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Dynamics of a differential-difference integrable (2+1)-dimensional system

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Abstract

Kadomtsev-Petviashvili (KP)-type equation appears in fluid mechanics, plasma physics and gas dynamics. In this paper, we propose an integrable semi-discrete analogue of the coupled $(2+1)$-dimensional system which is related to KP equation and Zakharov equation. N-soliton solutions of the discrete equation are presented. Some interesting examples of soliton resonance related to the two-soliton and three-soliton solutions are investigated. Numerical computations using the integrable semi-discrete equation are performed. It is shown that the integrable semi-discrete equation gives very accurate numerical results in the cases of one soliton evolution and soliton interactions.

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1 Introduction

Nonlinear evolution equations (NEEs) appear in almost all the physics branches, such as fluid mechanics, plasma physics, optical fibers and solid state physics. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. Since the concept of the solitons for the Korteweg-de Vries (KdV) equation was introduced, there has been considerable interest in this kind of special NEEs, such as Burgers equation, nonlinear Schrödinger (NLS) equation, Boussinesq equation.

Compared with the one-dimensional NEEs, the $(2+1)$-dimensional coupled systems are more attractive in describing the nonlinear phenomena in the real physical situations. Some $(2+1)$-dimensional NEEs exhibit not only localized coherent structures as the curved-line solitons, half-straight-line solitons and dromions [1, 2], but also the inelastic interactions, e.g., the resonance [3], reconnection [4], and annihilation [5].

Zakharov formulated the system of equations

$$\mathrm{i}E_t + \frac{1}{2}E_{xx} - \mathrm{n}E = 0, \tag{1}$$

$$\mathrm{n}_{tt} - \mathrm{n}_{xx} - 2(|E|^2)_{xx} = 0, \tag{2}$$

for the ion sound wave under the action of the ponderomotive force due to high-frequency field and for the Langmuir wave [6]. Here $Ee^{-\mathrm{i}\omega_p t}$ is the normalized electric field of the Langmuir oscillation, n is the normalized density perturbation, x is the normalized spatial variable, t the time variable and the subscripts denote the partial derivatives. For the ion sound wave propagating in only one-direction, for example, in the positive x-direction, one can suppose that

$$\mathrm{n}_t \cong -\mathrm{n}_x. \tag{3}$$

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Under this assumption Eq. (2) can be simplified as follows

$$\mathrm{n}_t + \mathrm{n}_x + (|E|^2)_x = 0. \tag{4}$$

Interaction of solitons of the system (1) and (4) were studied by the inverse scattering technique in [7].

In the present work, we consider the following $(2+1)$-dimensional soliton equation

$$\mathrm{i}u_t + u_{xx} + uv = 0, \tag{5a}$$

$$v_t + v_y + (|u|^2)_x = 0, \tag{5b}$$

where $\mathrm{i} = \sqrt{-1}$, u is a complex function of two scaled space coordinates x, y and time t, v is a real function. Eqs.(5) are similar to the integrable Zakharov equation (1) and (4) when $x = y$ in Eq. (5b). Maccari [8] obtained Eqs.(5) by an asymptotically exact reduction method based on Fourier expansion and spatiotemporal rescaling from the KP equation. He also constructed the Lax pair for the system. Painlevé property of the system (5) was investigated in [9] and its doubly-periodic solutions were given by using the extended Jacobian elliptic function expansion method [10]. Traveling wave solutions of the system were obtained in [11, 12]. The interaction dynamics between the two solitons, especially the soliton resonant interactions, was studied in [13]. However, to the best of our knowledge, $N-$soliton solutions of the system (5) have not been given by use of the Hirota method.

Over the decades, integrable discretizations of soliton equations have received considerable attention [14–17]. Ablowitz and Ladik proposed how to construct integrable discrete analogues of soliton equations based on Lax pairs [18, 19]. Hirota proposed bilinear method to construct integrable discrete analogues of soliton equations based on bilinear equations [20–22]. Applications of integrable discretizations of soliton equations were considered in various fields [23–27]. In our recent works, we proposed an integrable semi-discrete analogue of the coupled integrable dispersionless equations [28, 29]. The key step there is the discretization of bilinear differential operators under gauge invariance. Considering the physical background and potential application of the $(2+1)$-dimensional system (5), we aim to study its semi-discrete analogue and the dynamics of soliton solutions of the semi-discrete system.

The remainder of this paper is organized as follows. In section 2, we derive $N-$soliton solutions of the system (5) by using the Hirota method. In section 3, we present a semi-discrete analogue of the system in the spatial direction. In section 4, the numerical computations of the semi-discrete system are performed. Interactions of multi-soliton solutions, especially the resonance of two solitons, are investigated by means of asymptotic behaviors in section 5. Conclusions are given in section 6. Finally we present N-soliton solution of the semi-discrete system by pfaffian technique in the appendix.

2 Bilinear form and soliton solutions

Through the dependent variable transformations

$$u = \frac{g}{f}, \qquad v = 2(\ln f)_{xx}, \tag{6}$$

where g and f are the complex and real functions of x, y and t, respectively, the bilinear forms of system (5) is expressed as

$$(\mathrm{i}D_t + D_x^2)g \bullet f = 0, \tag{7}$$

$$(D_xD_t + D_xD_y)f \bullet f + gg^* = 0. \tag{8}$$

Here the bilinear differential operator is defined by [30]

$$D_{x_1}^{n_1} D_{x_2}^{n_2} a \bullet b \equiv \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_1'}\right)^{n_1} \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2'}\right)^{n_2} a(x_1, x_2) \bullet b(x_1', x_2')|_{x_1'=x_1, x_2'=x_2}. \tag{9}$$

In [13], one-soliton and two-soliton solution of (7)-(8) were found. We get that one-soliton solution can be expressed in the form

$$g = \exp(\eta_1), \tag{10}$$

$$f = 1 + a(1, 1^*)\exp(\eta_1 + \eta_1^*), \tag{11}$$

with $\eta_1 = k_1x + p_1y + \mathrm{i}k_1^2t$ and $a(1,1^*) = -\frac{1}{2(k_1+k_1^*)(\mathrm{i}k_1^2-\mathrm{i}k_1^{*2}+p_1+p_1^*)}$. Here k_1, p_1 are complex constants and η^* denotes complex conjugate of η.

Two-soliton solution is in the following form

$$\begin{aligned}
g =& \exp(\eta_1) + \exp(\eta_2) + a(1,2,1^*)\exp(\eta_1+\eta_2+\eta_1^*) \\
&+ a(1,2,2^*)\exp(\eta_1+\eta_2+\eta_2^*),
\end{aligned} \tag{12}$$

$$\begin{aligned}
f =& 1 + a(1,1^*)\exp(\eta_1+\eta_1^*) + a(1,2^*)\exp(\eta_1+\eta_2^*) \\
&+ a(2,1^*)\exp(\eta_2+\eta_1^*) + a(2,2^*)\exp(\eta_2+\eta_2^*) \\
&+ a(1,2,1^*,2^*)\exp(\eta_1+\eta_2+\eta_1^*+\eta_2^*),
\end{aligned} \tag{13}$$

with $\eta_j = k_jx + p_jy + \mathrm{i}k_j^2t, \quad (j=1,2)$. Here the coefficients are defined by formulas

$$a(j,l^*) = -\frac{1}{2(k_j+k_l^*)(\mathrm{i}k_j^2 - \mathrm{i}k_l^{*2} + p_j + p_l^*)}, \tag{14}$$

