Mechanisms for the clustering of inertial particles in the inertial range of isotropic turbulence

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Mechanisms for the clustering of inertial particles in the inertial range of isotropic turbulence

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In this paper, we consider the physical mechanism for the clustering of inertial particles in the inertial range of isotropic turbulence. We analyze the exact, but unclosed, equation governing the radial distribution function (RDF) and compare the mechanisms it describes for clustering in the dissipation and inertial ranges. We demonstrate that in the limit $\mathrm{St_r} \ll 1$, where $\mathrm{St_r}$ is the Stokes number based on the eddy turnover timescale at separation r, the clustering in the inertial range can be understood to be due to the preferential sampling of the coarse-grained fluid velocity gradient tensor at that scale. When $\mathrm{St_r} \gtrsim \mathcal{O}(1)$ this mechanism gives way to a non-local clustering mechanism. These findings reveal that the clustering mechanisms in the inertial range are analogous to the mechanisms that we identified for the dissipation regime (see *New J. Phys.* **16**:055013, 2014). Further, we discuss the similarities and differences between the clustering mechanisms we identify in the inertial range and the "sweep-stick" mechanism developed by Coleman & Vassilicos (Phys. Fluids 21:113301, 2009). We show that the idea that initial particles are swept along with acceleration stagnation points is only approximately true because there always exists a finite difference between the velocity of the acceleration stagnation points and the local fluid velocity. This relative velocity is sufficient to allow particles to traverse the average distance between the stagnation points within the correlation timescale of the acceleration field. We also show that the stick part of the mechanism is only valid for $\mathrm{St_r} \ll 1$ in the inertial range. We emphasize that our clustering mechanism provides the more fundamental explanation since it, unlike the sweep-stick mechanism, is able to explain clustering in arbitrary spatially correlated velocity fields. We then consider the closed, model equation for the RDF given in Zaichik & Alipchenkov (Phys. Fluids. 19:113308, 2007) and use this, together with the results from our analysis, to predict the analytic form of the RDF in the inertial range for $\mathrm{St_r} \ll 1$, which, unlike that in the dissipation range, is not scale-invariant. The results are in good agreement with direct numerical simulations, provided the separations are well within the inertial range.

I. INTRODUCTION

An initially uniform distribution of inertial particles in an incompressible turbulent fluid velocity field will develop dynamically evolving spatial clusters. Such clustering has important implications for aerosol processes such as gravitational settling [1, 2], turbulence modulation [3, 4] and particle collisions [5, 6]. These processes are relevant to industrial processes such as aerosol manufacturing [7], drug delivery [8] and spray combustion [9] as well as to natural processes such as sediment and plankton distribution in oceans [10] and even the formation of planets in the early universe [11].

In a recent paper [12], we considered in detail the physical mechanism responsible for the clustering of inertial particles in the dissipation range of isotropic turbulence. Formally, the dissipation range is defined as $r \ll \eta$, where r is the distance between two points in space and η is the Kolmogorov length scale, though it should be noted that experiments and numerical simulations of the Navier-Stokes equation suggest that the dissipation range actually extends to $r = \mathcal{O}(10\eta)$ [13]. Nevertheless, in what follows we define the dissipation range to be the limit $r \ll \eta$. In [12] we showed that in the regime $\mathrm{St} \ll 1$ (where $\mathrm{St} \equiv \tau_p/\tau_\eta$ is the Stokes number, τ_p is the particle response time and τ_η is the Kolmogorov timescale), the mechanism for clustering in the Zaichik & Alipchenkov theory [14–16] (hereafter this body of work is referred to as 'ZT') is the same as that in the Chun *et al.* theory [17] (hereafter referred to as 'CT'), which is essentially an extension of the classical argument of Maxey [1] that particles are centrifuged out of rotating regions of the fluid into regions of high strain rate. When $\mathrm{St} \gtrsim \mathcal{O}(1)$, we showed that the ZT describes an additional non-local contribution to the clustering mechanism that is discussed in greater detail in §II.

If the Taylor microscale Reynolds number, Re_λ, is sufficiently large, particles may also cluster in the inertial range of the turbulence, a scenario that has been considered in several works [18–24]. The inertial range is defined as $\eta \ll r \ll L$, where L is the integral lengthscale of the turbulence. In [18], they showed using direct numerical simulations (DNS) that particle clustering at $\eta \ll r \ll L$ is not scale-invariant, unlike for $r \ll \eta$. Furthermore, they also argued that the clustering is not simply characterized by $\mathrm{St_r}$, as would be predicted by a white-in-time flow analysis (e.g. [19]), but rather by a rescaled contraction rate, at least for $\mathrm{St_r} \ll 1$, where $\mathrm{St_r} \equiv \tau_p/\langle\epsilon\rangle^{-1/3}r^{2/3}$ is the scale-dependent particle Stokes number based on eddies of size r, and $\langle\epsilon\rangle$ is the average turbulent energy dissipation rate. In a series of articles [20–23], an explanation for clustering at

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$\eta \ll r \ll L$ was developed in terms of the "sweep-stick" mechanism, whereby inertial particles are argued to stick to stagnation points in the fluid acceleration field and are swept along with them by the local fluid velocity. Since the fluid acceleration stagnation points are clustered in Navier-Stokes turbulence, they argue that this leads to clustering of the inertial particles at $\eta \ll r \ll L$. Moreover, in [23], they argue that the clustering mechanisms operating at $r \ll \eta$ and $\eta \ll r \ll L$ are different, with the sweep-stick mechanism describing the clustering only for $\eta \ll r \ll L$. The break in scale-invariance of the clustering noted in [18] as one goes from the dissipation range to the inertial range is certainly consistent with their hypothesis of different clustering mechanisms operating in the two regimes.

The outline of the paper is as follows. In §II we examine the question of the clustering mechanism in the inertial range by analyzing an exact equation for the radial distribution function (RDF), and show that the mechanism is precisely analogous to that operating in the dissipation range. We show that the break in scale-invariance of the clustering does not arise from a change in the underlying mechanism. In §III, we contrast our findings with the sweep-stick model of Coleman & Vassilicos [23]. Finally, in §IV we apply our findings to the model equation for the RDF from Zaichik & Alipchenkov [15] and derive a prediction for the analytical form of the RDF in the inertial range for $\mathrm{St_r} \ll 1$, which we test against DNS data at $\mathrm{Re}_\lambda = 597$.

II. ANALYSIS OF THE CLUSTERING MECHANISM IN THE INERTIAL RANGE

We consider the relative motion between two identical point particles, a 'primary' particle and a 'satellite' particle. We make the approximations that the particles are subject to Stokes drag forces only, that they do not interact with each other through physical collisions or hydrodynamic interactions and that they are at low enough concentration to not affect the turbulence (i.e., 'one-way coupling'). Furthermore, we restrict our attention to statistically stationary, homogeneous and isotropic turbulence. One of the reasons for choosing such simplified turbulence and particle dynamics is that we want to compare our analysis with earlier studies that were based on the same simplifications [e.g. 18–24]. The equation governing the relative motion of the two particles is [25]

$$\dot{\boldsymbol{w}}^p(t) = (\mathrm{St}\tau_\eta)^{-1}\Big(\Delta\boldsymbol{u}(\boldsymbol{r}^p(t),t) - \boldsymbol{w}^p(t)\Big), \tag{1}$$

where $\boldsymbol{r}^p(t), \boldsymbol{w}^p(t), \dot{\boldsymbol{w}}^p(t)$ are the particle pair relative separation, relative velocity and relative acceleration vectors, respectively, and $\Delta\boldsymbol{u}(\boldsymbol{r}^p(t),t)$ is the difference in the fluid velocity evaluated at the positions of the two particles.

For the system governed by (1) the exact equation governing the probability density function (PDF) $p(\boldsymbol{r},\boldsymbol{w},t) \equiv \langle\delta(\boldsymbol{r}^p(t)-\boldsymbol{r})\delta(\boldsymbol{w}^p(t)-\boldsymbol{w})\rangle$ describing the distribution of $\boldsymbol{r}^p(t), \boldsymbol{w}^p(t)$ in the phase-space $\boldsymbol{r}, \boldsymbol{w}$ is

$$\begin{aligned}\partial_t p = &-\boldsymbol{\nabla_r}\cdot p\boldsymbol{w} + (\mathrm{St}\tau_\eta)^{-1}\boldsymbol{\nabla_w}\cdot p\boldsymbol{w}\\ &-(\mathrm{St}\tau_\eta)^{-1}\boldsymbol{\nabla_w}\cdot p\langle\Delta\boldsymbol{u}(\boldsymbol{r}^p(t),t)\rangle_{\boldsymbol{r},\boldsymbol{w}},\end{aligned} \tag{2}$$

where $\langle\cdot\rangle_{\boldsymbol{r},\boldsymbol{w}}$ denotes an ensemble average conditioned on $\boldsymbol{r}^p(t)=\boldsymbol{r}$ and $\boldsymbol{w}^p(t)=\boldsymbol{w}$. A commonly used statistical measure of particle clustering is the RDF [26], which is defined as the ratio of the number of particle pairs at separation $r=|\boldsymbol{r}|$ to the number that would be expected if the particles were uniformly distributed. An exact equation for the statistically stationary RDF, $g(\boldsymbol{r})$, can be constructed by multiplying the stationary form of (2) by $\boldsymbol{w}$ and then integrating over all $\boldsymbol{w}$ yielding

$$\boldsymbol{0} = g\langle\Delta\boldsymbol{u}(\boldsymbol{r}^p(t),t)\rangle_{\boldsymbol{r}} - \mathrm{St}\tau_\eta\boldsymbol{S}_2^p\cdot\boldsymbol{\nabla_r}g - \mathrm{St}\tau_\eta g\boldsymbol{\nabla_r}\cdot\boldsymbol{S}_2^p, \tag{3}$$

where

$$g(\boldsymbol{r}) = \frac{N(N-1)}{n^2V}\int_{\boldsymbol{w}} p(\boldsymbol{r},\boldsymbol{w})\,d\boldsymbol{w}, \tag{4}$$

N is the total number of particles lying within the control volume V, $n \equiv N/V$ is the number density of particles, and $\boldsymbol{S}_2^p(\boldsymbol{r}) \equiv \langle\boldsymbol{w}^p(t)\boldsymbol{w}^p(t)\rangle_{\boldsymbol{r}}$ is the second-order particle velocity structure function.

