Effect of zealotry in high-dimensional opinion dynamics models

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The effect of zealotry in high dimensional opinion dynamics models

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Most of the work on opinion dynamics models focuses on the case of two or three opinion types. We consider the case of an arbitrary number of opinions in the mean field case of the naming game model in which it is assumed the population is infinite and all individuals are neighbors. A particular challenge of the naming game model is that the number of variables, which correspond to the number of possible sets of opinions, grows exponentially with the number of possible opinions. We present a method for generating mean field dynamical equations for the general case of k opinions. We calculate the steady states in two important special cases in arbitrarily high dimension: the case in which there exist zealots of only one type, and the case in which there are an equal number of zealots for each opinion. We show that in these special cases a phase transition occurs at critical values p_c of the parameter p describing the fraction of zealots. In the former case the critical value determines the threshold value beyond which it is not possible for the opinion with no zealots to be held by more nodes than the opinion with zealots, and this point remains fixed regardless of dimension. In the latter case, the critical point p_c is the threshold value beyond which a stalemate between all k opinions is guaranteed, and we show that it decays precisely as a lognormal curve in k.

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I. INTRODUCTION

Opinions are influenced by exposure to different views; for example, this forms the basic tenet of political election and advertising campaigns [1], to name just two. Opinion formation is a dynamic process, with new information leading to changes in the beliefs of a society through both exogenous (e.g., media-driven) and endogenous (e.g., peer-influence) means. The field of *opinion dynamics* seeks to mathematically understand the evolution of opinions in a society.

Increasingly, individuals are facing a large number of discrete choices ("opinions") from which to choose. One common example is the choice of which of several social media platforms to engage in, such as Facebook, Twitter, and Google+ [2]. Another example is the several platforms for portable computing to choose from, such as Apple iPad, Amazon Kindle, and Samsung Galaxy Note. A third example is the choice of operating system, such as Microsoft Windows, Apple MAC O/S, and Linux (which itself has many choices, such as Ubuntu, Fedora, and Mint). Note that in these examples, the possible choices are not necessarily mutually exclusive; an individual may choose to affiliate with only one opinion, or with multiple opinions. Some of these individuals will be adaptive; that is, presented enough evidence to consider another opinion, they would adopt it (e.g., a Linux Ubuntu user also installing Microsoft Windows). However, some individuals are zealots (e.g., fiercely loyal Apple fans) who advocate a single opinion and refuse to consider any others. In addition to the examples discussed above, zealotry accounts for persistent disagreement in a wide variety of other disciplines, such as in politics [3] or religion [4].

The challenge of modeling opinion dynamics mathematically has been tackled by the research community over the last four decades. The work in Ref. [5] provides a comprehensive review. Much of the recent research uses the so-called sociophysics approach, which borrows fundamental ideas from physics in studying the macro-level (population) behaviors that emerge from various micro-level (individual) interactions. In practice, there are a huge number of variants of opinion dynamics models to consider. Each unique definition of a micro-level interaction process can lead to the emergence of different regimes of behavior at the population-level. Yet after a series of micro-level updates, we can ask several questions. Is the network converging to consensus on a particular opinion? If so, how quickly? If not, will one of the competing opinions win a plurality? What aspects of network structure and initial conditions affect the answers to the previous questions?

Virtually all the classical research in discrete opinion dynamics has considered the answers to the above questions in the context of a small number of competing opinions, usually two, and where agents are adaptive, i.e., not zealots. The phenomenon of zealotry has been formally studied only recently [6–11]. Both Ref. [12] and Ref. [13] consider the "naming game" model with competing zealots where there are two opinions, each with a zealot backing. In 1995, the naming game model was introduced as a method for agents to identify each other with names or spatial descriptions [14]. More recently, it has been adopted as a model decribing the evolution of language and of opinion dynamics in social networks. There have been several studies that have considered the naming game model (with two opinions) for opinion dynamics; for example, see Refs. [15–18]. In this work, we focus on the situation where there are a large number of opinions in the presence of zealots. In particular, we generalize some of the key results of Ref. [12] and Ref. [13] of zealotry in the binary naming game, to arbitrary dimensions. More than two opinions under cyclic dominance have been studied under the classical Rock-Paper-Scissors model, e.g., Ref. [19], the Rock-Paper-Scissors-Spock-Lizard variant, e.g., Ref. [20], and a cyclic Lotka - Volterra model [21].