$$a(i,j) = 2(k_i-k_j)(-\mathrm{i}k_i^2 + \mathrm{i}k_j^2 - p_i + p_j), \tag{15}$$

$$a(i^*,j^*) = 2(k_i^*-k_j^*)(\mathrm{i}k_i^{*2} - \mathrm{i}k_j^{*2} - p_i^* + p_j^*), \tag{16}$$

$$a(i,j,k^*) = a(i,j)a(i,k^*)a(j,k^*), \tag{17}$$

$$a(i,j^*,k^*) = a(i,j^*)a(i,k^*)a(j^*,k^*), \tag{18}$$

$$a(i,j,k^*,l^*) = a(i,j)a(i,k^*)a(i,l^*)a(j,k^*)a(j,l^*)a(k^*,l^*), \tag{19}$$

where k_j and p_j are complex constants. In the same way, we can construct the three-soliton solution,

$$\begin{aligned}
g =& \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) \\
&+ a(1,2,1^*)\exp(\eta_1+\eta_2+\eta_1^*) + a(1,3,1^*)\exp(\eta_1+\eta_3+\eta_1^*) + a(2,3,2^*)\exp(\eta_2+\eta_3+\eta_2^*) \\
&+ a(1,2,2^*)\exp(\eta_1+\eta_2+\eta_2^*) + a(1,3,3^*)\exp(\eta_1+\eta_3+\eta_3^*) + a(2,3,3^*)\exp(\eta_2+\eta_3+\eta_3^*) \\
&+ a(1,2,3^*)\exp(\eta_1+\eta_2+\eta_3^*) + a(1,3,2^*)\exp(\eta_1+\eta_3+\eta_2^*) + a(2,3,1^*)\exp(\eta_2+\eta_3+\eta_1^*) \\
&+ a(1,2,3,1^*,2^*)\exp(\eta_1+\eta_2+\eta_3+\eta_1^*+\eta_2^*) + a(1,2,3,1^*,3^*)\exp(\eta_1+\eta_2+\eta_3+\eta_1^*+\eta_3^*) \\
&+ a(1,2,3,2^*,3^*)\exp(\eta_1+\eta_2+\eta_3+\eta_2^*+\eta_3^*),
\end{aligned} \tag{20}$$

$$\begin{aligned}
f =& 1 + a(1,1^*)\exp(\eta_1+\eta_1^*) + a(2,2^*)\exp(\eta_2+\eta_2^*) + a(3,3^*)\exp(\eta_3+\eta_3^*) \\
&+ a(1,2^*)\exp(\eta_1+\eta_2^*) + a(2,1^*)\exp(\eta_2+\eta_1^*) + a(2,3^*)\exp(\eta_2+\eta_3^*) + a(3,2^*)\exp(\eta_3+\eta_2^*) \\
&+ a(1,3^*)\exp(\eta_1+\eta_3^*) + a(3,1^*)\exp(\eta_3+\eta_1^*) \\
&+ a(1,2,1^*,2^*)\exp(\eta_1+\eta_2+\eta_1^*+\eta_2^*) + a(1,3,1^*,3^*)\exp(\eta_1+\eta_3+\eta_1^*+\eta_3^*) \\
&+ a(1,2,1^*,3^*)\exp(\eta_1+\eta_2+\eta_1^*+\eta_3^*) + a(1,3,1^*,2^*)\exp(\eta_1+\eta_3+\eta_1^*+\eta_2^*) \\
&+ a(1,2,2^*,3^*)\exp(\eta_1+\eta_2+\eta_2^*+\eta_3^*) + a(2,3,1^*,2^*)\exp(\eta_2+\eta_3+\eta_1^*+\eta_2^*) \\
&+ a(1,3,2^*,3^*)\exp(\eta_1+\eta_3+\eta_2^*+\eta_3^*) + (2,3,1^*,3^*)\exp(\eta_2+\eta_3+\eta_1^*+\eta_3^*) \\
&+ a(2,3,2^*,3^*)\exp(\eta_2+\eta_3+\eta_2^*+\eta_3^*) + a(1,2,3,1^*,2^*,3^*)\exp(\eta_1+\eta_2+\eta_3+\eta_1^*+\eta_2^*+\eta_3^*),
\end{aligned} \tag{21}$$

where the coefficients are defined as (14)-(19) and in general

$$a(i_1,i_2,\cdots,i_n,j_1^*,\cdots,j_m^*) = \prod_{1\le k<l\le n} a(i_k,i_l) \prod_{1\le k\le n, 1\le l\le m} a(i_k,j_l^*) \prod_{1\le k<l\le m} a(j_k^*,j_l^*). \tag{22}$$

From the above expressions of the one-, two- and three- soliton solutions, we know that the exact N-soliton solution of Eqs. (5) is in the following form

$$f = \sum_{\mu=0,1}^{(e)} \exp\left[\sum_{j=1}^{N}\mu_j\eta_j + \sum_{j=N+1}^{2N}\mu_j\eta_{j-N}^* + \sum_{1\le i<j}^{2N}\mu_i\mu_jA_{ij}\right], \tag{23}$$

$$g = \sum_{\nu=0,1}^{(o)} \exp\left[\sum_{j=1}^{N}\nu_j\eta_j + \sum_{j=N+1}^{2N}\nu_j\eta_{j-N}^* + \sum_{1\le i<j}^{2N}\nu_i\nu_jA_{ij}\right], \tag{24}$$

where

$$\eta_j = k_j x + p_j y + \mathrm{i}k_j^2 t, \quad j = 1, 2, \cdots, N, \tag{25}$$

$$\eta_j^* = \text{conjugate of } \eta_j, \quad j = 1, 2, \cdots, N, \tag{26}$$

$$\exp(A_{i,j}) = a(i, j), \quad i < j = 2, 3, \cdots, N, \tag{27}$$

$$\exp(A_{i,N+j}) = a(i, j^*) \quad i, j = 1, 2, \cdots, N, \tag{28}$$

$$\exp(A_{N+i,N+j}) = a(i^*, j^*), \quad i < j = 2, 3, \cdots, N. \tag{29}$$

Here α_j, γ_j are both real parameters relating respectively to the amplitude and phase of the $i-$th soliton. The sum $\sum_{\mu=0,1}^{(e)}$ indicates the summation over all possible combinations of $\mu_i = 0, 1$ under the condition

$$\sum_{j=1}^{N} \mu_j = \sum_{j=1}^{N} \mu_{N+j}, \tag{30}$$

and $\sum_{\nu=0,1}^{(o)}$ indicates the summation over all possible combinations of $\nu_i = 0, 1$ under the condition

$$\sum_{j=1}^{N} \nu_j = \sum_{j=1}^{N} \nu_{N+j} + 1. \tag{31}$$

The form of the N-soliton solution (23)-(24) is the same as that of the combined Schrödinger-mKdV equation in [31]. The proof of the N-soliton solution here can be completed by induction and is similar to the one in [31]. The reader can find the details there.

It is known that soliton solutions of many integrable systems (e.g. Schrödinger type, BKP type) can be expressed in pfaffian form. In the appendix part, we construct $N-$soliton solution to the semi-discrete system (32) by using the pfaffian technique.

