, N-2$ are the time intervals of between the $N - 2$ previously entering particles.

In order to compute the correlation function associated with this rejection-free algorithm, we replace $\psi(t_i)$ in Eqs.(57,58) with

$$\psi(t_i - \max(\tau - \sum_{j=1}^{N-2} t_{i-j}, 0)). \qquad (60)$$

The partial correlation function $c(n, t)$ can be also expressed as the average over the event of having a first and $n + 1th$ particles separated by a time duration t

$$c(n, t) = \langle \delta(t - \sum_{i=1}^{n} t_i) \rangle. \qquad (61)$$

The conservation of the probability reads

$$\int_0^\infty dt c(n, t) = 1. \qquad (62)$$

By summing over n, the integral correlation function $C(t)$ is given at long time by

$$\int_0^t dt' C(t') = \langle n(t) \rangle, \qquad (63)$$

where $\langle n(t) \rangle$ is the mean number of particles along a trajectory for a time duration t. At large t, this quantity goes to a constant because we only consider trajectories that have survived. By using that $C(t)$ goes to a constant at long times (due to to the decay of the memory between particles that entered with a large time difference) ($C(t) \to C_\infty$), we infer that $C_\infty = 1/\bar{t}$ where $\bar{t}$ is the average separation in time between successive particles.

We now focus on $N = 2$ and $N = 3$ by using the rejection-free trajectories for which exact solutions can be obtained.

B. Explicit Solution for $N = 2$

The partial correlation function $c(n, t)$ is simply given as the product of integrals on each independent interval. Eq.(59) is very simple $t_n = \tau + \eta$, which means that $\psi(t)$ is replaced with $\psi(t - \tau)$ (see Eq.(60)). Therefore, the Laplace transform of $c(n, t)$ is given by

$$\tilde{c}(n, u) = \left(\int_\tau^\infty dt \psi(t - \tau) e^{-tu} \right)^n = \tilde{c}(1, u)^n. \qquad (64)$$

This results from the fact that successive events are not correlated.

Inserting Eq.(64) in Eq.(56) we obtain

$$\tilde{C}(u) = \frac{\tilde{c}(1, u)}{1 - \tilde{c}(1, u)}. \qquad (65)$$

At long times, $C(t)$ approaches a constant value corresponding to a constant mean density. By using the factorization property, $\tilde{c}(n, u) = \tilde{c}(1, u)^n$, and the expansion $\tilde{c}(1, u) = \tilde{c}(1, 0) + u \partial \tilde{c}(1, u)/\partial u|_{u=0} + O(u^2)$ one can show that $C(\infty) = \lim_{u \to 0} u\tilde{C}(u) = 1/\bar{t}$ where $\bar{t} = \int_0^\infty tc(1, t)dt = -\partial \tilde{c}(1, u)/\partial u|_{u=0}$ is the average interval between particles. That is, the smaller the average separation in time between successive particles, the larger the steady state value of the time correlation function.

For a Poisson distribution $\psi(t) = \lambda e^{-\lambda t}$ we find

$$\tilde{C}(u) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda + u} \right)^n e^{-nu\tau}. \qquad (66)$$

The inverse Laplace transform gives an explicit expression

$$C(t) = \lambda \sum_{n=1}^{\infty} \theta(\lambda(t - n\tau)) \frac{(\lambda(t - n\tau))^{n-1} e^{-\lambda(t-n\tau)}}{(n-1)!}. \qquad (67)$$

Figure 9(a) shows $C(t)$ for two values of $\lambda\tau$. As expected, $C(t)$ is strictly equal to 0 for $t < \tau$ since no particle can be inserted if the delay between two successive particles is less than τ, corresponding to a strong anti-correlation. $C(t)$ reaches a maximum at $t = \tau$ where $C(\tau) = \lambda$ and then, after a short oscillatory period, attains a stationary value. Therefore after a few τ there is no remaining correlation between entering and exiting particles. Note that a cusp is present at $t = 2\tau$, a similar behavior observed for the other quantities such as the flux and the survival probability. In the long time limit $C(t = \infty) = \lim_{u \to 0} u\tilde{C}(u) = \lambda/(1 + \lambda\tau)$.

It is also interesting to note that correlation function Eq.(67) corresponds to the density correlation function of the positions of the particle centers in a hard rod fluid of density ρ with $\lambda = \rho/(1 - \rho)$.

