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Comment on “Kinetic theory for a mobile impurity in a degenerate Tonks-Girardeau gas”

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In a recent paper [1] Gamayun, Lychkovskiy, and Cheianov studied the dynamics of a mobile impurity weakly coupled to a one-dimensional Tonks-Girardeau gas of strongly interacting bosons. Employing the Boltzmann equation approach they, in particular, arrived at the following conclusions: (i) a light impurity, being accelerated by a constant force, F , does *not* exhibit Bloch oscillations, which were predicted and studied in Refs. [2, 3]; (ii) a heavy impurity does undergo Bloch oscillations, accompanied by a drift with the velocity $v_D \propto \sqrt{F}$. In this comment we argue that the result (i) is an artifact of the classical Boltzmann approximation. The latter misses the formation of the (quasi) *bound-state* between the impurity and a hole. Its dispersion relation $E_b(P, \rho)$ is a *smooth* periodic function of momentum P with the period $2k_F = 2\pi\hbar\rho$, where ρ is a density of the host gas. Being accelerated by a small force, such a bound-state exhibits Bloch oscillations superimposed with the drift velocity $v_D = \mu F$. The mobility μ may be expressed *exactly* [3] in terms of $E_b(P, \rho)$. Result (ii), while not valid at exponentially small forces, indeed reflects an interesting intermediate-force behavior.

The origin of Bloch oscillations is most transparent for a weakly interacting Bose gas, described by the Gross-Pitaevskii (GP) equation. Its solution reveals [3] that a repulsive impurity binds to a dark soliton – a region of the size of the healing length ξ , where the host gas is depleted. The resulting composite object (the “depleton”) has a dispersion curve, which is periodic in momentum with the period $2\pi\hbar\rho$, where ρ is the 1d density of the host gas. Upon application of a sufficiently small external force, F , the impurity follows its dispersion curve and thus exhibits the Bloch oscillations. This occurs as a result of the time scale separation between the fast local equilibration time $\sim \xi/c$ of the depleton, where c is the sound velocity, and the slow period of the Bloch oscillations $\sim 2\pi\hbar\rho/F$.

One of the principal claims of Refs. [3, 4] is that the above features are not artifacts of the GP equation, but apply generally to mobile impurities in 1d quantum liquids. In particular, the periodicity of the impurity dispersion can be understood in the opposite limit of Tonks-Girardeau (TG) gas of impenetrable bosons. The main focus of this Comment is to *illustrate* this effect by formation of the bound-state between the impurity and the single hole in the fermionic representation of TG gas. Such a bound-state, crucial for Bloch oscillations to exist, is not captured by the perturbative Boltzmann approach of Ref. [1].

One may represent strongly interacting TG Bose gas of N hard-core bosons by free fermions created by $c_p^\dagger$, weakly coupled to a quantum impurity with the coordinate x_i through the density-density interaction ($\hbar = 1$):

$$\hat{H} = -\frac{1}{2m_i} \frac{\partial^2}{\partial x_i^2} + \sum_p \frac{p^2}{2m_h} c_p^\dagger c_p + \gamma \frac{\rho^2}{m_h N} \sum_{p,q} c_p^\dagger c_{p+q} e^{iqx_i} \quad (1)$$

where m_h is the mass of host particles, m_i is the impurity mass and $0 < \gamma \ll 1$ is a dimensionless coupling

constant. Consider a state of the system with total momentum $P > 0$. If $P < P_0 \equiv \min\{m_i v_F, k_F\}$, the low energy states are those where most of the momentum is carried by the impurity. Indeed, the impurity kinetic energy $P^2/(2m_i)$ is less than that of soft particle-hole excitations above the Fermi sea $\sim v_F P$. In the opposite limit $P > P_0$ the low energy states are those where *hole excitations* carry a significant fraction of the entire momentum P . The many-body ground state adiabatically connects between these two limits, signaling strong impurity-hole hybridization at $P \gtrsim P_0$. We show that in addition to the perturbative corrections discussed in Ref. [1], there exists a non-perturbative component of this hybridization which becomes important at $P > P_0$. Such processes are responsible for the smoothness of the impurity dispersion relation, which in turn gives rise to Bloch oscillations under application of an external force.

To illustrate this, consider a subspace of the full many-body space, which contains a single hole excitation with momentum $0 < k < 2k_F$ in addition to the impurity with momentum $P - k$ (this restriction is justified in the limit $\gamma \ll 1$). The basis vectors of this subspace are

$$|k; P\rangle = e^{i(P-k)x_i} c_{k_F}^\dagger c_{k_F-k} |\text{Fermi Sea}\rangle. \quad (2)$$

The corresponding Schrödinger equation $\sum_{k'} \langle k; P | \hat{H} | k'; P \rangle \psi_P(k') = E \psi_P(k)$ takes the form of the two-particle problem with the *attractive* delta-interaction (formally the attraction arises from anti-commuting the fermionic operators in the last term in Eq. (1)),

$$\left[\frac{(P-k)^2}{2m_i} + E_h(k) \right] \psi_P(k) - \frac{\gamma\rho}{m_h} \int_0^{2k_F} \frac{dk'}{2\pi} \psi_P(k') = E \psi_P(k), \quad (3)$$

where $E_h(k) = v_F k - k^2/(2m_h)$ is the hole kinetic energy (we measure E relative to $NE_F/3 + \gamma\rho^2/m_h$). This prob-

lem admits a unique bound-state solution, whose energy $E = E_b(P)$ is found from the integral equation

$$\int_0^{2k_F} \frac{dk'}{\frac{(P-k')^2}{2m_i} + E_h(k') - E_b(P)} = \frac{2\pi m_h}{\gamma\rho}. \quad (4)$$