3 Integrable discrete analogue of the (2+1)-dimensional system

We consider the discrete system

$$\left[\mathrm{i}D_t + \frac{4}{\epsilon^2}\sinh^2\left(\frac{D_n}{2}\right)\right] g_n \bullet f_n = 0, \tag{32a}$$

$$\frac{4}{\epsilon}(D_t + D_y) f_{n+1} \bullet f_n + g_{n+1} g_n^* + g_n g_{n+1}^* = 0, \tag{32b}$$

where the bilinear difference operator $\exp(\delta D_n)$ in sinh function is defined by

$$\exp(\delta D_n) a \bullet b \equiv a(n + \delta) b(n - \delta), \tag{33}$$

and the parameter ϵ can be regarded as spatial discrete step. With the variable transformation

$$u_n = \frac{g_n}{f_n}, \quad w_n = \ln \frac{f_{n+1}}{f_n}, \tag{34}$$

bilinear equations (32) can be cast into

$$\mathrm{i}u_{n,t}\epsilon^2 + (u_{n+1} + u_{n-1}) e^{w_n - w_{n-1}} - 2u_n = 0, \tag{35a}$$

$$4(w_{n,t} + w_{n,y}) + \epsilon(u_n^* u_{n+1} + u_n u_{n+1}^*) = 0. \tag{35b}$$

Setting

$$v_n = \frac{1}{\epsilon^2} \frac{2(f_{n+1} f_{n-1} - f_n^2)}{f_n^2} = \frac{2}{\epsilon^2}(e^{w_n - w_{n-1}} - 1) \tag{36}$$

and substituting it into (35a) result

$$\mathrm{i}u_t\epsilon^2 + (2u + \epsilon^2 u_{xx})(1 + \frac{\epsilon^2}{2}v) - 2u + \mathcal{O}(\epsilon^2) = 0. \tag{37}$$

The coefficient of the term ϵ^2 is

$$\mathrm{i}u_t + u_{xx} + uv = 0. \tag{38}$$

By shifting n to $n-1$ in (35b) and subtracting each other, we get

$$4(\partial_t + \partial_y)e^{w_n - w_{n-1}} + e^{w_n - w_{n-1}}\epsilon(\mathrm{E} - 1)(u_{n-1}u_n^* + u_{n-1}^* u_n) = 0, \tag{39}$$

or equivalently,

$$4(\partial_t + \partial_y)(1 + \frac{\epsilon^2}{2}v_n) + \epsilon(1 + \frac{\epsilon^2}{2}v_n)(\mathrm{E} - 1)(u_{n-1}u_n^* + u_{n-1}^* u_n) = 0. \tag{40}$$

Here E is the shift operator $\mathrm{E}a_n = a_{n+1}$. The continuum limit of (40) as $\epsilon \to 0$ is

$$v_t + v_y + (|u|^2)_x = 0. \tag{41}$$

Thus (35) gives a semi-discrete analogue of the system (5). From the derivation above, by eliminating w in (35), we obtain the following semi-discrete system for u and v,

$$\mathrm{i}u_{n,t} + \frac{u_{n+1} + u_{n-1} - 2u_n}{\epsilon^2} + \frac{(u_{n+1} + u_{n-1})v_n}{2} = 0, \tag{42a}$$

$$v_{n,t} + v_{n,y} + (1 + \frac{\epsilon^2}{2}v_n)\frac{u_n(u_{n+1}^* - u_{n-1}^*) + u_n^*(u_{n+1} - u_{n-1})}{2\epsilon} = 0. \tag{42b}$$

Remark 3.1. By multiplying u_n^* in the both sides of the equation (42a), employing the conjugate and then subtracting two equations each other, we have

$$\begin{aligned}\mathrm{i}(|u_n|^2)_t + &\frac{1}{\epsilon^2}\Big((u_{n+1} + u_{n-1})u_n^* - (u_{n+1}^* + u_{n-1}^*)u_n\Big) \\ &+ \frac{1}{2}\Big(u_n^* v_n(u_{n+1} + u_{n-1}) - u_n v_n(u_{n+1}^* + u_{n-1}^*)\Big) = 0.\end{aligned}$$

By summation of n, we get

$$\frac{d}{dt}\sum_{n=-\infty}^{+\infty}|u_n|^2 = 0,$$

which proves that the total energy $\sum_{n=-\infty}^{\infty}|u_n|^2$ is conserved. Numerical computation is given in the next section.

Remark 3.2. One can check that the first bilinear equation of (5), i.e. Eq. (7), is the same as the one of nonlinear Schrödinger (NLS) equation. It is well-known that Davey-Stewartson equation, a two-dimensional NLS equation that appeared as a shallow-water limit of the Benney-Roskes equation, arises from the two-component KP hierarchy [32]. It was pointed out in [33] that the discretization of NLS equation can be obtained from the reduction of two-component KP hierarchy. Hence we believe that the semi-discrete system (42) must have relation with the two-component KP hierarchy. Meanwhile, since the $(2+1)$-dimensional system (5) is derived from the KP equation via an asymptotically exact reduction method, the relation between the semi-discrete system (42) and the differential-difference KP equation [34, 35],

$$\Delta(u_t + 2u_y - 2uu_y) = (2 + \Delta)u_{yy}, \tag{43}$$

deserves further consideration. Here $u = u(y,t,n)$ and Δ denotes the forward difference operator defined by $\Delta f_n = f_{n+1} - f_n$.

One-soliton solution for (32) has the form

$$f_n = 1 + b(1,1^*)\exp(\eta_1+\eta_1^*), \qquad g_n = \exp(\eta_1), \tag{44}$$

with

$$\eta_1 = k_1 n + p_1 y + q_1 t, \qquad q_1 = \frac{\mathrm{i}}{\epsilon^2}\left(e^{k_1}+e^{-k_1}-2\right), \tag{45}$$

$$b(1,1^*) = -\frac{\epsilon(e^{k_1}+e^{k_1^*})}{4(e^{k_1+k_1^*}-1)(p_1+p_1^*+q_1+q_1^*)}, \tag{46}$$

and p_1, k_1 are complex constants and η_1^* is the complex conjugate of η_1. If we set $x = n\epsilon$ and $k_1 = \epsilon\tilde{k_1}$, we get the following asymptotic relation,

$$\eta_1 = \tilde{k}_1 x + p_1 y + q_1 t, \tag{47}$$

$$q_1 = \frac{\mathrm{i}}{\epsilon^2}(e^{\tilde{k}_1\epsilon}+e^{-\tilde{k}_1\epsilon}-2) = \mathrm{i}\tilde{k}_1^2 + \mathcal{O}(\epsilon), \tag{48}$$

$$b(1,1^*) = \frac{-1}{2(\tilde{k}_1+\tilde{k}_1^*)(p_1+p_1^*+q_1+q_1^*)} + \mathcal{O}(\epsilon). \tag{49}$$

This shows that the one-soliton solution of the semi-discrete equation yields the one of the continuous equation through the continuum limit $\epsilon \to 0$.