C. Explicit Solution for $N = 3$

For $N = 3$, the discrete stochastic equation, Eq.(59), becomes

$$t_n = \max(\tau - t_{n-1}, 0) + \eta, \qquad (68)$$

where t_n denotes the time interval between the $n-1$ and n particles and η is a random number chosen with an exponential probability distribution $\lambda e^{-\lambda t}$. In queuing theory this equation is known as the Lindley-type equation [29–31].
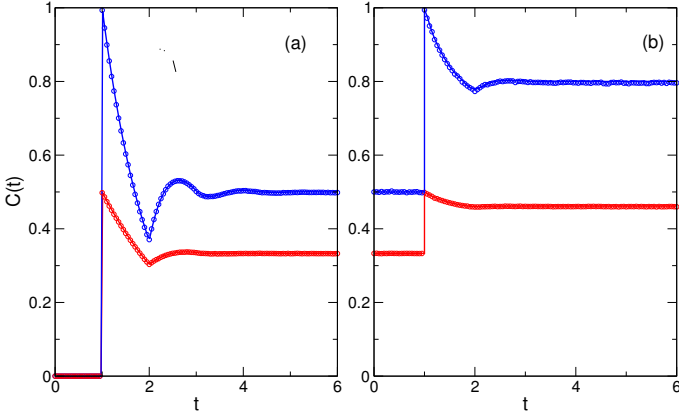
FIG. 9. (Color online) Correlation functions for (a) $N = 2$ and (b) $N = 3$ for $\lambda = 1$ and 0.5 (lower curves). The solid lines correspond to Eq. (67) ($N = 2$) and Eq. (79) ($N = 3$) and circles to numerical simulations.

For the Poisson distribution $\psi(t)$, Eqs.(57,58) with Eq.(60) gives

$$c(n,t) = \int_0^\infty dt_0 c(1,t_0)\delta\left(t - \sum_{i=1}^n t_i\right)$$
$$\prod_{i=1}^n \left(\int_{\max(\tau-t_{i-1},0)}^\infty dt_i \lambda e^{-\lambda(t_i - \max(\tau-t_{i-1},0))}\right) \tag{69}$$

and

$$c(1,t) = \int_{Max(\tau-t,0)}^\infty dt_1 c(1,t_1)\lambda e^{-\lambda(t - Max(\tau-t_1,0))}. \tag{70}$$

Note that the constraint applies to two consecutive intervals, i.e the arrival time between three consecutive particles is greater than τ. Consequently, the partial correlation $\tilde{c}(n,u)$ is never the product of smaller correlation functions, as for the $N = 2$ model.

Because the kinetics were obtained exactly in the previous section only for the Poisson distribution, we restrict our analysis to this distribution.

From Eq.(70), one easily shows that $c(1,t)$ is constant for $t > \tau$. For $t < \tau$, by taking the derivative of Eq.(70), one obtains

$$\frac{dc(1,t)}{dt} = \lambda(-c(1,t) + \theta(\tau - t)c(1, \tau - t)) \tag{71}$$

whose solution is

$$c(1,t) = \frac{\lambda}{1 + \lambda\tau}(\theta(\tau - t) + e^{-\lambda(t-\tau)}\theta(t - \tau)). \tag{72}$$

One can easily obtain the average time between two consecutively particles in a trajectory.

$$\bar{t} = \int_0^\infty dt c(1,t)t = \frac{(\lambda\tau + 1)^2 + 1}{2\lambda(\lambda\tau + 1)}. \tag{73}$$

As might be expected, when the intensity $\lambda\tau$ is high, the probability distribution is uniform within the first interval $[0,\tau]$ and equal to $\frac{1}{2}$. Conversely, when $\lambda\tau$ tends to 0 the effect of the constraint is negligible, $\bar{t}$ diverges as $\frac{1}{\lambda}$, corresponding to the Poisson distribution.

It is easy to calculate the first few partial correlation functions by direct integration of Eq.(69): for instance, the probability $c(2,t)$ is given by

$$c(2,t) = \frac{\lambda^2 t}{\lambda\tau + 1}e^{-\lambda(t-\tau)}\theta(t - \tau). \tag{74}$$

To obtain a general expression of $c(n,t)$, we first take the Laplace transforms of Eq.(69)

$$\tilde{c}(n,u) = \int_0^\infty dt c(1,t)e^{-ut}m(n,t), \tag{75}$$

where $m(n,t)$ is auxiliary function given by

$$m(n,t) = \int_{\max(\tau-t,0)}^\infty dt' \lambda e^{-((u+\lambda)t' - \lambda\max(\tau-t,0))}m(n-1,t'). \tag{76}$$

The initial condition is obviously, $m(1,t) = 1$.