The drift mechanisms that generate clustering are associated with the term $\mathrm{St}\tau_\eta\boldsymbol{\nabla_r}\cdot\boldsymbol{S}_2^p$. The contribution from $g\langle\Delta\boldsymbol{u}(\boldsymbol{r}^p(t),t)\rangle_{\boldsymbol{r}}$ may also contain drift contributions in addition to diffusion effects (see [12]), and this term is unclosed. It is not necessary at this stage to consider closure approximations for $g\langle\Delta\boldsymbol{u}(\boldsymbol{r}^p(t),t)\rangle_{\boldsymbol{r}}$ since its physical interpretation is known, namely it describes a flux arising from correlations between $\Delta\boldsymbol{u}$ and $\boldsymbol{r}^p(t)$ that is associated with preferential sampling effects. Hence for this qualitative discussion, we will focus on understanding the physical mechanisms described by the term $\mathrm{St}\tau_\eta\boldsymbol{\nabla_r}\cdot\boldsymbol{S}_2^p$.

We begin by reviewing the findings from [12] on the meaning and behavior of $\mathrm{St}\tau_\eta\boldsymbol{\nabla_r}\cdot\boldsymbol{S}_2^p$ in the dissipation range. In [12] we showed that for $r \ll \eta$ and $\mathrm{St} \ll 1$

$$\mathrm{St}\tau_\eta\boldsymbol{\nabla_r}\cdot\boldsymbol{S}_2^p = \frac{\mathrm{St}\tau_\eta}{3}\boldsymbol{r}(\mathcal{A}-\mathcal{B}), \tag{5}$$

where $\mathcal{A} \equiv \langle\mathcal{S}^2(\boldsymbol{x}^p(t),t)\rangle$ and $\mathcal{B} \equiv \langle\mathcal{R}^2(\boldsymbol{x}^p(t),t)\rangle$ are averages of the second invariants of the strain-rate $\boldsymbol{\mathcal{S}}$ and rotation-rate $\boldsymbol{\mathcal{R}}$ tensors evaluated along the inertial particle trajectory $\boldsymbol{x}^p(t)$. This drift mechanism is identical to the one derived in the CT using perturbation theory, and is associated with the traditional centrifuge mechanism. For $\mathrm{St} \gtrsim \mathcal{O}(1)$, the particle velocity dynamics become increasingly non-local, and this fundamentally changes the clustering mechanism described by $\mathrm{St}\tau_\eta\boldsymbol{\nabla_r}\cdot\boldsymbol{S}_2^p$. The physical interpretation of the non-local drift is as follows. Particle pairs arriving at separation $\boldsymbol{r}$ coming from larger separations carry a memory of larger fluid velocity differences in their path-history as compared with pairs arriving at $\boldsymbol{r}$ from smaller separations. This path-history bias breaks the symmetry of the particle inward and outward motions, creating a net inward drift and clustering.

In order to analyze the clustering mechanism in the inertial range, we consider the limit $\mathrm{Re}_\lambda \to \infty$, such that the inertial range is unbounded. Furthermore, we define a scale-dependent Stokes number as $\mathrm{St_r} \equiv \tau_p/\tau_r$, where τ_r is the eddy turnover timescale defined as $\tau_r \equiv \langle\epsilon\rangle^{-1/3} r^{2/3}$ for $\eta \ll r \ll L$, where $\langle\epsilon\rangle$ is the average turbulent kinetic energy dissipation rate and L is the (asymptotically large) integral length scale. For arbitrary Stokes numbers, St, the regime $\mathrm{St_r} \ll 1$ corresponds to $r \gg \eta \mathrm{St}^{3/2}$. We can analyze this regime in much the same way as CT did for $r \ll \eta$ and $\mathrm{St} \ll 1$.

Introducing the coarse-grained strain-rate $\widetilde{\boldsymbol{\mathcal{S}}}$ and rotation-rate $\widetilde{\boldsymbol{\mathcal{R}}}$ tensors, with coarse-graining length scale r, we can write the fluid velocity difference as $\Delta\boldsymbol{u}(\boldsymbol{r},t) \approx (\widetilde{\boldsymbol{\mathcal{S}}} + \widetilde{\boldsymbol{\mathcal{R}}}) \cdot \boldsymbol{r}$ [27–29]. In the regime $\mathrm{St_r} \ll 1$, $\boldsymbol{w}^p(t) \approx \Delta\boldsymbol{u}(\boldsymbol{r}^p(t),t) + \mathcal{O}(\mathrm{St_r})$ and therefore to leading order $\mathrm{St}\tau_\eta \boldsymbol{\nabla_r} \cdot \boldsymbol{S}_2^p$ is $\mathrm{St}\tau_\eta \boldsymbol{\nabla_r} \cdot \langle \Delta\boldsymbol{u}(\boldsymbol{r}^p(t),t)\Delta\boldsymbol{u}(\boldsymbol{r}^p(t),t)\rangle_{\boldsymbol{r}}$. We can derive an expression for the latter quantity using the coarse-graining and the scaling from Kolmogorov's 1941 theory (K41, see [30]), yielding

$$\mathrm{St}\tau_\eta \boldsymbol{\nabla_r} \cdot \boldsymbol{S}_2^p \approx \frac{\mathrm{St}\tau_\eta}{3}\boldsymbol{r}\Big[(2r/5)\nabla_r \widetilde{\mathcal{A}} + \widetilde{\mathcal{A}} - \zeta\widetilde{\mathcal{B}}\Big], \qquad (6)$$

where

$$\widetilde{\mathcal{A}} \equiv \Big\langle \widetilde{\boldsymbol{\mathcal{S}}^p(t)} : \widetilde{\boldsymbol{\mathcal{S}}^p(t)} \Big\rangle, \quad \widetilde{\mathcal{B}} \equiv \Big\langle \widetilde{\boldsymbol{\mathcal{R}}^p(t)} : \widetilde{\boldsymbol{\mathcal{R}}^p(t)} \Big\rangle,$$

$\widetilde{\boldsymbol{\mathcal{S}}^p(t)}$ and $\widetilde{\boldsymbol{\mathcal{R}}^p(t)}$ denote $\boldsymbol{\mathcal{S}}(\boldsymbol{x}^p(t),t)$ and $\boldsymbol{\mathcal{R}}(\boldsymbol{x}^p(t),t)$ coarse-grained over the scale r, $\zeta(r \ll \eta) = 1$ and $\zeta(\eta \ll r \ll L) = 7/15$ [31]. For $\eta \ll r \ll L$ (6) becomes

$$\mathrm{St}\tau_\eta \boldsymbol{\nabla_r} \cdot \boldsymbol{S}_2^p = \frac{7\mathrm{St}\tau_\eta}{45}\boldsymbol{r}(\widetilde{\mathcal{A}} - \widetilde{\mathcal{B}}), \qquad (7)$$

and for $r \ll \eta$, (6) reduces to (5). Preferential sampling of the inertial range eddies will lead to $\widetilde{\mathcal{A}} > \widetilde{\mathcal{B}}$, which is associated with centrifuging out of eddies of size $\approx r$. Note that any drift contribution coming from the unclosed term $\langle \Delta\boldsymbol{u}(\boldsymbol{r}^p(t),t)\rangle_{\boldsymbol{r}}$ in (3) has a similar interpretation.

At separations $r \lesssim \mathcal{O}(\eta \mathrm{St}^{3/2})$, corresponding to $\mathrm{St_r} \gtrsim \mathcal{O}(1)$, and so long as $\Delta\boldsymbol{u}(\boldsymbol{r},t)$ is statistically dependent upon $\boldsymbol{r}$, the non-local, path-history symmetry breaking contribution to $\mathrm{St}\tau_\eta \boldsymbol{\nabla_r} \cdot \boldsymbol{S}_2^p$ is important. This transition is analogous to the one that occurs in the dissipation range (i.e., $r \ll \eta$) for particles with $\mathrm{St} \gtrsim \mathcal{O}(1)$. However, the relative magnitude of the transition from the local to the non-local mechanisms is more pronounced in the dissipation range than in the inertial range. The reason for this is that although the particle relative velocities have a non-local contribution when $\mathrm{St_r} \gtrsim \mathcal{O}(1)$, the non-locality is much weaker in the inertial range because $\Delta\boldsymbol{u}(\boldsymbol{r},t)$ varies with $\boldsymbol{r}$ more weakly than in the dissipation range. In this case, at $\mathrm{St_r} \gtrsim \mathcal{O}(1)$ the filtering effect of the particle inertia (see [32]) can dominate the non-local contribution to the particle relative velocities leading to $\boldsymbol{S}_2^p/\langle \Delta\boldsymbol{u}(\boldsymbol{r},t)\Delta\boldsymbol{u}(\boldsymbol{r},t)\rangle < 1$. DNS results show that whereas $\boldsymbol{S}_2^p/\langle \Delta\boldsymbol{u}(\boldsymbol{r},t)\Delta\boldsymbol{u}(\boldsymbol{r},t)\rangle \gg 1$ for $\mathrm{St} \gtrsim \mathcal{O}(1)$ in the dissipation range, $\boldsymbol{S}_2^p/\langle \Delta\boldsymbol{u}(\boldsymbol{r},t)\Delta\boldsymbol{u}(\boldsymbol{r},t)\rangle < 1$ for $\mathrm{St_r} \gtrsim \mathcal{O}(1)$ in the inertial range [33]. However, the latter result is sensitive to the Reynolds number. In particular, in the limit $\mathrm{Re}_\lambda \to \infty$, where the filtering effect of particle inertia at the large scales of the flow becomes weak, the non-local clustering mechanism would dominate at $\eta \ll r \ll L$ for $\mathrm{St_r} \gtrsim \mathcal{O}(1)$.