In this manuscript "dimension" refers to the number of different opinions, but in previous work, discussed next, dimensionality refers to the dimension of the lattice on which the dynamics occur. A few previous works have considered higher-dimensional analogs of some opinion dynamics models. For example, in the continuous-valued case, Ref. [22] studies higher-dimensional analogs of the standard consensus problem under linear update. The authors consider the time to consensus as a function of dimension. Ref. [23] shows through detailed simulations that the chance of a "vast-majority" consensus increases with dimension, but so do the number of minority opinions. There is even less work on higher dimensional analogs in the discrete opinion case. For example, Ref. [24] considers higher dimensional versions of the majority-rule model. The work in [25] also studies the majority rule model in higher dimensional lattices, finding deviations from predicted mean-field behavior for $d = 4$ and uses simulations to establish the approximate values of the critical exponent for up to $d = 7$, showing that these values agree well with mean field theory.

The rest of this paper is organized as follows. We first begin in Sec. II by presenting the basic update mechanism and present the concrete rule table for the case of the two opinion naming game. The case of k opinions, $k > 2$, is a straightforward extension. In Sec. III we provide a derivation of the general mean field equations for arbitrary k, which is one of the contributions of this work. Sec. IV details our primary contribution, which is the computation of steady states and critical points in two important special cases for arbitrarily high values of k. In Sec. V, we introduce a low-dimensional model which is very similar to the naming game yet more amenable to analysis.

II. MODEL, ASSUMPTIONS, PROBLEM FORMULATION

We study the naming game using the same interaction model presented in Xie *et al.* [12] and using notation from Ref. [13]. Consider the general situation of a discrete opinion space with k opinions where individuals can hold multiple opinions. Define $\mathcal{O} = \{O_m, m = 1, ..., k\}$ as the set of all possible opinions. Let $I_\ell(t)$ denote the set of opinions held by node ℓ at time t, and let $\mathcal{N}_\ell(t)$ denote its neighbors, i.e., with whom it can communicate or interact directly at time t (note that in the mean field case that we study here, $\mathcal{N}_\ell(t)$ is the set of all nodes in the graph). At each discrete timestep, an agent, say i, is selected randomly and randomly selects one of its neighbors, say j, with which to interact. Given node i is selected at time t, the probability it chooses j is thus $1/|\mathcal{N}_i(t)|$. Node i randomly chooses one opinion O_m from its set of opinions $I_i(t)$ with uniform probability $1/|I_i(t)|$ and chooses it for discussion with j. If j already has O_m in its own set, then both agree upon O_m and both of them discard the rest of the opinions from their sets; otherwise j adds O_m to its set. If the initiator is a zealot, it does not change its opinion; similarly, if the responder is a zealot, it does not change it's opinion. The interaction model for the general case is described by:

$$(I_i, I_j) \overset{O_m}{\to} (O_m, O_m), \quad \text{if } O_m \in I_j \tag{1}$$

$$\overset{O_m}{\to} (I_i, I_j \cup \{O_m\}), \quad \text{if } O_m \notin I_j \quad \text{and j is not a zealot} \tag{2}$$

$$\overset{O_m}{\to} (I_i, I_j), \quad \text{otherwise} . \tag{3}$$

In particular, note that when a common opinion is found between speaker and listener, it becomes the sole opinion adopted by both.

The dynamics of the naming game model can be approximated in the mean field by a system of differential equations. For example in the two opinion case (i.e., the binary naming game) in the equations below, x and y refer to the fraction of nodes which have opinion A and B respectively, and z is the fraction of nodes which have both opinions A and B. The zealotry parameters p and q describe the fraction of zealots having opinions A and B respectively. Note that $x + y + z = 1$, so the zealots are included in the variables x and y. These equations describe the mean field evolution over time of the naming game, and can be used to determine the expected values of x, y, and z over time as described in Sec. III C.

$$\begin{aligned} x' &= z(x + z + \frac{p}{2}) - y(x - p) \\ y' &= z(y + z + \frac{q}{2}) - x(y - q) \\ z &= 1 - x - y \end{aligned}$$

It is straightforward to calculate the steady states by setting $x' = y' = 0$, and there are two special cases in which the steady state solutions can be expressed concisely.