Let us introduce the generating function $G_m(z,t,u)$ of the auxiliary functions $m(n,t)$

$$G_m(z,t,u) = \sum_{n=1}^\infty z^{n-1}m(n,t). \tag{77}$$

After some calculation, one obtains $G_m(z,t,u)$ (see Appendix B), and using Eq.(56), the Laplace transform of the correlation function is given

$$\tilde{C}(u) = \int_0^\infty dt c(1,t)G_m(1,t,u)e^{-ut}$$
$$= \frac{\lambda}{1 + \lambda\tau}\left(\int_0^\tau dt G_m(1,t,u)e^{-ut} + G_m(1,\tau,u)\frac{e^{-u\tau}}{u+\lambda}\right). \tag{78}$$

By inserting Eq.(B7) in Eq.(78) we obtain

$$\tilde{C}(u) = \frac{\lambda}{(u+\lambda)(1+\lambda\tau)}\left[A_1(1,u)\left(-\frac{e^{-s_2\tau}(\lambda+s_1) - u - \lambda}{s_2}\right)\right.$$
$$+ B_1(1,u)\left(-\frac{e^{-s_1\tau}(\lambda+s_2) - u - \lambda}{s_1}\right)$$
$$\left.+ \frac{(u+\lambda)^2 + e^{-u\tau}\lambda(u^2\tau + \lambda(u\tau - 1))}{u(u+\lambda-\lambda e^{-u\tau})}\right]. \tag{79}$$

Figure 9(b) displays the correlation function $C(t)$ for $N = 3$ versus time (with $\tau = 1$). As expected for $t \leq \tau$, $C(t)$ is constant and is equal to $\frac{\lambda}{1+\lambda\tau}$, because $c(n,t) = 0, n > 1$, and $c(1,t)$ is given by Eq.(72), which is constant and different from 0 in this time interval. One also observes a discontinuity at $t = \tau$ and a long time limit equal to $\frac{2\lambda(1+\lambda\tau)}{2+2\lambda\tau+\lambda^2\tau^2}$. We verify that, as for $N = 2$, this is equal to $1/\bar{t}$ with $\bar{t}$ given by Eq.(73).

Comparing the correlation functions for $N = 2$ and $N = 3$ for the same values of λ we note that the steady state values are higher for $N = 3$ corresponding to a shorter time interval between particles in the steady state. The oscillations are less pronounced for $N = 3$ and decay more quickly than for $N = 2$ due to the greater constraint imposed by the channel. In general the time correlation is flatter for larger N corresponding to weaker two-body correlation.

IV. DISCUSSION

In this article we have developed a theoretical approach to describe blocking that occurs in channels with limited carrying capacity. These models are relevant for a variety of applications spanning a range of lengthscales including vehicular and pedestrian traffic flow, filtration of particulate matter and the flow of molecules through nanotubes. The results presented here generalize the model introduced by Gabrielli et al. [18, 19] in which blocking is triggered by the simulataneous presence of only two particles in a channel. It was also assumed that the particles enter according to a Poisson distribution.

The results presented here go beyond these initial models by considering an arbitrary threshold where blocking is triggered by the simultaneous presence of $N > 2$ particles. In addition, the particle ingress is no longer limited to a Poisson distribution, but follows a general distribution of entry times. This necessitated the introduction of a novel integral representation of the n particle survival probabilities.

For $N = 3$, we have presented exact solutions of the model for the Poisson distribution and obtained expressions for all physically relevant quantities including the survival probability, the mean time to blockage, Eqs.(40,41), the statistics of exiting particles and the exiting flux. In addition we have also investigated the interparticle time correlation functions.

For $N \geq 4$ obtaining an exact solution appears to be very challenging, but we have analyzed the generic features of the model using numerical simulation. One of the key results is that the mean time to blockage for small intensity and arbitrary N diverges as a power of N, Eq.(54). This results from the fact that as N increases, the channel exerts a weaker constraint on the incoming stream and blocking is less likely. A similar trend occurs for the exiting flux that remains nearly constant for times comparable to the mean blocking time.

Future directions include the development of a multichannel model, which can be applicable to filtration phenomenon [1], and to consider systems with diffusive motion that are relevant for transport through biological or synthetic nanotubes [32].

V. ACKNOWLEDGEMENTS

Appendix A: $G_r(z, t, u)$ for the $N = 3$ model

Multiplying Eq.(32) by z^{n-1} and summing over n, one obtains that

$$G_r(z, t, u) = 1 + z \int_{\max(\tau - t, 0)}^{\infty} dt' \psi(t') e^{-ut'} G_r(z, t', u). \tag{A1}$$

For $t > \tau$ $G_r(z, t, u)$ is constant, i.e. $G_r(z, t, u) = G_r(z, \tau, u)$. For $t < \tau$, it is convenient to express the time evolution of $G_r(z, t, u)$ as follows: taking the first two partial derivatives of $G(z, t, u)$ with respect to t, one obtains the ordinary differential equation

$$\frac{\partial^2 G_r(z, t, u)}{\partial t^2} = \left(-\frac{\dot{\psi}(\tau - t)}{\psi(\tau - t)} + u \right) \frac{\partial G_r(z, t, u)}{\partial t} - z^2 \psi(\tau - t) \psi(t) e^{-u\tau} G_r(z, t). \tag{A2}$$