The one-soliton solutions for u_n and v_n are expressed as

$$u_n = \frac{g_n}{f_n} = \frac{e^{\eta_1}}{1+b_{11}e^{\eta_1+\eta_1^*}} = \frac{1}{2\sqrt{b_{11}}}e^{\mathrm{i}\mathrm{Im}(\eta_1)}\mathrm{sech}(\mathrm{Re}(\eta) + \frac{\ln b_{11}}{2}), \tag{50}$$

$$v_n = \frac{2}{\epsilon^2}\Big(\frac{f_{n+1}f_{n-1}}{f_n^2}-1\Big) = \frac{2}{\epsilon^2}\Big[\frac{(1+b_{11}e^{2\mathrm{Re}(\eta_1+k_1)})(1+b_{11}e^{2\mathrm{Re}(\eta_1-k_1)})}{(1+b_{11}e^{2\mathrm{Re}(\eta_1)})^2}-1\Big], \tag{51}$$

with $b_{11} = b(1,1^*)$. Two-soliton solution of the semi-discrete system has the form

$$\begin{aligned} f_n =& 1 + b(1,1^*)\exp(\eta_1+\eta_1^*) + b(1,2^*)\exp(\eta_1+\eta_2^*) \\ &+ b(2,1^*)\exp(\eta_2+\eta_1^*) + b(2,2^*)\exp(\eta_2+\eta_2^*) \\ &+ b(1,2,1^*,2^*)\exp(\eta_1+\eta_2+\eta_1^*+\eta_2^*), \end{aligned} \tag{52}$$

$$\begin{aligned} g_n =& \exp(\eta_1) + \exp(\eta_2) + b(1,2,1^*)\exp(\eta_1+\eta_2+\eta_1^*) \\ &+ b(1,2,2^*)\exp(\eta_1+\eta_2+\eta_2^*), \end{aligned} \tag{53}$$

with the coefficients

$$b(i,j^*) = -\frac{\epsilon(e^{k_i}+e^{k_j^*})}{4(e^{k_i+k_j^*}-1)\left(p_i+p_j^*+q_i+q_j^*\right)}, \tag{54}$$

$$b(i,j) = -\frac{4(e^{k_i}-e^{k_j})(p_i-p_j+q_i-q_j)}{\epsilon(e^{k_i+k_j}+1)}, \tag{55}$$

$$b(i^*,j^*) = -\frac{4(e^{k_i^*}-e^{k_j^*})(p_i^*-p_j^*+q_i^*-q_j^*)}{\epsilon(e^{k_i^*+k_j^*}+1)}, \tag{56}$$

where $\eta_j = k_j n + p_j y + q_j t$, $1 \le i,j \le 2$ with complex constants k_1, k_2, p_1 and p_2 and the dispersion relation $q_j = \mathrm{i}\frac{e^{k_j}+e^{-k_j}-2}{\epsilon^2}$. Setting $x = n\epsilon, k_j = \epsilon\tilde{k}_j$, in the continuum limit $\epsilon \to 0$, we obtain

$$\eta_j = \tilde{k}_j x + p_j y + q_j t, \quad q_j \to \mathrm{i}\tilde{k}_j^2, \tag{57}$$

$$b(i,j^*) \to -\frac{1}{2(\tilde{k}_i+\tilde{k}_j^*)\left(p_i+p_j^*+q_i+q_j^*\right)} = a(i,j^*), \tag{58}$$

$$b(i,j) \to -2(\tilde{k}_i-\tilde{k}_j)(p_i-p_j+q_i-q_j) = a(i,j), \tag{59}$$

$$b(i^*,j^*) \to -2(\tilde{k}_i^*-\tilde{k}_j^*)(p_i^*-p_j^*+q_i^*-q_j^*) = a(i^*,j^*). \tag{60}$$

Thus we conclude that the two-soliton solutions of the semi-discrete system reduce to the ones of the continuous system through the continuum limit $\epsilon \to 0$. Substituting (52)-(53) into (34) and (36), we obtain the two-soliton solutions u_n and v_n respectively.

The exact $N-$soliton solutions to Eqs.(32) have the form

$$f_n = \sum_{\mu=0,1}^{(e)} \exp\left[\sum_{j=1}^{N} \mu_j \eta_j + \sum_{j=N+1}^{2N} \mu_j \eta_{j-N}^{*} + \sum_{1\le i<j}^{2N} \mu_i \mu_j B_{ij}\right], \tag{61}$$

$$g_n = \sum_{\nu=0,1}^{(o)} \exp\left[\sum_{j=1}^{N} \nu_j \eta_j + \sum_{j=N+1}^{2N} \nu_j \eta_{j-N}^{*} + \sum_{1\le i<j}^{2N} \nu_i \nu_j B_{ij}\right], \tag{62}$$

where

$$\eta_j = k_j n + p_j y + q_j t, \quad q_j = \mathrm{i}\left(e^{k_j} + e^{-k_j} - 2\right)/\epsilon^2, \tag{63}$$

$$\eta_j^{*} = \text{complex conjugate of } \eta_j, \quad j = 1, 2, \cdots, N, \tag{64}$$

$$\exp(B_{i,j}) = b(i,j), \quad i < j = 2, 3, \cdots, N, \tag{65}$$

$$\exp(B_{i,N+j}) = b(i,j^{*}), \quad i,j = 1, 2, \cdots, N, \tag{66}$$

$$\exp(B_{N+i,N+j}) = b(i^{*},j^{*}), \quad i < j = 2, 3, \cdots, N. \tag{67}$$

Following the proof of one- and two-soliton solution, one can show that the exact $N-$soliton solutions of the semi-discrete system reduce to those of the continuous system in the continuum limit.

4 Numerical computations

In this section, two examples will be illustrated to show that the integrable semi-discretization is a powerful scheme for the numerical solutions of the system (5). They include (1) propagation of the one-soliton solution, (2) interaction of the 2-soliton solutions. We employ the Crank-Nicholson scheme for the system (42), the central difference scheme in the y-direction and the Dirichlet condition. We choose the exact one-soliton solution and two-soliton solutions of the system (5) as the initial and boundary values.

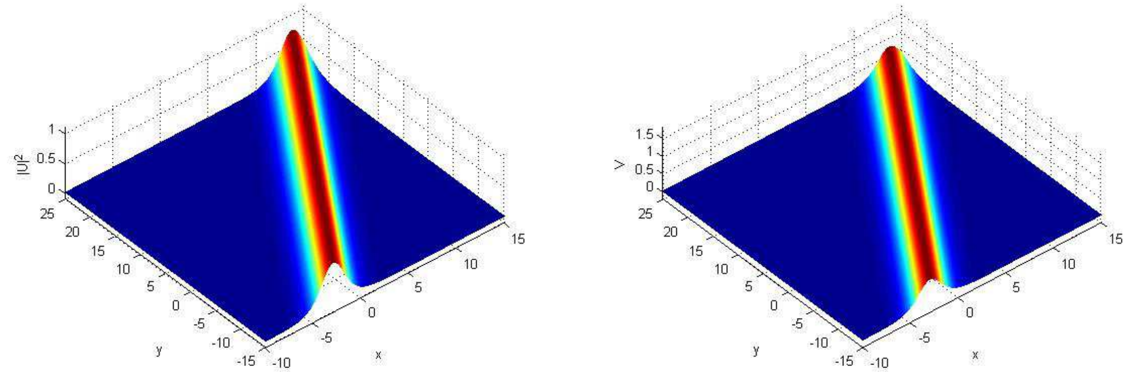
Example 1. one-soliton propagation. The parameters taken for the one-soliton solution are $k_1 = 0.6 + 0.3\mathrm{i}, p_1 = -0.25 - 0.4\mathrm{i}$. The number of grid is taken as 250 in an interval of width 25 in the x-domain, which implies a mesh size of $\epsilon = 0.1$. The number of grid is 400 in an interval of width 40 in the y-direction. The time-step size is taken as $\Delta t = 0.05$. Figure 1 displays the numerical solution of the one-soliton solution at $t = 4$. The L_∞ norm is 0.0385 for $|u|^2$ and 0.0422 for v at $t = 4$. It is noted that the numerical error is mainly due to the error of the dispersion relation. In other words, even after a fairly long time, the numerical solution of a soliton preserves its shape very well except for a phase shift.