1. The case in which $q = 0$ and hence there are no zealots representing opinion B. In this case, a phase transition occurs as p is increased at a critical point $p_c \approx 0.1$. In the subcritical case where $p < p_c$ it is possible to have $y > x$ in steady state. In the supercritical case where $p > p_c$, it is guaranteed that $x > y$ in steady state [9].

2. The case in which $p = q$ and hence the number of zealots representing opinions A and B are equal. A phase transition also occurs in this case at a different critical point $p_c = \sqrt{10} - 3$. There is always

a steady state solution in which $x = y$, but when $p < p_c$ this solution is unstable and two other stable solutions exist in which respectively $x > y$ and $x < y$. When $p > p_c$ the only steady state solution is $x = y$, which is stable [9].

A. The Challenge of High Dimensionality

Extending the binary naming game model to higher dimensions leads to an exponential blowup in the dimensionality of the opinion space, primarily because of the undecideds, i.e., those holding mutiple opinions. To see this, note that for k unique opinions, the "decideds" can only be of k unique types, but the undecideds may have *any* subset of the k opinions that has cardinality greater than or equal to 2. Thus, when there are k possible distinct opinions the total number of possible unique opinion states (decided plus undecided) is 2^k-1. For example, if the number of opinions is $k = 4$, the possible opinion states are A, B, C, D, AB, AC, AD, BC, BD, CD, ABC, ABD, ACD, BCD, and $ABCD$ where the first four are decideds and the remaining are undecideds. Explicitly deriving the governing system of mean field equations and then computing the resulting steady state values for all possible states for even moderate k is tedious at best. One significant contribution of this paper is to provide general mean field equations for all k. However, since the number of variables is exponential it still quickly becomes computationally intractable to find the steady states and critical points from these mean field equations. We address this challenge in Sec. V.

B. Mean Field Analysis

To simplify analysis, we make the mean field assumption that each node can interact with every other node. We refer to this as mean field analysis, and it is equivalent to assuming that the underlying graph is complete. To assess the validity of the mean field assumption, we ran the process with $k = 3$ and 3% zealots of each type on several different online social networks in comparison to the mean field case and Erdős-Rényi random graphs. In order to compute the steady states on these networks with respect to the naming game process we ran the process for a total of $200n$ timesteps, where n is the number of nodes in the system, and averaged over 10 realizations. In the table below, A, B, and C refer to the fraction of the network following the three decided opinions, with Z representing all other nodes. We see a remarkable similarity in steady state achieved in real world networks (which have a heterogeneous degree distribution) compared to the mean field case and to Erdős-Rényi random graphs with the same number of nodes and edges (which have a homogeneous degree distribution). It is reasonable to conjecture that results which hold in the mean field case may also hold more generally on a large number of social networks. Since $p_c > 3\%$ for $k = 3$, we expect that, depending upon initial conditions, different steady state results are possible with the mean-field approach. Table I presents the steady state in which A dominates.

Network	Mean Degree	System Size	A	B	C	Z
Mean field	50,000	50,000	0.879	0.032	0.032	0.057
brightkite	7.35	58,000	0.832	0.068	0.068	0.032
Erdős-Rényi	7.35	58,000	0.815	0.058	0.058	0.069
gowalla	9.6	196,000	0.839	0.064	0.064	0.033
Erdős-Rényi	9.6	196,000	0.83	0.049	0.049	0.072
facebook	25.6	63,000	0.847	0.052	0.052	0.049
Erdős-Rényi	25.6	63,000	0.864	0.037	0.037	0.062

TABLE I. Steady states for the fraction of nodes with opinions A,B, and C obtained via a direct implementation of the opinion dynamics process described at the beginning of Sec. II on real-world networks compared to the mean field case and to Erdős-Rényi random graphs of the same edge density. Z refers to the portion of undecided nodes, that is, nodes with opinion sets containing more than one opinion. The intent is to measure the effect of network topology on the naming game process in the specific case of real-world social networks, and although the real-world networks differ significantly in structure from both the Erdős-Rényi random graphs and the mean field case, the broad agreement is striking.