By using Eq.(A1), the differential is supplemented by two boundary conditions

$$\begin{cases} G_r(z, 0, u) & = 1 + z G_r(z, \tau, u) \int_\tau^\infty dt' \psi(t') e^{-ut'} \\ \left. \frac{\partial G_r(z, t, u)}{\partial t} \right|_{t=\tau} & = z \psi(0) G_r(z, 0, u). \end{cases} \tag{A3}$$

Eq.(A2) cannot be solved analytically in general but for a Poisson distribution it becomes

$$\frac{\partial^2 G_r(z, t, u)}{\partial t^2} = (\lambda + u) \frac{\partial G_r(z, t, u)}{\partial t} - (z\lambda)^2 e^{-(u+\lambda)\tau} G_r(z, t, u), \tag{A4}$$

with the boundary condition given by Eq.(A3) with $\psi(t) = e^{-\lambda t}$.

The solutions of the characteristic equation of Eq.(A4) are

$$s_{1,2}(z, u) = \frac{(\lambda + u) \pm \sqrt{(\lambda + u)^2 - 4(z\lambda)^2 e^{-(\lambda+u)\tau}}}{2} \tag{A5}$$

and the generating function is given by $G_r(z, t, u) = A(z, u) e^{s_1(z,u)t} + B(z, u) e^{s_2(z,u)t}$ where $A(z, u)$ and $B(z, u)$ are determined by Eq.(A3) adapted to a Poisson process.

Appendix B: $G_m(z, t, u)$ for the $N = 3$ model

Inserting Eq.(76) in Eq.(77), we obtain

$$G_m(z, t, u) = 1 + z \int_{\max(\tau-t, 0)}^{\infty} dt' G_m(z, t', u)$$
$$\lambda e^{-((u+\lambda)t' - \lambda \max(\tau-t, 0))}. \tag{B1}$$

For $t > \tau$ the generating function is constant, $G_m(z, t, y) = G_m(z, \tau, u)$. For $t < \tau$, by taking the two partial derivatives of the integral equation Eq.(B1), one obtains

$$\frac{\partial^2 G_m(z, t, u)}{\partial^2 t} = z\lambda u e^{-u(\tau-t)} \left(G_m(z, \tau - t, u) \right.$$
$$\left. + \frac{\partial G_m(z, \tau - t, u)}{\partial t} \right) - \lambda \frac{\partial G(z, t, u)}{\partial t}. \tag{B2}$$

Simplifying we obtain

$$\frac{\partial^2 G_m(z, t, u)}{\partial t^2} = u \frac{\partial G_m(z, t, u)}{\partial t} + (u\lambda + \lambda^2 - (\lambda z)^2 e^{-u\tau}) \times$$
$$G_m(z, t, u) - u\lambda - \lambda^2 - \lambda^2 z e^{-u(\tau-t)}. \tag{B3}$$

with boundary conditions (from Eq.(B1)).

$$\begin{cases} G_m(z, 0, u) & = 1 + z G_m(z, \tau, u) \frac{\lambda e^{-u\tau}}{u+\lambda} \\ \frac{\partial G_m(z, t, u)}{\partial t}\Big|_{t=\tau} & = z\lambda G_m(z, 0, u) - \lambda[G_m(z, \tau, u) - 1]. \end{cases} \tag{B4}$$

whose solution is given by

$$G_m(z, t, u) = A_1(z, u) e^{s_1 t} + B_1(z, u) e^{s_2 t}$$
$$+ \frac{(u\lambda + \lambda^2 + \lambda^2 z e^{-u(\tau-t)})}{u\lambda + \lambda^2 - (\lambda z)^2 e^{-u\tau}}, \tag{B5}$$

where $s_{1,2}$ are the roots of the characteristic equation

$$s_{1,2} = \frac{1}{2}(u \pm \sqrt{(u+2\lambda)^2 - 4z^2\lambda^2 e^{-u\tau}}). \tag{B6}$$

Finally we have

$$G_m(z, t, u) = \left(A_1(z, u) e^{s_1 t} + B_1(z, u) e^{s_2 t} \right.$$
$$+ \frac{(u\lambda + \lambda^2 + \lambda^2 z e^{-u(\tau-t)})}{u\lambda + \lambda^2 - (\lambda z)^2 e^{-u\tau}} \bigg) \theta(\tau - t)$$
$$+ G_m(z, \tau, u)\theta(t - \tau), \tag{B7}$$

where $A(z, u)$ and $B(z, u)$ are determined by the boundary conditions, Eq.(B4).

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