Example 2. two-soliton interaction. The parameters taken for the two-soliton solution are $k_1 = 0.6 + 0.3\mathrm{i}, \mathrm{k}_2 = -0.5 + 0.5\mathrm{i}, \mathrm{p}_1 = -0.25 - 0.4\mathrm{i}, \mathrm{p}_2 = 0$. Figure 2 shows the exact two-soliton solution of $|u|^2$ and v. Figure 3 displays the numerical solution for the collision of the two-soliton solution. The profiles show that the collision of two solitons is well simulated.

5 Dynamic properties

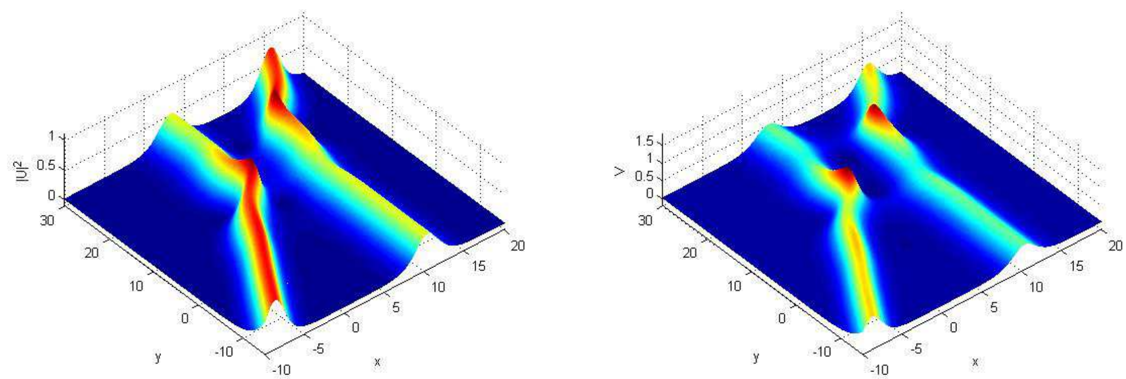
In the following discussion, we fix the discrete step $\epsilon = 1$ in the solutions (61)-(62). For $b(1,2) \neq 0$ in (55), namely $b(1,2,1^{*},2^{*}) \neq 0$ in (52), the two solitons possess four arms and display regular interaction as shown in figure 4. One can see that the two obliquely moving solitons walk through each other unaffectedly and keep their original shapes and velocities invariant during the whole propagation. Therefore, the regular interaction between the solitons is completely elastic. Elastic interaction is found in figure 4. Parameters are chosen as $k_1 = 0.6 + 0.3\mathrm{i}, k_2 = 0.37 - 0.02\mathrm{i}, p_1 = -0.0185 - 0.192\mathrm{i}, p_2 = -0.2406903076 - 0.1197442234\mathrm{i}$.

When $b(1,2,1^{*},2^{*}) = 0$, that is, $b(1,2) = 0$ in (55), resonant interactions can happen. The resonant interactions in this case are called the "minus resonance" [36, 37], namely, after the solitons interact with each other, the amplitudes decrease, sometimes the amplitudes can even reach zero. The resonant situation



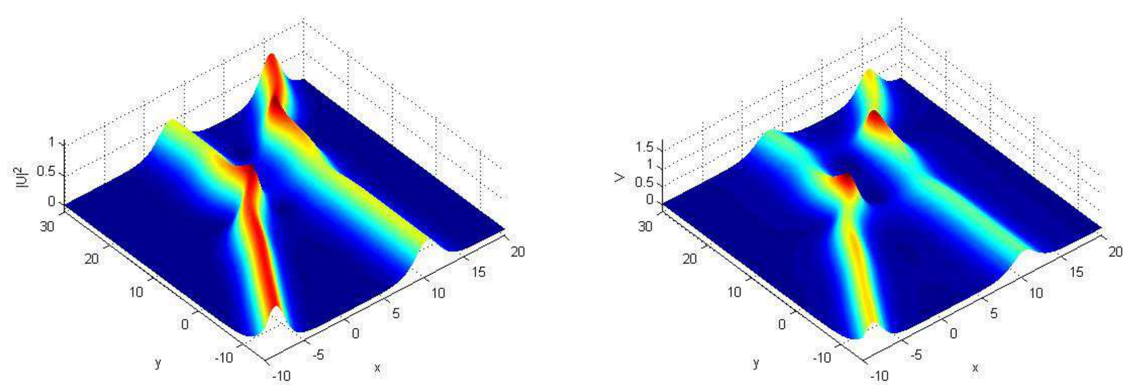
(a) (b)

Figure 1: (Color online) Numerical solution of the one-soliton solution at $t = 4$.



(a) (b)

Figure 2: (Color online) Exact solution for the collision of the two-soliton solution at $t = 4$.



(a) (b)

Figure 3: (Color online) Numerical solution for the collision of the two-soliton solution at $t = 4$.

here is similar as the continuous case. In order to analyze the amplitudes of the resonant solitons, we rewrite two-soliton solutions as follows:

$$u \to \begin{cases} u_1 = \frac{1}{2\sqrt{b(1,1^*)}} e^{\mathrm{i}\eta_{1I}} \operatorname{sech}(\eta_{1R} + \frac{\ln b(1,1^*)}{2}), & \eta_{2R} \to -\infty, \eta_{1R} \sim 0, \\ u_2 = \frac{1}{2\sqrt{b(2,2^*)}} e^{\mathrm{i}\eta_{2I}} \operatorname{sech}\left(\eta_{2R} + \frac{\ln b(2,2^*)}{2}\right), & \eta_{1R} \to -\infty, \eta_{2R} \sim 0, \\ u_3 = 0, & \eta_{1R} \to +\infty, \eta_{1R} - \eta_{2R} \sim 0, \end{cases} \tag{68}$$

and in the second case when $|b(1,2)| \ll 1$,

$$v \to \begin{cases} v_1 = \frac{1}{4}\left(e^{2k_{1R}} + e^{-2k_{1R}} - 2\right) \operatorname{sech}^2\left(\eta_1 + \frac{\ln b(1,1^*)}{2}\right), \eta_{2R} \to -\infty, \eta_{1R} \sim 0, \\ v_2 = \frac{1}{4}\left(e^{2k_{2R}} + e^{-2k_{2R}} - 2\right) \operatorname{sech}^2\left(\eta_2 + \frac{b(2,2^*)}{2}\right), \eta_{1R} \to -\infty, \eta_{2R} \sim 0, \\ v_3 = \frac{B_1 + \sqrt{b(1,2^*)b(2,1^*)b(1,1^*)b(2,2^*)}\operatorname{Re}\left(\left(e^{k_1^*-k_2^*} + e^{k_2^*-k_1^*} - 2\right)\cosh\left(\eta_1 - \eta_2 + \frac{1}{2}\ln\frac{b(1,2^*)b(1,1^*)}{b(2,1^*)b(2,2^*)}\right)\right)}{\left(2\sqrt{b(1,1^*)b(2,2^*)}\cosh\left(\eta_{1R} - \eta_{2R} + \frac{1}{2}\ln\frac{b(1,1^*)}{b(2,2^*)}\right) + 2\sqrt{b(1,2^*)b(2,1^*)}\cosh\left(\eta_{1I} - \eta_{2I} + \frac{1}{2}\ln\frac{b(1,2^*)}{b(2,1^*)}\right)\right)^2}, \\ \qquad \eta_{1R} \to +\infty, \eta_{1R} - \eta_{2R} \sim 0, \end{cases} \tag{69}$$

where

$$B_1 = b(1,1^*)b(2,2^*)\left(e^{2k_{1R}-2k_{2R}} + e^{2k_{2R}-2k_{1R}} - 2\right) + b(1,2^*)b(2,1^*)\left(e^{2k_{1I}-2k_{2I}} + e^{2k_{2I}-2k_{1I}} - 2\right).$$