We therefore conclude that the clustering mechanisms operating in the inertial range are analogous to those operating in the dissipation range. When $\mathrm{St_r} \ll 1$ preferential sampling of the coarse-grained fluid velocity gradient tensor at scale $\approx r$ generates the inward drift and clustering, and when $\mathrm{St_r} \gtrsim \mathcal{O}(1)$ the non-local, path-history symmetry breaking mechanism contributes to the clustering.

III. RELATIONSHIP TO THE SWEEP-STICK MECHANISM

As noted earlier, there is an alternative description of inertial particle clustering known as the "sweep-stick" mechanism [20–23]. The sweep-stick mechanism was motivated by the observation that the instantaneous particle positions $\boldsymbol{x}^p(t)$ are correlated with the positions of the stagnation points of the acceleration field of the fluid, $\boldsymbol{s}_a(t)$, defined such that $\boldsymbol{a}(\boldsymbol{s}_a(t),t) \equiv \boldsymbol{0}$, where $\boldsymbol{a}(\boldsymbol{x},t)$ is the fluid acceleration field. Chen *et al.* [20] used K41 scaling to obtain

$$\Big\langle |\dot{\boldsymbol{s}}_a(t) - \boldsymbol{u}(\boldsymbol{s}_a(t),t)|^2 \Big\rangle \approx u'u'\Big(L/\eta\Big)^{-2/3}, \qquad (8)$$

where $\boldsymbol{u}(\boldsymbol{s}_a(t),t)$ is the fluid velocity at $\boldsymbol{s}_a(t)$, $u' \equiv \sqrt{\langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle/3}$ and L is the integral lengthscale of the flow. In the limit we are considering, namely $\mathrm{Re}_\lambda \to \infty$, (8) suggests that $\dot{\boldsymbol{s}}_a(t) = \boldsymbol{u}(\boldsymbol{s}_a(t),t)$, i.e. stagnation points are swept by the local fluid velocity. In [23] they use DNS to consider the joint PDF of $\dot{\boldsymbol{s}}_a(t)$ and $\boldsymbol{u}(\boldsymbol{s}_a(t),t)$ and do find a strong correlation, even at the modest values of Reynolds numbers in the study, $\mathrm{Re}_\lambda < 200$. For $\mathrm{St} \ll 1$, $\boldsymbol{v}^p(t) \approx \boldsymbol{u}(\boldsymbol{x}^p(t),t) - \mathrm{St}\tau_\eta \boldsymbol{a}(\boldsymbol{x}^p(t),t)$ where $\boldsymbol{v}^p(t)$ is the particle velocity and $\boldsymbol{u}(\boldsymbol{x}^p(t),t)$, $\boldsymbol{a}(\boldsymbol{x}^p(t),t)$ are the fluid velocity and acceleration at the particle position, respectively. According to this expression, when $\boldsymbol{x}^p(t) = \boldsymbol{s}_a(t)$ the co-located particle moves with the fluid velocity $\boldsymbol{u}(\boldsymbol{x}^p(t),t)$. This is statistically the same velocity with which the $\boldsymbol{a} = \boldsymbol{0}$ points move, and therefore it is argued that the particle sticks to $\boldsymbol{s}_a(t)$ and is swept along by $\boldsymbol{u}$. Although the above explanation for the stick part of the mechanism is technically valid only for $\mathrm{St} \ll 1$, in [23] they present results from DNS which, they argue, show that even for $\mathrm{St} = \mathcal{O}(1)$, particles at acceleration stagnation points move, statistically, with the same velocity as the local fluid.

The conceptual framework of the sweep-stick mechanism is interesting and since particles do cluster near $\boldsymbol{a} = \boldsymbol{0}$ points, it provides a reasonable argument for inertial particle clustering. However, there is a confounding conceptual problem that occurs when applying the sweep-stick mechanism to stochastic flows such as kinematic simulations (KS). In KS, the acceleration stagnation points are uniformly distributed, yet the inertial particles still cluster. Chen *et al.* [20] argued that clustering in this instance is due to the repelling action of the *velocity* stagnation points (taken in the stationary frame of reference), which are clustered in KS. However, we would argue that, rather than concluding that the clustering mechanisms operating in DNS and KS are different, a more convincing conclusion would be that the sweep-stick mechanism is, in fact, not the underlying cause of the clustering in the inertial range.

The argument we presented in §II explains clustering in both KS and DNS. In particular, our argument states that the cause of the particle clustering lies in the nature of the interaction of the inertial particles with the fields $\widetilde{\boldsymbol{\mathcal{S}}}$ and $\widetilde{\boldsymbol{\mathcal{R}}}$. This applies to both DNS and KS since it does not depend upon the dynamics of the underlying system governing $\widetilde{\boldsymbol{\mathcal{S}}}$ and $\widetilde{\boldsymbol{\mathcal{R}}}$. However, it is possible that the sweep-stick mechanism provides a valid explanation for clustering in DNS, but not KS, because of a relationship that exists between $\widetilde{\boldsymbol{\mathcal{S}}}$, $\widetilde{\boldsymbol{\mathcal{R}}}$ and $\boldsymbol{s}_a(t)$ that is specific to Navier-Stokes turbulence. For example, at $\mathrm{St} \ll 1$

$$\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{v}(\boldsymbol{x}^p(t), t) \approx -\mathrm{St}\tau_\eta \Big(\mathcal{S}^2(\boldsymbol{x}^p(t), t) - \mathcal{R}^2(\boldsymbol{x}^p(t), t) \Big),$$

which applies to any fluid velocity field that has spatial structure. However, in Navier-Stokes turbulence

$$\mathcal{S}^2(\boldsymbol{x}^p(t), t) - \mathcal{R}^2(\boldsymbol{x}^p(t), t) = -\boldsymbol{\nabla}^2_{\boldsymbol{x}} p^f(\boldsymbol{x}^p(t), t),$$

such that in DNS one may speak of the behavior of $\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{v}(\boldsymbol{x}^p(t), t)$ in terms of either the particles interaction with $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{R}}$, *or equivalently* in terms of their interaction with the fluid pressure field p^f. Yet, since such a relationship is particular to the Navier-Stokes equations, one ought (for the sake of generality) to describe the behavior of $\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{v}(\boldsymbol{x}^p(t), t)$ in terms of the particles interaction with $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{R}}$.

It may well be the case that in an analogous way, a relationship exists in Navier-Stokes turbulence between $\widetilde{\boldsymbol{\mathcal{S}}}$,$\widetilde{\boldsymbol{\mathcal{R}}}$ and $\boldsymbol{s}_a(t)$. A consequence of this could be that the explanations of inertial particle clustering in terms of either the clustering of $\boldsymbol{s}_a(t)$ points (as in the sweep-stick mechanism) or in terms of the particles preferential sampling of $\widetilde{\boldsymbol{\mathcal{S}}}$ over $\widetilde{\boldsymbol{\mathcal{R}}}$ (as in our explanation) are equivalent.

To consider this possibility we will analyze the sweep-stick mechanism to see if it provides a relationship between $\boldsymbol{s}_a(t)$ and $\boldsymbol{x}^p(t)$. We will then derive a relationship between $\widetilde{\boldsymbol{\mathcal{S}}}$, $\widetilde{\boldsymbol{\mathcal{R}}}$ and $\boldsymbol{s}_a(t)$ and demonstrate that in Navier-Stokes turbulence $\boldsymbol{s}_a(t)$ points cluster in regions where $\widetilde{\mathcal{A}} - \widetilde{\mathcal{B}} > 0$, i.e., precisely the regions where the particles are predicted to cluster by the analysis in §II.

A. Sweeping of acceleration stagnation points

In [20] it is argued that (8) implies that $\dot{\boldsymbol{s}}_a(t) = \boldsymbol{u}(\boldsymbol{s}_a(t), t)$ in the limit $\mathrm{Re}_\lambda \to \infty$. In the process of deriving (8), they used $u'u'(L/\eta)^{-2/3} = u_\eta^2$. Therefore the result from (8) that $\dot{\boldsymbol{s}}_a(t) = \boldsymbol{u}(\boldsymbol{s}_a(t), t)$ in the limit $\mathrm{Re}_\lambda \to \infty$ follows simply from the fact that $\lim_{\mathrm{Re}_\lambda \to \infty} u_\eta \to 0$. However, relative to u_η, $|\dot{\boldsymbol{s}}_a(t) - \boldsymbol{u}(\boldsymbol{s}_a(t), t)|$ is finite and independent of Re_λ, and this has important implications since the fluid acceleration field $\boldsymbol{a}$ fluctuates on a timescale $\mathcal{O}(\tau_\eta)$, and the average separation between stagnation points is $\mathcal{O}(\eta)$ [20]. Supposing for the moment that the stick part of the mechanism is valid, then $|\dot{\boldsymbol{s}}_a(t) - \boldsymbol{v}^p(t|\boldsymbol{x}^p(t) = \boldsymbol{s}_a(t))| = \mathcal{O}(u_\eta)$ which implies that the particle may traverse the average distance between two stagnation points during the timescale τ_η. Consequently, the idea that inertial particles stick to particular stagnation points and are swept along with them is probably not be valid. However, (8) does nevertheless demonstrate that compared to their individual speeds, the speed of stagnation points relative to the local fluid velocity is small for $\mathrm{Re}_\lambda \to \infty$ and thus in a significant way the stagnation points are swept by the velocity field. Assuming again the validity of the stick mechanism, this could then imply that although inertial particles may not be swept together with particular stagnation points, they may be swept along with clusters of stagnation points, with the particles having the freedom to move between stagnation points within the cluster as they are collectively swept by the large scale motions of the turbulence. Essentially the same alternative interpretation of the sweep-stick mechanism was in fact suggested in [34].