III. GENERAL MEAN FIELD EQUATIONS

Previously, we defined the naming game model with zealots, and below in Sec. III A we introduce mean field equations for the three-opinion case, which demonstrates how unwieldy it is to explicitly write out these equations when $k > 2$. In Sec. III B we address this problem by providing a method for quickly deriving mean field equations for general k.

A. The Three Opinion Case

A natural generalization of the binary naming game is to consider the case in which there are three opinions rather than two. Instead of three variables, this results in seven:

1. The decideds with a single opinion: x, y, w.
2. The undecideds with two opinions: z_{xy}, z_{xw}, z_{yw}
3. The undecideds with three opinions: z_{xyw}

For convenience, define:

$$z_x = z_{xy} + z_{xw}$$
$$z_y = z_{xy} + z_{yw}$$
$$z_w = z_{xw} + z_{yw}$$

Finally, we define p, q, r, as the zealot fractions backing the three opinions modeled by x,y,and w. We can then derive the following mean field equations:

$$
\begin{aligned}
x' &= z_x\left(x + z_x + \frac{p}{2}\right) - (y + w + z_{yw})(x-p) + \frac{1}{3}z_{xyw}(2x + 2p + 2z_{xyw} + 5z_x) \\
y' &= z_y(y + z_y + \frac{q}{2}) - (x + w + z_{xw})(y-q) + \frac{1}{3}z_{xyw}(2y + 2q + 2z_{xyw} + 5z_y) \\
w' &= z_w(w + z_w + \frac{r}{2}) - (x + y + z_{xy})(w-r) + \frac{1}{3}z_{xyw}(2w + 2r + 2z_{xyw} + 5z_w) \\
z'_{xy} &= (x + \frac{1}{2}z_x + \frac{1}{3}z_{xyw})(y-q) + (y + \frac{1}{2}z_y + \frac{1}{3}z_{xyw})(x-p) \\
&\quad - z_{xy}(\frac{4+p+q}{2} - x - y - z - \frac{1}{2}z_w) \\
z'_{xw} &= (x + \frac{1}{2}z_x + \frac{1}{3}z_{xyw})(w-r) + (w + \frac{1}{2}z_w + \frac{1}{3}z_{xyw})(x-p) \\
&\quad - z_{xw}(\frac{4+p+r}{2} - x - y - z - \frac{1}{2}z_y) \\
z'_{yw} &= (w + \frac{1}{2}z_x + \frac{1}{3}z_{xyw})(y-q) + (y + \frac{1}{2}z_y + \frac{1}{3}z_{xyw})(w-r) \\
&\quad - z_{yw}(\frac{4+q+r}{2} - x - y - z - \frac{1}{2}z_x) \\
z'_{xyw} &= -(x' + y' + w' + z'_{xy} + z'_{xw} + z'_{yw})
\end{aligned}
$$

The last equation follows from the identity $x+y+w+z_{xy}+z_{xw}+z_{yw}+z_{xyw}=1$.

B. The General Case

Suppose that there are k opinions, with the fraction of inividuals holding opinion i denoted by x_i for $i = 1, 2, ..., k$. More generally, z_S is the fraction of nodes whose opinion set is S. Note that, in this notation, $x_i = z_i$. For convenience, we also define T_i as the probability that a random opinion from the opinion set of a random node is i. In other words, T_i is the probability that opinion i is transmitted. T_i is the sum over all $\frac{1}{|S|}z_S$ where the set S contains i. In symbols, $T_i = \sum_{\{i\}\subseteq S\subseteq\{1,2,...,k\}} \frac{1}{|S|}z_S$.

Similarly, C_i is the probability that a randomly chosen node's opinion set contains i. C_i is the sum over all z_S where the set S contains i, or $C_i = \sum_{\{i\}\subseteq S\subseteq\{1,2,...,k\}} z_S$. These variables make it much easier to define general mean field equations for any number of opinions k, shown in equations (4)-(6) below.