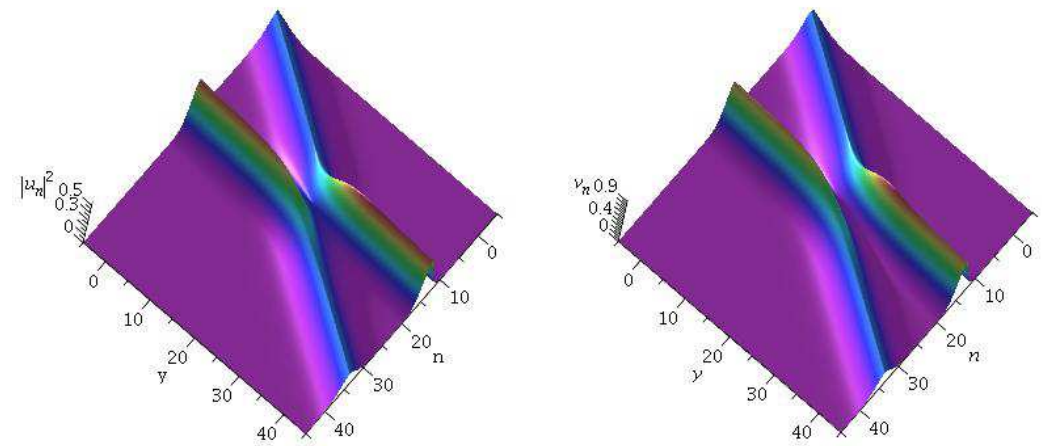
According to (68) and (69), the interaction between two solitons is investigated in figure 5. Parameters are chosen as $k_1 = 0.5 + 0.20\mathrm{i}, k_2 = 0.37 - 0.02\mathrm{i}, p_1 = -0.0185 - 0.192\mathrm{i}, p_2 = -0.2406903076 - 0.1197442234\mathrm{i}$. We can find that the two solitons possess three branches extending to infinity, which is called the triple wave structure [37]. Therefore, one can see that for the potential $|u_n|^2$, the amplitude of the third branch is zero. The third branch has high and steep wave hump for the potential v_n. The phenomena can also be found in the continuous case.

When $b(1,2,1^*,2^*) \to 0$, similar as the continuous case, another type of the resonance is shown in figure 6. The parameters are chosen as $k_1 = 0.50000001 + 0.20000001\mathrm{i}, k_2 = 0.37 - 0.02\mathrm{i}, p_1 = -0.0185 - 0.192\mathrm{i}, p_2 = -0.2406903076 - 0.1197442234\mathrm{i}$ and now $b(1,2) = 5.222108512 * 10^{-9} + 4.762014844 * 10^{-9}\mathrm{i}$. In figure 6(a), the two solitons generate a small amplitude soliton (in fact the amplitude is close to zero here) in the vicinity of the crossing point, which is different from those in figure 5(a). It looks like that the two solitons separate from each other to two parts. But in figure 6(b), the line solitons interact to create a particularly high and steep wave hump in the vicinity of the crossing point, which is also different from those in figure 5(b). Compared with the continuous case, the resonant interaction under the situation $b(1,2,1^*,2^*) \to 0$ is similar to each other.

Three-soliton solutions can be obtained from (61)-(62) by setting $N = 3$. The elastic interaction among three solitons is shown in figure 7 with parameters chosen as $k_1 = 0.6 + 0.3\mathrm{i}, k_2 = 0.37 - 0.02\mathrm{i}, k_3 = 0.2 + 0.1\mathrm{i}, p_1 = -0.0185 - 0.192\mathrm{i}, p_2 = -0.2406903076 - 0.1197442234\mathrm{i}, p_3 = -0.1 - 0.25\mathrm{i}$. The resonant interaction among three solitons is much more complicated than the one of two solitons. Here only one case is depicted in figure 8 with parameters $k_1 = 0.5 + 0.2\mathrm{i}, k_2 = 0.37 + 0.02\mathrm{i}, k_3 = 0.24 + 0.257\mathrm{i}, p_1 = 0.0185 + 0.092\mathrm{i}, p_2 = -0.1734123148 + 0.1642557766\mathrm{i}, p_3 = -0.6 - 0.3\mathrm{i}$.

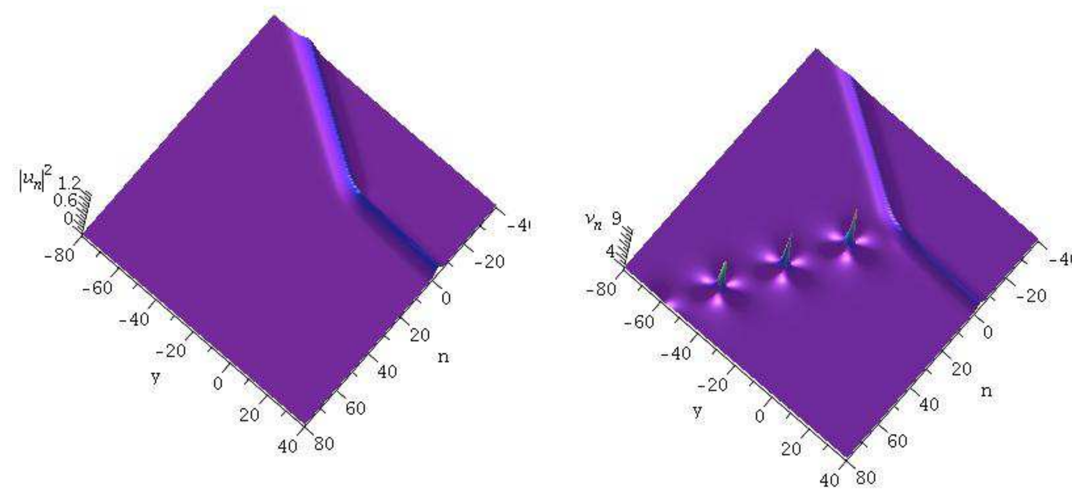
6 Conclusion

To summarize, we presented here a semi-discrete integrable version for the $(2+1)$-dimensional system and derived their $N-$soliton solutions by using pfaffian technique. Based on the asymptotic behavior of two-soliton solutions (52)-(53) and graphical analysis, we analyzed the dynamics of the interactions. It is shown that the regular interaction is completely elastic (i.e., figure 4), and two types of resonance occur between two solitons, both of which are non-completely elastic (i.e., figure 5 and 6). A triple structure (figure 5) in the procedure of interactions and a high wave hump in the vicinity of the crossing point (i.e., figure 6), are observed. Based on the results obtained, it is natural to further consider integrability of the differential-difference system, such as Bäcklund transformation, Lax pair and infinite conservation laws.