B. Is the "stick" mechanism valid for all Stokes numbers?

The next concern is with the "stick" part of the mechanism and whether it is really valid when $\mathrm{St} \gtrsim \mathcal{O}(1)$. The stick mechanism was formulated by appealing to the $\mathrm{St} \ll 1$ expression $\boldsymbol{v}^p(t) = \boldsymbol{u}(\boldsymbol{x}^p(t), t) - \mathrm{St}\tau_\eta \boldsymbol{a}(\boldsymbol{x}^p(t), t)$. Since this expression is not valid for $\mathrm{St} \gtrsim \mathcal{O}(1)$, in [23] they appeal to DNS results to show that $\boldsymbol{v}^p(t) = \boldsymbol{u}(\boldsymbol{x}^p(t), t)$ when $\boldsymbol{x}^p(t) = \boldsymbol{s}_a(t)$. Specifically, in [23] they show that $\langle \boldsymbol{v}^p(t) - \boldsymbol{u}(\boldsymbol{x}^p(t), t) \rangle_{\boldsymbol{a}} = \boldsymbol{0}$, when $\boldsymbol{a} = \boldsymbol{0}$, where $\langle \cdot \rangle_{\boldsymbol{a}}$ denotes an ensemble average conditioned on $\boldsymbol{a}(\boldsymbol{x}^p(t), t) = \boldsymbol{a}$. On this basis, they conclude that the stick mechanism is valid even for $\mathrm{St} \gtrsim \mathcal{O}(1)$. However, this result does not validate the stick mechanism, nor does it explain the relationship between $\boldsymbol{v}^p(t)$ and $\boldsymbol{u}(\boldsymbol{x}^p(t), t)$ at $\boldsymbol{a} = \boldsymbol{0}$ points.

From the particle equation of motion we have

$$-\mathrm{St}\tau_\eta \Big\langle \dot{\boldsymbol{v}}^p(t) \Big\rangle = \Big\langle \boldsymbol{v}^p(t) - \boldsymbol{u}(\boldsymbol{x}^p(t), t) \Big\rangle, \tag{9}$$

and using this we may write

$$-\mathrm{St}\tau_\eta\Big\langle\dot{\boldsymbol{v}}^p(t)\Big\rangle \equiv -\mathrm{St}\tau_\eta\int_{-\infty}^{+\infty}\rho(\boldsymbol{a})\Big\langle\dot{\boldsymbol{v}}^p(t)\Big\rangle_{\boldsymbol{a}}\,d\boldsymbol{a}$$
$$=\int_{-\infty}^{+\infty}\rho(\boldsymbol{a})\Big\langle\boldsymbol{v}^p(t)-\boldsymbol{u}(\boldsymbol{x}^p(t),t)\Big\rangle_{\boldsymbol{a}}\,d\boldsymbol{a}, \tag{10}$$

where $\rho(\boldsymbol{a})\equiv\langle\delta(\boldsymbol{a}(\boldsymbol{x}^p(t),t)-\boldsymbol{a})\rangle$. In statistically stationary, isotropic turbulence, $\rho(\boldsymbol{a})=\rho(-\boldsymbol{a})$ and $\langle\dot{\boldsymbol{v}}^p(t)\rangle=\boldsymbol{0}$. In the limit of strong local fluid acceleration, $\lim_{\boldsymbol{a}\to\boldsymbol{\infty}}\langle\dot{\boldsymbol{v}}^p(t)\rangle_{\boldsymbol{a}}\to\boldsymbol{a}$, i.e. when the local fluid acceleration is very strong it is the dominant contribution to $\dot{\boldsymbol{v}}^p(t)$. Together with the condition $\int\rho(\boldsymbol{a})\langle\dot{\boldsymbol{v}}^p(t)\rangle_{\boldsymbol{a}}\,d\boldsymbol{a}=\boldsymbol{0}$ this implies that $\langle\dot{\boldsymbol{v}}^p(t)\rangle_{\boldsymbol{a}}$ is an odd function of $\boldsymbol{a}$, and hence $\langle\boldsymbol{v}^p(t)-\boldsymbol{u}(\boldsymbol{x}^p(t),t)\rangle_{\boldsymbol{a}=\boldsymbol{0}}=\boldsymbol{0}$ for all St. The DNS results in Fig. 12 of [23] do in fact confirm that $\langle\dot{\boldsymbol{v}}^p(t)\rangle_{\boldsymbol{a}}$ is an odd function of $\boldsymbol{a}$.

There are two implications following from this analysis. First, $\langle\boldsymbol{v}^p(t)-\boldsymbol{u}(\boldsymbol{x}^p(t),t)\rangle_{\boldsymbol{a}=\boldsymbol{0}}=\boldsymbol{0}$ is not dynamically significant since with respect to the particle dynamics it follows simply from the fact that $\langle\boldsymbol{v}^p(t)\rangle=\langle\boldsymbol{u}(\boldsymbol{x}^p(t),t)\rangle=\boldsymbol{0}$, yet two variables with equal means may be entirely independent of one another. Second, since (10) implies $\langle\boldsymbol{v}^p(t)-\boldsymbol{u}(\boldsymbol{x}^p(t),t)\rangle_{\boldsymbol{a}=\boldsymbol{0}}=\boldsymbol{0}$ for all St, then if $\langle\boldsymbol{v}^p(t)-\boldsymbol{u}(\boldsymbol{x}^p(t),t)\rangle_{\boldsymbol{a}=\boldsymbol{0}}=\boldsymbol{0}$ were sufficient to demonstrate the stick mechanism, then it would imply that $\mathrm{St}\to\infty$ particles should cluster through the action of the sweep-stick mechanism, which is clearly invalid [35].

In order to demonstrate that the stick mechanism is valid for $\mathrm{St}\gtrsim\mathcal{O}(1)$ one must consider an alternative statistic such as

$$\mathcal{Q}\equiv\Big\langle|\boldsymbol{v}^p(t)-\boldsymbol{u}(\boldsymbol{x}^p(t),t)|^2\Big\rangle_{|\boldsymbol{a}|^2}=(\mathrm{St}\tau_\eta)^2\Big\langle|\dot{\boldsymbol{v}}^p(t)|^2\Big\rangle_{|\boldsymbol{a}|^2}, \tag{11}$$

which cannot vanish for any trivial reason at $\boldsymbol{a}=\boldsymbol{0}$ since the particle and fluid velocity variances are, unlike their mean values, non-zero. In the regime $\mathrm{St}\ll 1$, $\mathcal{Q}=(\mathrm{St}\tau_\eta)^2|\boldsymbol{a}|^2$, which is consistent with the stick mechanism. However, there is no reason to expect that $\mathcal{Q}(\boldsymbol{a}=\boldsymbol{0})=0$ in the regime $\mathrm{St}\gtrsim\mathcal{O}(1)$. Nevertheless, in order for the stick mechanism to be valid one does not necessarily require that $\mathcal{Q}(\boldsymbol{a}=\boldsymbol{0})=0$ precisely but rather that $\mathcal{Q}(\boldsymbol{a}=\boldsymbol{0})$ is in some sense small. For example, the sweep part of the mechanism suggests that the velocity with which the $\boldsymbol{s}_a(t)$ points are swept is related to u'. In this case, if $\mathcal{Q}(\boldsymbol{a}=\boldsymbol{0})\ll u'u'$, then although the particles do not precisely stick to the stagnation points, they remain close enough to follow them in a significant way.

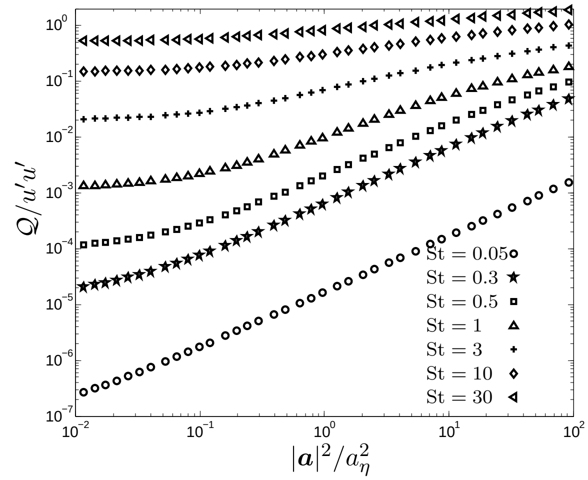

FIG. 1. DNS data for $\mathcal{Q}$ at various St, plotted as a function of $|\boldsymbol{a}|^2/a_\eta^2$, where a_η is the Kolmogorov acceleration.

In Fig. 1 we show results for $\mathcal{Q}$ computed from DNS at $\mathrm{Re}_\lambda=597$. Details on the DNS used throughout this paper can be found in [33]. As expected, the results show that $\mathcal{Q}=(\mathrm{St}\tau_\eta)^2|\boldsymbol{a}|^2$ for $\mathrm{St}\ll 1$, implying $\mathcal{Q}(\boldsymbol{a}\to\boldsymbol{0})\to 0$, consistent with the stick mechanism. For $\mathrm{St}=\mathcal{O}(1)$, while $\mathcal{Q}(\boldsymbol{a}\to\boldsymbol{0})\not\to 0$, $\mathcal{Q}(\boldsymbol{a}\to\boldsymbol{0})\ll u'u'$, implying that although the particles do not precisely stick to $\boldsymbol{s}_a(t)$ points, they remain close enough to follow them in a significant way. For $\mathrm{St}=\mathcal{O}(10)$, $\mathcal{Q}(\boldsymbol{a}\to\boldsymbol{0})$ remains quite small relative to $u'u'$. However, for $\mathrm{St}=\mathcal{O}(10)$ the variation of $\mathcal{Q}$ with $\boldsymbol{a}$ for $|\boldsymbol{a}|^2/a_\eta^2\leq\mathcal{O}(1)$ is weak. This implies that although $\mathcal{Q}(\boldsymbol{a}\to\boldsymbol{0})$ is still smaller than $u'u'$ at $\mathrm{St}=\mathcal{O}(10)$, the significance of $\boldsymbol{s}_a(t)$ points for the particle motion becomes small. This follows from noting that if $\mathcal{Q}(\boldsymbol{a})$ were constant for a given St, then it would imply that the particle motion is entirely uncorrelated with $\boldsymbol{a}(\boldsymbol{x}^p(t),t)$. Nevertheless, our DNS data shows that $\mathrm{St}=\mathcal{O}(10)$ particles cluster, and in fact cluster more strongly than $\mathrm{St}=\mathcal{O}(1)$ particles in the inertial range (see [33]), indicating the breakdown of the sweep-stick mechanism as the explanation for clustering when $\mathrm{St}=\mathcal{O}(10)$. In our DNS at $\mathrm{Re}_\lambda=597$, $\mathrm{St}\lesssim\mathcal{O}(1)\implies\mathrm{St_r}\ll 1$, and $\mathrm{St}\gtrsim\mathcal{O}(10)\implies\mathrm{St_r}\gtrsim\mathcal{O}(1)$ for r in the inertial range.