Its solution, shown in Fig. 1, is a smooth periodic function of the total momentum, which is split from the scattering continuum $E_h(k) + (P-k)^2/2m_i$ by the gap Δ . The hard gap between the bound-state and the continuum is an artifact of restricting the particle in Eq. (2) to be created right at the Fermi momentum k_F . Allowing for slight deviation $c_{k_F}^\dagger \rightarrow c_{k_F+p}^\dagger$, enlarges the Hilbert space to include, in addition to the bound-state, low energy, $\sim v_F p$, particle-hole excitations. It is well known [6–8] that interactions with these excitations transforms the bound-state into the *quasi* bound-state with the power-law (instead of the pole) correlation function [5]. These low energy excitations are responsible for radiation losses and thus for finite mobility μ . Below we argue that they do not destroy the Bloch oscillations, associated with the impurity following the quasi bound-state upon application of a small external force.

It is worth noticing that the one-hole bound-state solution (4) is in *quantitative* agreement with the available exact results. For example, for $m_i = m_h$, the integrability of Eq. (1) provides with the exact ground-state energy in the vicinity of $P \approx P_0 = k_F$ [6], $E_0(P \sim k_F) = E_F - \frac{2\pi^2}{3\gamma} \frac{(P-k_F)^2}{2m_h}$. One may verify from Eq. (4): $E_b(P \sim k_F) + \gamma\rho^2/m_h \approx E_0(P \sim k_F)$, up to perturbative corrections of $\mathcal{O}(\gamma^2)$ to the effective mass which are subleading for $\gamma \ll 1$. This shows that the single hole binding to the impurity is indeed the leading physical effect in the weak coupling limit.

To estimate a characteristic force $F_{\max}$, below which the Bloch oscillations exist, one needs to resort to the exact scattering phase shifts, $\delta_\pm(P)$ [8], of the left/right moving low energy excitations on the impurity. The local equilibration takes place if the rate of change of these parameters is smaller than the inverse characteristic time required for phonons to traverse the vicinity of the depletion, $\sim c/\xi$. One demands thus $\partial_t \delta_\pm < c/\xi$ and using $\partial_t \delta_\pm = F \partial_P \delta_\pm$, finds $F < (\partial_P \delta_\pm)^{-1} c/\xi$. The right hand side is smallest when the phases $\delta_\pm$ change rapidly with the momentum, which occurs near P_0 . Near this point the phase shifts change by approximately π in the narrow interval of momenta $\delta P \sim \Delta/v$, where $v = v_F \min\{1, 1/\eta\}$ and $\eta = m_i/m_h$. As a result we obtain $F_{\max} \sim c\Delta/v\xi$ and, using Eq. (4) and the fact that in the Tonks-Girardeau regime $c = v_F$, $\xi = \pi/k_F$, one finds in the two limiting cases

$$F_{\max} \propto \frac{k_F^3 \eta}{m_h} \begin{cases} \gamma^2, & \eta \ll 1 \\ e^{-\pi^2/\gamma}, & \eta \gg 1 \end{cases} \quad (5)$$

For $F < F_{\max}$ both light and heavy impurities exhibit Bloch oscillations along with the drift [3], whose velocity scales linearly with the force $v_D = \mu F$. The finite

FIG. 1. (Color online) The bound-state $E_b(P)$, Eq. (4), (thick black line) and scattering continuum $\frac{(P-k)^2}{2m_i} + E_h(k)$ for a set of k (thin gray lines) for the light impurity $\eta = 1/2$ (a) and heavy impurity $\eta = 2$ (b). In both cases $\gamma = 0.7$.

curvature of the dispersion relation, associated with the binding energy Δ , is the crucial physical insight to reach this conclusion.

In Refs. [4, 6] it was shown that for a heavy impurity *away* from the Tonks-Girardeau limit, there exists a phase transition at a critical value of the impurity mass: for $m_i < M_c$ the ground-state is a smooth function of momentum, while for $m_i > M_c$ the ground-state exhibits a cusp singularity at momenta $P = (1+2n)k_F$ for integer n (in the Tonks-Girardeau limit $M_c \rightarrow \infty$). In the latter case the impurity “overshoots” the intersection points at $P = (1+2n)k_F$ and has to emit phonons to reach the ground state. This leads to an enhanced dissipation [4] and thus to super-linear drift velocity

$$v_D \propto F^{1/(1+\alpha)}, \quad (6)$$

where $\alpha \approx 2K - 1$ for $\gamma \ll 1$ and K is the Luttinger parameter of the host.

Notice that in the Tonks-Girardeau limit the validity of the $v_D = \mu F$ response for $\eta > 1$ is limited to an exponentially small force given by Eq. (5). This scale originates from the exponentially narrow region of momenta, where the bound-state exhibits the avoided crossing behavior,

Fig. 1(b). For $F > F_{\max}$ the impurity overshoots the avoided crossing and follows the “wrong” parabola before emitting phonons and returning to the ground state. Thus, for $F > F_{\max}$ one may apply Eq. (6) with $K = 1$ – appropriate for the Tonks gas. This leads to $v_D \propto \sqrt{F}$, in full agreement with Ref. [1]. An important exten-

sion of results in Ref. [1] is that the super-linear drift (6) is to be expected for moderately heavy impurities $m_h < m_i < M_c$ in an intermediate range of forces where the linear drift $v_D = \mu F$ is inapplicable.

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