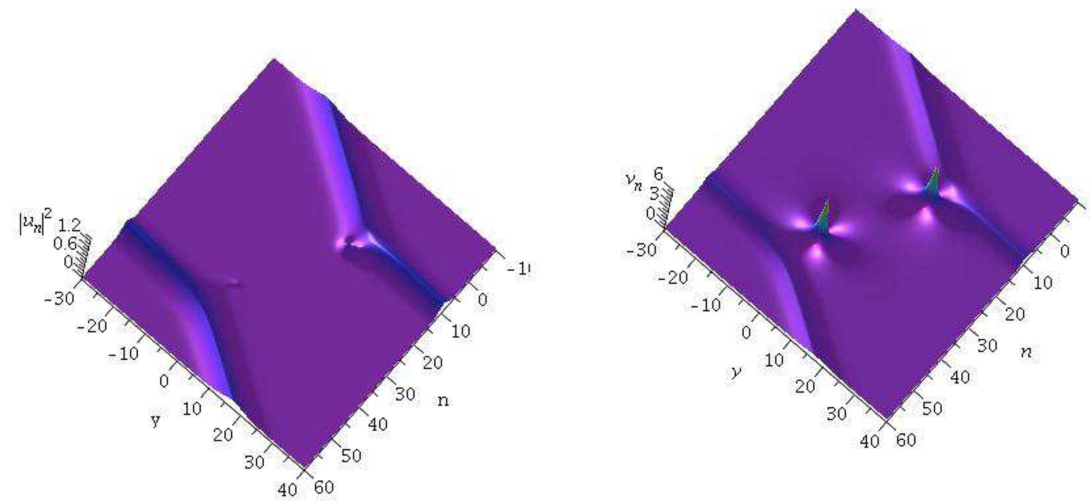
(a) (b)

Figure 4: (Color online) The elastic interaction of two solitons at $t = 20$.



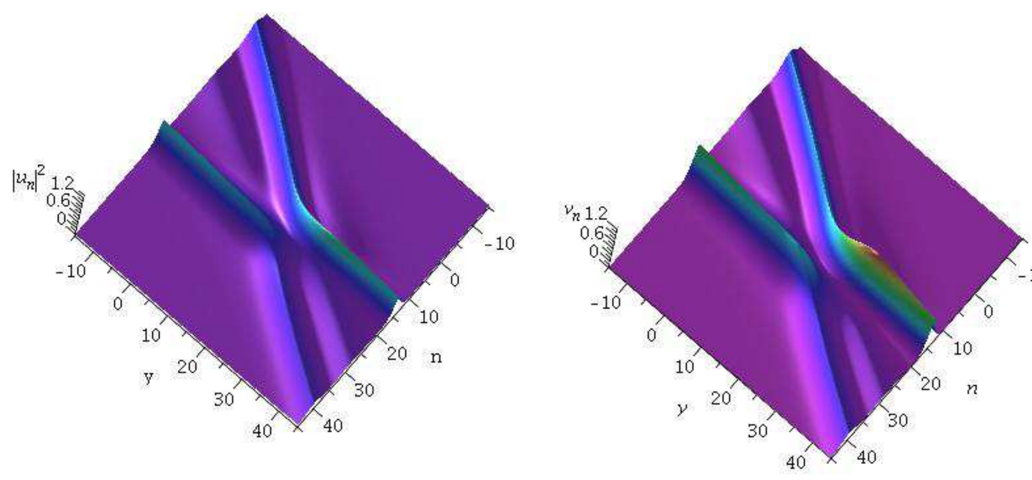
(a) (b)

Figure 5: (Color online) Resonant interactions between two solitons at $t = -10$ when $b(1, 2) = 0$.



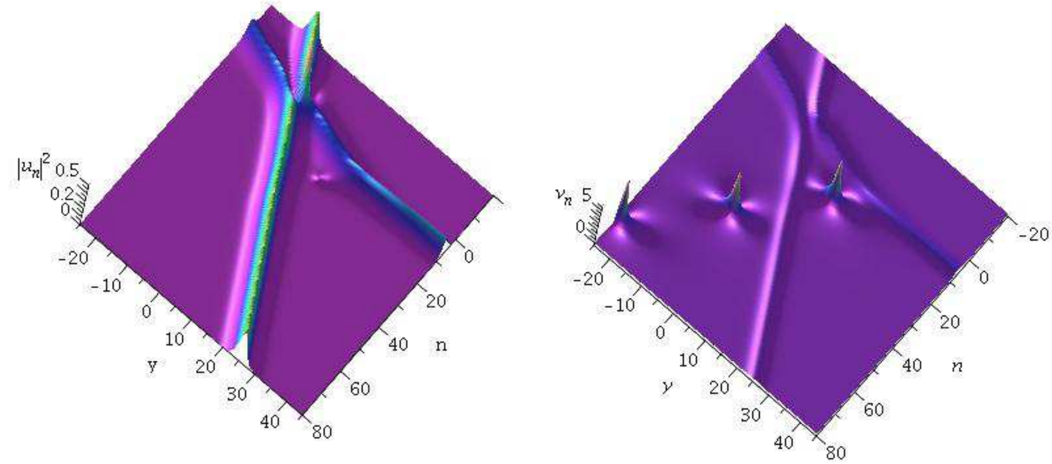
(a) (b)

Figure 6: (Color online)Resonant interactions between two solitons at $t = 20$ when $b(1, 2) \to 0$.



(a) (b)

Figure 7: (Color online) Elastic interactions between three solitons at $t = 20$.



(a) (b)

Figure 8: (Color online) Resonant interactions between three solitons at $t = 20$.

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7 Appendix

A pfaffian is the square root of a skew-symmetric determinant of order $2n$, and consequently the properties of pfaffians are closely related to those of determinants [30]. Let $A = \det(a_{j,k})(1 \le j, k \le 2n)$, where $a_{j,k} = -a_{k,j}$. The pfaffian expression of A is

$$A = [\mathrm{pf}(1, 2, 3, \cdots, 2n)]^2.$$

For example, if $n = 1$, we have

$$\begin{vmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{vmatrix} = a_{12}^2 = [\mathrm{pf}(1, 2)]^2. \tag{70}$$

If $n = 2$, we get

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{21} & 0 & a_{23} & a_{24} \\ -a_{31} & -a_{32} & 0 & a_{34} \\ -a_{41} & -a_{42} & -a_{43} & 0 \end{vmatrix} = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2 = [\mathrm{pf}(1, 2, 3, 4)]^2. \tag{71}$$

We rewrite original Eqs. (32) as follows

$$\mathrm{i}D_t g_n \bullet f_n + g_{n+1} \bullet f_{n-1} + g_{n-1} f_{n+1} - 2 g_n f_n = 0, \tag{72a}$$

$$4(D_t + D_y) f_{n+1} \bullet f_n + g_{n+1} g_n^* + g_n g_{n+1}^* = 0, \tag{72b}$$

and define the pfaffian elements:

$$\text{pf}(a_i, a_j) = \text{i}\frac{e^{k_i} - e^{k_j}}{e^{k_i+k_j} + 1}e^{\eta_i+\eta_j},\ \text{pf}(a_i^*, a_j^*) = -\text{i}\frac{e^{k_i^*} - e^{k_j^*}}{e^{k_i^*+k_j^*} + 1}e^{\eta_i^*+\eta_j^*}, \tag{73a}$$

$$\text{pf}(a_i, a_j^*) = -\text{i}\frac{e^{k_i} + e^{k_j^*}}{e^{k_i+k_j^*} - 1}e^{\eta_i+\eta_j^*}, \tag{73b}$$

$$\text{pf}(a_i, b_j^*) = (a_i^*, b_j) = 0,\ \text{pf}(a_i, b_j) = \text{pf}(a_i^*, b_j^*) = \delta_{ij}, \tag{73c}$$