The conclusion to be drawn is that the sweep-stick mechanism provides a valid explanation for clustering in the inertial range of Navier-Stokes turbulence when $\mathrm{St_r}\ll 1$, but it is not valid when $\mathrm{St_r}\gtrsim\mathcal{O}(1)$. This is not surprising since the sweep-stick mechanism is essentially a local mechanism. It is also not surprising since the correlation timescale of $\boldsymbol{a}(\boldsymbol{x},t)$ is $\mathcal{O}(\tau_\eta)$, the lifetime of $\boldsymbol{s}_a(t)$ points is typically too short to cause the clustering of particles with $\mathrm{St}=\mathcal{O}(10)$ (i.e. $\mathrm{St_r}\gtrsim\mathcal{O}(1)$ in the inertial range).

Thus, in the regime $\mathrm{St_r}\ll 1$, the sweep-stick mechanism provides an essentially equivalent explanation for clustering to the mechanism we presented in §II (i.e. centrifuging by eddies of size $\approx r$) if the particles are suspended in Navier-Stokes turbulence. This would be analogous to the case in the dissipation regime where for St $\ll 1$, particle clustering may be described either in terms of their interaction with $\boldsymbol{\mathcal{S}}$,$\boldsymbol{\mathcal{R}}$ or with the pressure field p^f, if the particles are in Navier-Stokes turbulence.

C. Where do $\boldsymbol{s}_a(t)$ points cluster?

Irrespective of the validity of the sweep-stick mechanism in providing a causal connection between the clustering of $\boldsymbol{s}_a(t)$ and $\boldsymbol{x}^p(t)$ points, the fact remains that DNS results reveal that there is a striking correlation between the distribution of the two sets of points. The mechanism we have argued for in §II predicts that the particles should cluster in regions of high coarse-grained strain. If our mechanism is correct then it is important to demonstrate that $\boldsymbol{s}_a(t)$ points also cluster in regions of high coarse-grained strain in Navier-Stokes turbulence. In order to demonstrate this we begin by defining the PDF

$$\begin{aligned}\mathcal{P}(\boldsymbol{r},\Delta\boldsymbol{u},\boldsymbol{a}_1,\boldsymbol{a}_2,t)\equiv\Big\langle&\delta(\boldsymbol{r}^f(t)-\boldsymbol{r})\delta(\Delta\boldsymbol{u}^f(t)-\Delta\boldsymbol{u})\\&\times\delta(\boldsymbol{a}_1^f(t)-\boldsymbol{a}_1)\delta(\boldsymbol{a}_2^f(t)-\boldsymbol{a}_2)\Big\rangle,\end{aligned}\tag{12}$$

which describes the probability density of a pair of fluid particles having relative separation $\boldsymbol{r}$, relative velocity $\Delta\boldsymbol{u}$ and accelerations $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ at time t. The evolution equation for $\mathcal{P}$ is

$$\begin{aligned}\partial_t\mathcal{P}=&-\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\mathcal{P}\Delta\boldsymbol{u}-\boldsymbol{\nabla}_{\Delta\boldsymbol{u}}\cdot\mathcal{P}(\boldsymbol{a}_2-\boldsymbol{a}_1)\\&-\boldsymbol{\nabla}_{\boldsymbol{a}_1}\cdot\mathcal{P}\Big\langle\dot{\boldsymbol{a}}_1^f(t)\Big\rangle_{\boldsymbol{r},\Delta\boldsymbol{u},\boldsymbol{a}_1,\boldsymbol{a}_2}\\&-\boldsymbol{\nabla}_{\boldsymbol{a}_2}\cdot\mathcal{P}\Big\langle\dot{\boldsymbol{a}}_2^f(t)\Big\rangle_{\boldsymbol{r},\Delta\boldsymbol{u},\boldsymbol{a}_1,\boldsymbol{a}_2}.\end{aligned}\tag{13}$$

By multiplying this equation by $\Delta\boldsymbol{u}$ and integrating over $\Delta\boldsymbol{u}$ we obtain the equation governing $\varrho(\boldsymbol{r},\boldsymbol{a}_1,\boldsymbol{a}_2,t)=\int\mathcal{P}(\boldsymbol{r},\Delta\boldsymbol{u},\boldsymbol{a}_1,\boldsymbol{a}_2,t)\,d\Delta\boldsymbol{u}$, and in the stationary state the equation is

$$\begin{aligned}\boldsymbol{0}=&-\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\varrho\Big\langle\Delta\boldsymbol{u}^f(t)\Delta\boldsymbol{u}^f(t)\Big\rangle_{\boldsymbol{r},\boldsymbol{a}_1,\boldsymbol{a}_2}\\&-\boldsymbol{\nabla}_{\boldsymbol{a}_1}\cdot\varrho\Big\langle\dot{\boldsymbol{a}_1}^f(t)\Delta\boldsymbol{u}^f(t)\Big\rangle_{\boldsymbol{r},\boldsymbol{a}_1,\boldsymbol{a}_2}\\&-\boldsymbol{\nabla}_{\boldsymbol{a}_2}\cdot\varrho\Big\langle\dot{\boldsymbol{a}_2}^f(t)\Delta\boldsymbol{u}^f(t)\Big\rangle_{\boldsymbol{r},\boldsymbol{a}_1,\boldsymbol{a}_2}\\&+\varrho(\boldsymbol{a}_2-\boldsymbol{a}_1).\end{aligned}\tag{14}$$

We now introduce into this the coarse-graining approximation $\Delta\boldsymbol{u}^f(t)\approx\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}^f(t)$, where $\boldsymbol{\Gamma}^f(t)\equiv\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x}^f(t),t)$ is the fluid velocity gradient tensor evaluated at $\boldsymbol{x}^f(t)$, yielding

$$\begin{aligned}\boldsymbol{0}\approx&-\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\varrho\Big\langle\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big\rangle_{\boldsymbol{r},\boldsymbol{a}_1,\boldsymbol{a}_2}\\&-\boldsymbol{\nabla}_{\boldsymbol{a}_1}\cdot\varrho\Big\langle\dot{\boldsymbol{a}_1}^f(t)\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big\rangle_{\boldsymbol{r},\boldsymbol{a}_1,\boldsymbol{a}_2}\\&-\boldsymbol{\nabla}_{\boldsymbol{a}_2}\cdot\varrho\Big\langle\dot{\boldsymbol{a}_2}^f(t)\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big\rangle_{\boldsymbol{r},\boldsymbol{a}_1,\boldsymbol{a}_2}\\&+\varrho(\boldsymbol{a}_2-\boldsymbol{a}_1).\end{aligned}\tag{15}$$

In the case of inertial particles, when St $\ll 1$ the clustering is weak and the leading order behavior of the clustering can be approximated using $\langle\cdot\rangle_{\boldsymbol{r}}\approx\langle\cdot\rangle$ in the drift and diffusion tensor expressions (see [12]). Since the stagnation points are weakly clustered in the inertial range [20] then we may also use this approximation, with which we obtain (with the understanding that this is only accurate for $|\boldsymbol{r}|\gg\eta$, $\boldsymbol{a}_1\approx\boldsymbol{a}_2\approx\boldsymbol{0}$)

$$\begin{aligned}\boldsymbol{0}=&-\Big\langle\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big\rangle_{\boldsymbol{a}_1,\boldsymbol{a}_2}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}\varrho\\&-\varrho\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\Big\langle\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big\rangle_{\boldsymbol{a}_1,\boldsymbol{a}_2}\\&-\Big\langle\dot{\boldsymbol{a}_1}^f(t)\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big\rangle_{\boldsymbol{a}_1,\boldsymbol{a}_2}\cdot\boldsymbol{\nabla}_{\boldsymbol{a}_1}\varrho\\&-\varrho\boldsymbol{\nabla}_{\boldsymbol{a}_1}\cdot\Big\langle\dot{\boldsymbol{a}_1}^f(t)\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big\rangle_{\boldsymbol{a}_1,\boldsymbol{a}_2}\\&-\Big\langle\dot{\boldsymbol{a}_2}^f(t)\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big\rangle_{\boldsymbol{a}_1,\boldsymbol{a}_2}\cdot\boldsymbol{\nabla}_{\boldsymbol{a}_2}\varrho\\&-\varrho\boldsymbol{\nabla}_{\boldsymbol{a}_2}\cdot\Big\langle\dot{\boldsymbol{a}_2}^f(t)\Big(\widetilde{\boldsymbol{\Gamma}^f(t)}\cdot\boldsymbol{r}\Big)\Big\rangle_{\boldsymbol{a}_1,\boldsymbol{a}_2}\\&+\varrho(\boldsymbol{a}_2-\boldsymbol{a}_1).\end{aligned}\tag{16}$$

The second, fourth and sixth terms on the rhs of (16) represent drift fluxes in the phase-space. The fourth and sixth terms terms on the rhs of (16) represent drift fluxes describing a change in ϱ because of the movement of the particles in $\boldsymbol{a}_1,\boldsymbol{a}_2$ space. The behavior of these fluxes can be understood by considering the behavior of the distribution they govern, namely $\vartheta(\boldsymbol{a}_1,\boldsymbol{a}_2|\boldsymbol{r})$, where $\varrho\equiv\vartheta(\boldsymbol{a}_1,\boldsymbol{a}_2|\boldsymbol{r})\phi(\boldsymbol{r})$. For $\boldsymbol{r}$ in the inertial range, and under the weak clustering approximation $\vartheta(\boldsymbol{a}_1,\boldsymbol{a}_2|\boldsymbol{r})\approx\rho(\boldsymbol{a}_1)\rho(\boldsymbol{a}_2)$ (since the correlation lengthscale of $\boldsymbol{a}(\boldsymbol{x},t)$ is $\mathcal{O}(\eta)$). The fluid acceleration PDF in stationary, isotropic turbulence is symmetric with zero mean; under the approximation $\vartheta(\boldsymbol{a}_1,\boldsymbol{a}_2|\boldsymbol{r})\approx\rho(\boldsymbol{a}_1)\rho(\boldsymbol{a}_2)$ the symmetry of $\vartheta(\boldsymbol{a}_1,\boldsymbol{a}_2|\boldsymbol{r})$ at $\boldsymbol{a}_1=\boldsymbol{0}$, $\boldsymbol{a}_2=\boldsymbol{0}$ implies that the drift flux of probability in $\boldsymbol{a}_1,\boldsymbol{a}_2$ space is zero at $\boldsymbol{a}_1=\boldsymbol{0}$, $\boldsymbol{a}_2=\boldsymbol{0}$, and so the fourth and sixth terms on the rhs of (16) are zero at $\boldsymbol{a}_1=\boldsymbol{0}$, $\boldsymbol{a}_2=\boldsymbol{0}$. Furthermore, the acceleration PDF maxima at at $\boldsymbol{a}_1=\boldsymbol{0}$, $\boldsymbol{a}_2=\boldsymbol{0}$ means that the third and fifth terms in (16) are also zero at $\boldsymbol{a}_1=\boldsymbol{0}$, $\boldsymbol{a}_2=\boldsymbol{0}$. The only non-zero drift contribution governing the distribution of the stagnation points is therefore the second term on the rhs of (16). Following essentially the same procedure as was used for the inertial particles to simplify the coarse-grained statistics, we then obtain for

$\boldsymbol{r}$ in the inertial range and at $\boldsymbol{a}_1 = \boldsymbol{0}$, $\boldsymbol{a}_2 = \boldsymbol{0}$

$$\begin{aligned}\boldsymbol{0} =& - \Big\langle \Big(\widetilde{\boldsymbol{\Gamma}^f(t)} \cdot \boldsymbol{r}\Big)\Big(\widetilde{\boldsymbol{\Gamma}^f(t)} \cdot \boldsymbol{r}\Big)\Big\rangle_{\boldsymbol{0},\boldsymbol{0}} \cdot \boldsymbol{\nabla_r}\varphi^0 \\ & - \frac{7}{45}\varphi^0 \boldsymbol{r}\Big(\widetilde{\mathscr{A}}_0 - \widetilde{\mathscr{B}}_0\Big),\end{aligned} \tag{17}$$

where $\widetilde{\mathscr{A}}_0$ and $\widetilde{\mathscr{B}}_0$ are the averages of the second invariants of the coarse-grained strain-rate and rotation-rate tensors, respectively, evaluated at $\boldsymbol{s}_a(t)$ points. In (17), φ^0 describes the spatial distribution of acceleration stagnation points, where $\varrho \equiv \varphi(\boldsymbol{r}|\boldsymbol{a}_1, \boldsymbol{a}_2)\Theta(\boldsymbol{a}_1, \boldsymbol{a}_2)$ and $\varphi^0 \equiv \varphi(\boldsymbol{r}|\boldsymbol{0}, \boldsymbol{0})$.

Up until this point the arguments have been mainly kinematic in nature. However, stagnation point clustering is a result of dynamics not kinematics, occurring in Navier-Stokes turbulence but not in KS. The result in (17) does not itself demonstrate or explain the stagnation point clustering. For example, in KS where the stagnation points are not clustered, $\widetilde{\mathscr{A}}_0 - \widetilde{\mathscr{B}}_0 = 0$. Developing an explanation for the dynamical cause of stagnation point clustering in Navier-Stokes turbulence is beyond the scope of this paper. The value of (17) is that it tells us that if there is stagnation point clustering, i.e. $\boldsymbol{\nabla_r}\varphi^0 < \boldsymbol{0}$, then it must be that $\widetilde{\mathscr{A}}_0 - \widetilde{\mathscr{B}}_0 > 0$ (noting that $\langle(\widetilde{\boldsymbol{\Gamma}^f(t)} \cdot \boldsymbol{r})(\widetilde{\boldsymbol{\Gamma}^f(t)} \cdot \boldsymbol{r})\rangle_{\boldsymbol{0},\boldsymbol{0}}$ is positive-definite in isotropic turbulence). This implies that in Navier-Stokes turbulence where the stagnation points are clustered, they must preferentially cluster in regions of high coarse-grained strain.

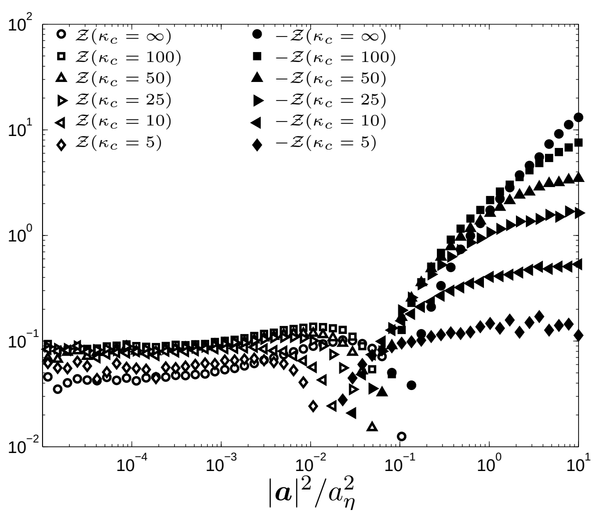

FIG. 2. DNS data for $\mathscr{Z}$ at various cut-off wavenumbers κ_c, plotted as a function of $|\boldsymbol{a}|^2/a_\eta^2$.

In order to confirm this prediction that $\boldsymbol{s}_a(t)$ points tend to cluster in regions where $\widetilde{\boldsymbol{\mathcal{S}}} : \widetilde{\boldsymbol{\mathcal{S}}} - \widetilde{\boldsymbol{\mathcal{R}}} : \widetilde{\boldsymbol{\mathcal{R}}} > 0$ we computed the quantity $\mathscr{Z} \equiv \langle \widetilde{\boldsymbol{\mathcal{S}}} : \widetilde{\boldsymbol{\mathcal{S}}} - \widetilde{\boldsymbol{\mathcal{R}}} : \widetilde{\boldsymbol{\mathcal{R}}}\rangle_{|\boldsymbol{a}|^2}$ using DNS. The coarse-graining was performed using a sharp spectral cut-off at wavenumber κ_c. The results in Fig. 2 confirm the prediction since they show that regions where the fluid acceleration is low ($\boldsymbol{a} \to \boldsymbol{0}$) are associated with regions where the coarse-grained strain exceeds the coarse-grained rotation ($\mathscr{Z} > 0$).

Our clustering mechanism is therefore consistent with the sweep-stick mechanism in the inertial range when $\mathrm{St_r} \ll 1$ in the sense that they both predict that particles will cluster in the same regions, namely near clusters of $\boldsymbol{s}_a(t)$ points. In light of the results in this section we may therefore suppose that particles being swept along with clusters of stagnation points is equivalent to particles being clustered in regions of high-coarse grained strain which are themselves swept by the largest scales of the turbulence. But we stress again that this equivalence only holds for $\mathrm{St_r} \ll 1$ in Navier-Stokes turbulence.

In closing this section we note that the prediction in §II that the inertial particles cluster in regions where $\widetilde{\mathscr{A}} - \widetilde{\mathscr{B}} > 0$ is only guaranteed for $\mathrm{St_r} \ll 1$, where the drift velocity is given by (7). When $\mathrm{St_r} \gtrsim \mathcal{O}(1)$ the non-local clustering mechanism contributes, and indeed dominates the centrifuge mechanism in the inertial range in the limit $Re_\lambda \to \infty$. When the non-local clustering mechanism dominates it is much more complicated to predict theoretically where the particles will cluster in the flow. However, recent work has shown that the non-local clustering mechanism in the dissipation range causes the particles to accumulate in the same high-strain, low-rotation regions of the turbulence as the local mechanism [36]. The analysis can be ported over to the inertial range, but now using the coarse-grained fluid velocity gradient field, to show that in the limit $Re_\lambda \to \infty$ and when $\mathrm{St_r} \gtrsim \mathcal{O}(1)$, the particles cluster in regions where $\widetilde{\mathscr{A}} - \widetilde{\mathscr{B}} > 0$.

IV. PREDICTING THE RDF IN THE INERTIAL RANGE

In §II we analyzed the exact equation governing $g(r)$ in order to consider the mechanism generating clustering when $\eta \ll r \ll L$. In this section we use a closed model equation for $g(r)$ in order to predict $g(r)$ when $\eta \ll r \ll L$ and $\mathrm{St_r} \ll 1$.