$$\text{pf}(b_i, b_j) = 0,\ \text{pf}(b_i^*, b_j^*) = 0,\ \text{pf}(b_i, b_j^*) = \frac{\text{i}}{4\left(p_i + p_j^* + q_i + q_j^*\right)}, \tag{73d}$$

$$\text{pf}(d_0, \beta) = \text{pf}(a_j, \beta) = \text{pf}(a_i^*, \beta) = \text{pf}(b_j^*, \beta) = 0,\ \text{pf}(b_i, \beta) = 1, \tag{73e}$$

$$\text{pf}(d_0, a_j) = e^{\eta_j},\ \text{pf}(d_0, a_j^*) = e^{\eta_j^*},\ \text{pf}(d_0, b_i) = \text{pf}(d_0, b_i^*) = 0, \tag{73f}$$

where δ_{ij} is the Kronecker delta function and

$$\eta_i = k_i n + p_i y + q_i t, \qquad q_i = \text{i}\left(e^{k_i} + e^{-k_i} - 2\right), \quad i = 1, 2, \cdots, N.$$

Theorem 7.1. The N-soliton solution to equations (32) can be expressed in the pfaffian form

$$f_n = \text{pf}(a_1, a_2, \cdots, a_N, a_1^*, a_2^*, \cdots, a_N^*, b_1, b_2, \cdots, b_N, b_1^*, b_2^*, \cdots, b_N^*) = \text{pf}(\bullet), \tag{74}$$

$$g_n = \text{pf}(d_0, \beta, a_1, a_2, \cdots, a_N, a_1^*, a_2^*, \cdots, a_N^*, b_1, b_2, \cdots, b_N, b_1^*, b_2^*, \cdots, b_N^*) = \text{pf}(d_0, \beta, \bullet), \tag{75}$$

where we use the notation $(\bullet)$ for the sake of simplicity.

Proof. We introduce the pfaffian elements c_p, c_m as

$$\text{pf}(d_0, c_p) = 0,\ \text{pf}(c_p, a_i) = \left(-\text{i}e^{k_i} - 1\right)e^{\eta_i},\ \text{pf}(c_p, a_i^*) = \left(\text{i}e^{k_i^*} - 1\right)e^{\eta_i^*}, \tag{76}$$

$$\text{pf}(d_0, c_m) = 0,\ \text{pf}(c_m, a_i) = \left(\text{i}e^{-k_i} - 1\right)e^{\eta_i},\ \text{pf}(c_m, a_i^*) = \left(-\text{i}e^{-k_i^*} - 1\right)e^{\eta_i^*}. \tag{77}$$

In what follows, we denote pf$(\bullet)$ by $(\bullet)$ for the sake of simplicity. From properties of pfaffians, we get the following differential and difference formulae for f_n and g_n,

$$f_{n+1} = (d_0, c_p, \bullet) + (\bullet), \tag{78a}$$

$$g_{n+1} = \text{i}\,(d_0, \beta, \bullet) + \text{i}\,(c_p, \beta, \bullet), \tag{78b}$$

$$f_{n-1} = (d_0, c_m, \bullet) + (\bullet), \tag{78c}$$

$$g_{n-1} = -\text{i}\,(d_0, \beta, \bullet) - \text{i}\,(c_m, \beta, \bullet), \tag{78d}$$

$$f_{n,t} = -(c_m, c_p, \bullet) + \text{i}\,(\bullet) + (d_0, c_m, \bullet) - (d_0, c_p, \bullet), \tag{78e}$$

$$g_{n,t} = -(d_0, c_m, c_p, \beta, \bullet) - \text{i}\,(d_0, \beta, \bullet) - (c_p, \beta, \bullet) + (c_m, \beta, \bullet). \tag{78f}$$

The substitution of (78) into (72a) implies:

$$\begin{aligned}&-\text{i}\,(d_0, c_m, c_p, \beta, \bullet)(\bullet) + \text{i}\,(d_0, c_m, \bullet)(c_p, \beta, \bullet)\\ &\quad -\text{i}\,(d_0, c_p, \bullet)(c_m, \beta, \bullet) + \text{i}\,(d_0, \beta, \bullet)(c_m, c_p, \bullet) = 0,\end{aligned}$$

that vanishes due to the pfaffian identity [30]:

$$\begin{aligned}&(a_1, a_2, a_3, a_4, 1, 2, \cdots, 2m)(1, 2, \cdots, 2m)\\ &-(a_1, a_2, 1, 2, \cdots, 2m)(a_3, a_4, 1, 2, \cdots, 2m)\\ &+(a_1, a_3, 1, 2, \cdots, 2m)(a_2, a_4, 1, 2, \cdots, 2m)\\ &-(a_1, a_4, 1, 2, \cdots, 2m)(, a_2, a_3 1, 2, \cdots, 2m)\\ &= 0.\end{aligned} \tag{79}$$

Thus we proved that (74)-(75) satisfy Eq.(72a). Furthermore, in order to confirm that (74)-(75) satisfy (72b), we introduce a new auxiliary element β^* and define new pfaffian entries as following,

$$(d_0, \beta^*) = (a_j, \beta^*) = (a_i^*, \beta^*) = (b_j, \beta^*) = 0,\ (b_i^*, \beta^*) = 1. \tag{80}$$

It is easy to verify that

$$
\begin{aligned}
&f_{n,t}+f_{n,y}=-\frac{\mathrm{i}}{4}\,(\beta,\beta^*,\bullet)\,, &(81a)\\
&f_{n+1,t}+f_{n+1,y}=-\frac{\mathrm{i}}{4}\,[(\beta,\beta^*,d_0,c_p,\bullet)+(\beta,\beta^*,\bullet)]\,, &(81b)\\
&g_n^*=(d_0,\beta^*,\bullet)\,, &(81c)\\
&g_{n+1}^*=-\mathrm{i}\,(d_0,\beta^*,\bullet)-\mathrm{i}\,(c_p,\beta^*,\bullet)\,. &(81d)
\end{aligned}
$$

(see the appendix in [38] for reference). By substituting (81) and (78) into (72b), Eq. (72b) is reduced to the pfaffian identity

$$
\begin{aligned}
&\mathrm{i}\Big[(d_0,c_p,\bullet)+(\bullet)\Big](\beta,\beta^*,\bullet)-\mathrm{i}\Big[(\beta,\beta^*,d_0,c_p,\bullet)+(\beta,\beta^*,\bullet)\Big](\bullet)\\
&\quad+\Big[\mathrm{i}\,(d_0,\beta,\bullet)+\mathrm{i}\,(c_p,\beta,\bullet)\Big](d_0,\beta^*,\bullet)-\Big[\mathrm{i}\,(d_0,\beta^*,\bullet)+\mathrm{i}\,(c_p,\beta^*,\bullet)\Big](d_0,\beta,\bullet)\\
&=-\mathrm{i}\,(\beta,\beta^*,d_0,c_p,\bullet)(\bullet)+\mathrm{i}\,(d_0,c_p,\bullet)(\beta,\beta^*,\bullet)\\
&\quad+\mathrm{i}\,(c_p,\beta,\bullet)(d_0,\beta^*,\bullet)-\mathrm{i}\,(c_p,\beta^*,\bullet)(d_0,\beta,\bullet)\\
&=0. \qquad (82)
\end{aligned}
$$

Thus the bilinear Eq. (72b) is established and so the theorem is proven. □

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