For isotropic turbulence (3) may be re-written as

$$\begin{aligned}0 =& g\langle \Delta u_\parallel(r^p(t), t)\rangle_r - \mathrm{St}\tau_\eta S^p_{2\parallel}\nabla_r g \\ & - \mathrm{St}\tau_\eta g\Big(\nabla_r S^p_{2\parallel} + 2r^{-1}[S^p_{2\parallel} - S^p_{2\perp}]\Big),\end{aligned} \tag{18}$$

where the subscripts $\parallel$ and $\perp$ denote the longitudinal and perpendicular projections of the tensors and $r^p(t) = |\boldsymbol{r}^p(t)|$. In [15] the term $\langle \Delta u_\parallel(r^p(t), t)\rangle_r$ is closed by approximating $\Delta \boldsymbol{u}(\boldsymbol{r}, t)$ as a spatio-temporally correlated Gaussian field and by using the Furutsu-Novikov closure method. The result they obtain is

$$\langle \Delta u_\parallel(r^p(t), t)\rangle_r \approx -\frac{1}{g}\mathrm{St}\tau_\eta \lambda_\parallel \nabla_r g, \tag{19}$$

and for $\mathrm{St_r} \ll 1$, $\eta \ll r \ll L$

$$\lambda_\parallel = (\mathrm{St}\tau_\eta)^{-1}\gamma C_2\langle\epsilon\rangle^{1/3} r^{4/3}, \tag{20}$$

where $C_2 = 2.1$ [37], $\gamma = \tau_{\mathcal{S}}(15C_2\tau_\eta^2)^{-1/2}$ [14] and $\tau_{\mathcal{S}}$ is the Lagrangian timescale of $\boldsymbol{\mathcal{S}}$. In our DNS $\tau_{\mathcal{S}} = 2.02\tau_\eta$.

It is well known that in turbulence $\Delta u_\parallel(r,t)$ can be strongly non-Gaussian, which calls into question the closure result in (19). However, results in [12] indicate that even for $r \ll \eta$, neglecting the non-Gaussian features of $\Delta u_\parallel(r,t)$ in the closure of $\langle \Delta u_\parallel(r^p(t),t)\rangle_r$ has a negligible effect on $g(r)$. This is likely a consequence of the fact that $g(r)$ is a low-order moment of the particle phase-space dynamics and therefore that it is only weakly affected by the strongly non-Gaussian features of $\Delta u_\parallel(r,t)$, which predominantly manifest themselves in the tails of the distribution. Therefore, for the present purposes of using the closure in (19) for $\eta \ll r \ll L$, the neglect of the non-Gaussianity of $\Delta u_\parallel(r,t)$ in the closure should be of even smaller importance since the non-Gaussianity of $\Delta u_\parallel(r,t)$ is weaker in the inertial range than in the dissipation range [13].

In deriving the closed expression for $\lambda_\parallel$ given in (20), ZT approximated the Lagrangian autocovariances of $\Delta\boldsymbol{u}(\boldsymbol{r}^p(t),t)$ as having an exponential decay in time with the timescale given by $\tau_r^{ZT} = \gamma\langle\epsilon\rangle^{-1/3}r^{2/3}$. This however appears to be in conflict with the behavior one would expect based on K41 arguments, namely

$$\Big\langle \Delta\boldsymbol{u}(\boldsymbol{r}^p(0),0)\cdot\Delta\boldsymbol{u}(\boldsymbol{r}^p(t'),t')\Big\rangle_{\boldsymbol{r}} \propto \langle\epsilon\rangle t', \tag{21}$$

for $\mathrm{St}=0$, according to which the autocovariances should grow indefinitely in the inertial range when $\mathrm{Re}_\lambda \to \infty$. However, it is known that applications of K41 scaling arguments to Lagrangian statistics can be in significant error, even for low order moments [38]. In Fig. 3 we show results computed from our DNS for

$$\mathcal{H}(r,t') \equiv \frac{\langle \Delta\boldsymbol{u}(\boldsymbol{r}^p(0),0)\cdot\Delta\boldsymbol{u}(\boldsymbol{r}^p(t'),t')\rangle_r}{\langle \Delta\boldsymbol{u}(\boldsymbol{r}^p(0),0)\cdot\Delta\boldsymbol{u}(\boldsymbol{r}^p(0),0)\rangle_r},$$

for $\mathrm{St}=0$ particles at $\eta \ll r \ll L$. The results show that $\mathcal{H}$ is in fact a decaying function of t' at $\eta \ll r \ll L$ and therefore demonstrate that (21) is fundamentally incorrect. We expect that the failure of the prediction in (21) is due to the fact that such a simple scaling argument does not capture the effect of the spatio-temporal decorrelation of the velocity field along the pair trajectory, and only accounts for the fact that as the pair separates, the two-point, one-time fluid velocity increments increase along the pair trajectory. In the inset of Fig. 3 we compare $\mathcal{T}_r \equiv \int_0^\infty \mathcal{H}dt'$ with the ZT prediction $\tau_r^{ZT} = \gamma\langle\epsilon\rangle^{-1/3}r^{2/3}$ which is used in their closure for $\lambda_\parallel$. The results show a remarkable agreement between τ_r^{ZT} and $\mathcal{T}_r$ and confirm the validity of the closure approximation made in the ZT for $\lambda_\parallel$ when $\eta \ll r \ll L$.

If we now substitute (19) into (18) and also use the result in (7) for the isotropic form of $\mathrm{St}\tau_\eta\boldsymbol{\nabla_r}\cdot\boldsymbol{S}_2^p$ for $\mathrm{St_r} \ll 1$ and $\eta \ll r \ll L$, we obtain the solution

$$g(r) = \exp\left[-\frac{7\mathrm{St}\tau_\eta}{45\gamma C_2\langle\epsilon\rangle^{1/3}}\int_0^r \mathfrak{r}^{-1/3}(\widetilde{\mathcal{A}}-\widetilde{\mathcal{B}})d\mathfrak{r}\right]. \tag{22}$$

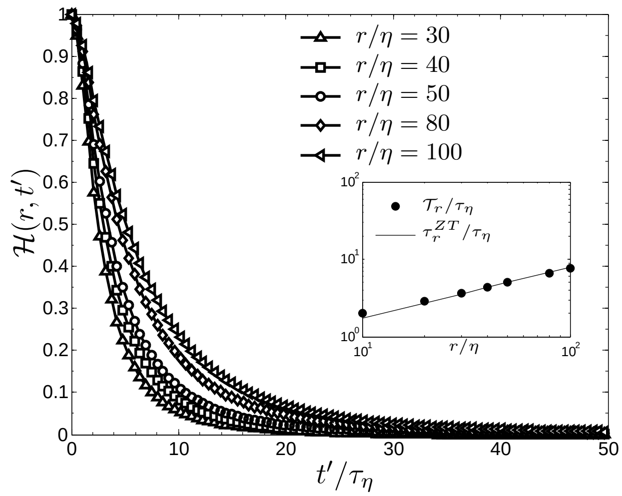

FIG. 3. DNS data for $\mathcal{H}$ for various r as a function of t'. The inset shows a comparison of the timescale $\mathcal{T}_r \equiv \int_0^\infty \mathcal{H}dt'$ with the ZT prediction $\tau_r^{ZT} = \gamma\langle\epsilon\rangle^{-1/3}r^{2/3}$.

The expression in (22) requires knowledge of $\widetilde{\mathcal{A}}-\widetilde{\mathcal{B}}$, which is difficult to predict. However, we can obtain an approximation for its r dependence in the regime $\mathrm{St_r} \ll 1$, which allows us through (22) to determine the r dependence of $g(r)$ over the range $\eta\mathrm{St}^{3/2} \ll r \ll L$. In this limit, we introduce a perturbation expansion for $\widetilde{\mathcal{A}}-\widetilde{\mathcal{B}}$ in $\mathrm{St_r}$

$$\widetilde{\mathcal{A}}-\widetilde{\mathcal{B}} = [\widetilde{\mathcal{A}}-\widetilde{\mathcal{B}}]^{[0]} + \mathrm{St_r}[\widetilde{\mathcal{A}}-\widetilde{\mathcal{B}}]^{[1]} + \mathcal{O}(\mathrm{St_r}^2), \tag{23}$$

where the superscript $[\cdot]$ denotes the order of the perturbation term. The zeroth-order term, $[\widetilde{\mathcal{A}}-\widetilde{\mathcal{B}}]^{[0]}$, which represents $\widetilde{\mathcal{A}}-\widetilde{\mathcal{B}}$ measured along fluid particle trajectories, is zero. Based on K41, we expect that to leading order in $\mathrm{St_r}$, $[\widetilde{\mathcal{A}}-\widetilde{\mathcal{B}}]^{[1]} \propto r^{-4/3}$, and using this together with the definition for $\mathrm{St_r}$, which can be re-expressed as $\mathrm{St_r} \equiv \mathrm{St}(r/\eta)^{-2/3}$, we obtain

$$\begin{aligned}\widetilde{\mathcal{A}}-\widetilde{\mathcal{B}} &= \mathrm{St}(r/\eta)^{-2/3}[\widetilde{\mathcal{A}}-\widetilde{\mathcal{B}}]^{[1]} + \mathcal{O}(\mathrm{St_r}^2)\\ &\propto r^{-2}.\end{aligned} \tag{24}$$

Substituting this into (22), we arrive at the following expression for $g(r)$ in the limit $\mathrm{St_r} \ll 1$

$$g(r) = \exp[\mathcal{D}r^{-4/3}], \tag{25}$$

where $\mathcal{D}$ is an unknown positive coefficient that is independent of r, but dependent on Stokes number, satisfying $\mathcal{D}(\mathrm{St}=0)=0$. The r dependance of $g(r)$ described by (25) is in fact the same as the result derived in [39]. This is somewhat surprising since their result was derived under the assumption that the flow is delta-correlated in time, yet our result is derived for finite correlation timescales of the flow and $\mathrm{St_r} \ll 1$. That the fundamentally different approaches lead to the same scaling prediction for $g(r)$ is because the r dependence of $g(r)$ in the inertial range reflects the shared assumption of K41 scaling in the inertial range. The physical mechanism responsible for the clustering is reflected in $\mathcal{D}$ rather than the scaling with r. The coefficient $\mathcal{D}$ in (25) arises from the dual assumptions of a finite time-correlated flow and that $\mathrm{St_r} \ll 1$ (noting that $\mathcal{D} \propto [\widetilde{\mathcal{A}} - \widetilde{\mathcal{B}}]^{[1]}$). For $\mathrm{St_r} \gtrsim \mathcal{O}(1)$, the particle relative velocities can no longer be described as power-law functions and consequently no simple analytical form for $g(r)$ can be derived. Our approach therefore makes it clear that the functional form of $g(r)$ described by (25) is only valid for $\mathrm{St_r} \ll 1$, as opposed to the functional form of $g(r)$ in the dissipation range, which is the same for all St. In contrast to our approach, [39] assumes a delta-correlated flow, for which $\mathcal{D}$ would be zero. As shown in Fig. 3, there is clear DNS support for the finite time correlations used in this analysis.

Equation (25) implies that even for $\mathrm{St_r} \ll 1$, clustering at $\eta \ll r \ll L$ is not scale-invariant [18, 24][40], in contrast to clustering at $r \ll \eta$ for $\mathrm{St} \ll 1$. This may seem surprising given that we argued that the mechanism generating the clustering in the inertial range is completely analogous to the mechanism in the dissipation range (cf. §II). The difference in the form of the clustering does not arise from a difference in the mechanism generating the clustering. Note also that according to our analysis the break in the scale-invariance of the particle clustering in the inertial range has nothing to do with the breakdown of the scale-invariance of $\Delta\boldsymbol{u}(\boldsymbol{r},t)$ in the inertial range [41] since our analysis used K41 scaling. The break in the scale-invariance of the clustering going from the dissipation to the inertial range is actually simply a consequence of the fact that τ_r is dependent on r in the inertial range, but is independent of r in the dissipation range. The final steady state form of $g(r)$ depends upon the way the drift and diffusion processes depend upon r, and their relative scaling with r is different in the dissipation and inertial ranges precisely because of the behavior of τ_r.

In Fig. 4, we use DNS data to test the prediction in (25) by plotting $r^{4/3}\ln[g(r)]$. The results show that the predicted form in (25) is quite accurate for $\mathrm{St} \lesssim 0.3$ and $10\eta \lesssim r \lesssim 200\eta$. Deviations from (25) for $\mathrm{St} > 0.3$ when $10\eta \lesssim r \lesssim 200\eta$ are due to the breakdown of the predicted scaling $\widetilde{\mathcal{A}} - \widetilde{\mathcal{B}} \propto r^{-2}$. If we assume in general $\widetilde{\mathcal{A}} - \widetilde{\mathcal{B}} \propto r^{-\alpha}$ then $g(r)$ would take the form $g(r) = \exp[\mathcal{D}r^{(2-3\alpha)/3}]$. Our data indicates that over the range of r that we have access to in our DNS, $\alpha \leq 2$, and this explains why the results in Fig. 4 show that for $\mathrm{St} > 0.3$ and $10\eta \lesssim r \lesssim 200\eta$, $\nabla_r(r^{4/3}\ln[g(r)]) > 0$.

The results in Fig. 4 for $200\eta \lesssim r \lesssim L$ show that for all St, $\nabla_r(r^{4/3}\ln[g(r)]) < 0$. This deviation of $g(r)$ from the form predicted in (25) cannot be due to a breakdown of the validity of the perturbation analysis used to derive (25), as this approximation should improve as r increases.

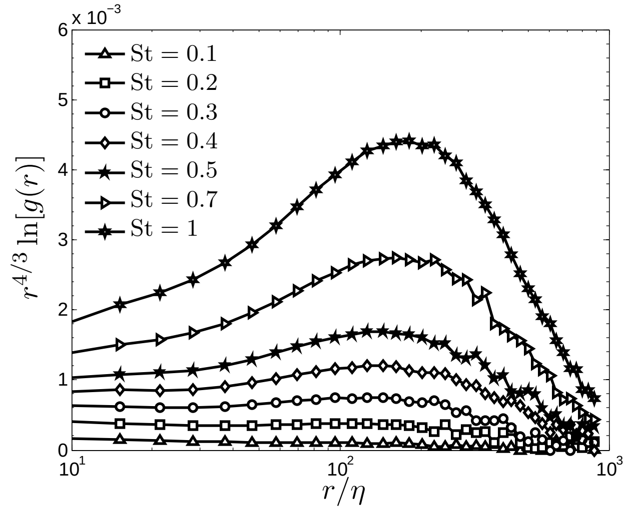

FIG. 4. DNS data for $r^{4/3}\ln[g(r)]$ for various St as a function of r.

The cause is actually the influence of the large scales. The DNS data shows that $\Delta\boldsymbol{u}(\boldsymbol{r},t)$ begins to depart from its inertial range scaling at $r \approx 200\eta$, which is somewhat surprising since in the DNS $L \approx 800\eta$. Naturally this limitation is removed in the limit $\mathrm{Re}_\lambda \to \infty$.

In order to test the quantitative accuracy of (22) we evaluate $\widetilde{\mathcal{A}} - \widetilde{\mathcal{B}}$ from the DNS using a sharp spectral cut-off at wavenumber $\kappa_c = 2\pi/r$ for the coarse-graining. Figure 5 compares $g(r)$ directly computed from the DNS with that obtained from (22) using DNS data for $\widetilde{\mathcal{A}} - \widetilde{\mathcal{B}}$. The results demonstrate the accuracy of (22) at $\eta \ll r \ll L$ when $\mathrm{St_r} \ll 1$. At this Re_λ, $\mathrm{St} > 3$ particles do not satisfy the $\mathrm{St_r} \ll 1$ requirement for (22) at $\eta \ll r \ll L$.

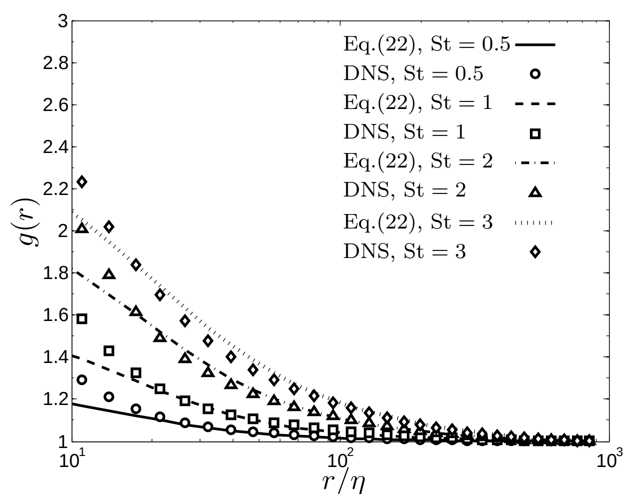

FIG. 5. Plot of DNS data and the predictions of (22) for $g(r)$.

Finally, we consider the behavior of $g(r)$ in the limit

$\mathrm{Re}_\lambda \to \infty$ as r decreases. For $\mathrm{St} \lesssim \mathcal{O}(1)$, $g(r)$ will transition from (22) to the scale-invariant form $g(r) \propto r^{-\xi(\mathrm{St})}$ at $r \ll \eta$, where $\xi(\mathrm{St}) \geq 0$. For $\mathrm{St} \gg 1$, $g(r)$ will deviate from (22) at $\eta \ll r = \mathcal{O}(\mathrm{St}^{3/2}\eta) \ll L$. At $r = \mathcal{O}(\mathrm{St}^{3/2}\eta)$, $\mathrm{St_r} = \mathcal{O}(1)$ at which point the path-history symmetry breaking effect dominates the clustering mechanism. We cannot derive a prediction for the analytic form of $g(r)$ in this regime because the particle relative velocity structure function in this regime does not have a known functional form (e.g. it is not a simple power law). As r decreases further, the particles enter a ballistic regime, where $g(r) \approx$ constant [24, 42]. All of these trends can be seen in [33]. The theoretical question of the existence of a transition to $g(r) \approx$ constant for $\mathrm{St} \lesssim \mathcal{O}(1)$ at $r \lll \eta$ remains an open question [42].

V. CONCLUSIONS

In this paper, we have considered the mechanism for the clustering of inertial particles in the inertial range of isotropic turbulence. By analyzing the exact equation governing the RDF we have demonstrated that the clustering mechanisms in the inertial range are completely analogous to the mechanisms in the dissipation range. For any separation r which is less than the integral lengthscale of the flow, the clustering mechanism for $\mathrm{St_r} \ll 1$ is related to the preferential sampling of the coarse-grained fluid velocity gradient tensor at scale $\approx r$, which is associated with centrifuging out of eddies at that scale. When $\mathrm{St_r} \gtrsim \mathcal{O}(1)$ a non-local mechanism contributes to the inward drift that generates the clustering through the statistical asymmetry of the path-history of approaching and separating particle pairs.

This claim regarding the universality of the clustering mechanism across the range of scales in turbulence is in disagreement with other explanations in the literature that in the inertial range a completely different mechanism generates the clustering, namely the sweep-stick mechanism. However, we have argued that the sweep-stick mechanism is essentially equivalent to our mechanism in the inertial range of Navier-Stokes turbulence when $\mathrm{St_r} \ll 1$. Since our mechanism reveals that the clustering mechanism is analogous at all scales in turbulence, we may therefore conclude that the sweep-stick mechanism does not really imply a basic change in the clustering mechanism as one goes from dissipative to inertial scales. We also showed that the sweep-stick mechanism is only valid for $\mathrm{St_r} \ll 1$ in the inertial range. Since our mechanism can explain clustering in any spatially correlated velocity field, whereas the sweep-stick mechanism can only explain clustering in Navier-Stokes turbulence, we conclude that our mechanism provides the more fundamental explanation.

Finally, we applied our results for the form of the drift velocity in the regime $\mathrm{St_r} \ll 1$ in the inertial range to the model equation for the RDF from [15]. Using this we obtained a prediction for the analytic form of the RDF in the inertial range when $\mathrm{St_r} \ll 1$. Comparisons with DNS data demonstrated the accuracy of the prediction.

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