$$
\begin{aligned}
x'_{\{i\}} &= 2(T_i - x_{\{i\}})(C_i - x_{\{i\}}) + (T_i - x_{\{i\}})x_{\{i\}} \\
&\quad + x_{\{i\}}(C_i - x_{\{i\}}) - \sum_{j\neq i} T_j(x_{\{i\}} - p_i) \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
z'_{\{i,j\}} &= T_i(x_{\{j\}} - p_j) + T_j(x_{\{i\}} - p_i) \\
&\quad - (1 - z_{\{i,j\}})z_{\{i,j\}} - 2z^2_{\{i,j\}} \\
&\quad - z_{\{i,j\}}(\frac{1}{2}(C_i - z_{\{i,j\}} + C_j - z_{\{i,j\}}) \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
z'_S &= \sum_{j=1}^{|S|} T_{i_j} z_{S\setminus i_j} - (1 - z_S)z_S - 2z_S^2 \\
&\quad - z_S(\frac{1}{|S|}\sum_{j=1}^{|S|}(C_{i_j} - z_S)) \qquad (6)
\end{aligned}
$$

C. Numerical Methods

For each choice of k, the set of solutions to system (4)-(6) comprise the steady state solutions of the naming game process with k opinions. However, because the number of variables in the system increases exponentially in k, it is computationally difficult to solve this system. Another method of finding steady state solutions is to simply run the process until it has essentially converged to a steady state from particular initial conditions. That is, we estimate $\lim_{t\to\infty} z_S(t)$ for each set S through direct simulation. By "direct simulation," we mean a direct implementation of the opinion dynamics process described at the beginning of Sec. II. By definition, we can write:

$$\langle z_S(t)\rangle = \langle z_S(t)\rangle + \langle z_S(t-1)\rangle - \langle z_S(t-1)\rangle$$

Thus:

$$\langle z_S(t)\rangle = \langle z_S(t-1)\rangle + z'_S(t)$$

By iterating this equation for all variables we may calculate the expected value of each variable at each step of the process. We terminate the process when $|z'_S(t)| < 10^{-9}$ for every variable z_S. This yields a more accurate estimation of steady state solutions compared to direct simulation of the process, as illustrated in Fig. 1 where we see there is still some amount of variance in the curves due to the stochastic nature of the process. These minor fluctuations persist even after reaching steady state.

However, this method will not find all steady state solutions, because the solution found depends on the initial point. In particular, note that if two variables describing decideds x_i and x_j are initially equal and $p_i = p_j$, then by symmetry they will remain equal in expected value. In an actual run of the process however, the symmetry will almost surely be broken and the process may converge to a steady state in which $x_i \neq x_j$.

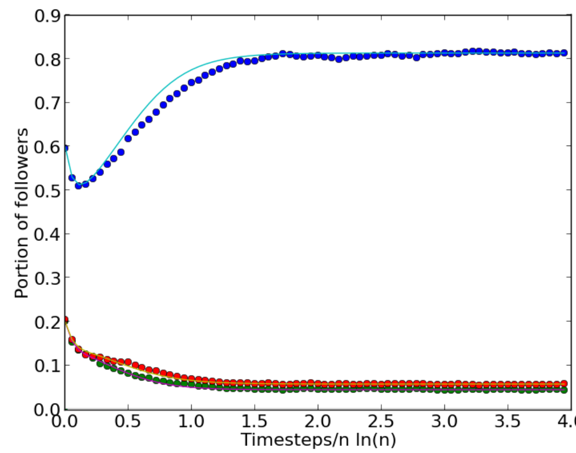

FIG. 1. The fraction of followers for each of three opinions: A,B, and C, with system size $n = 1,000,000$. Initially, A has 60% of the followers, B and C both have 20%, and there are no initial undecideds. The solid line indicates expected value obtained from equations (4)-(6) and the dots indicate direct simulation. Note that direct simulation yields a less accurate approximation of steady states due to minor fluctuations in the stochastic process. The x axis is scaled by $n\ln(n)$ and not by n as might seem more natural, due to the mixing time of the process, which appears to be of order $n\ln(n)$ in both the subcritical and supercritical regimes.

IV. PHASE TRANSITION IN HIGHER DIMENSIONS

As stated in Sec. II A, the primary challenge of extending the model to higher dimensions is the exponentially increasing number of variables. Although we have provided a means for explicitly writing the mean field equations in any dimension, the large number of variables makes it intractable to analyze these equations in higher dimensions. Here we reduce the number of variables by "glueing" most of them together in certain special instances, reducing the total number of variables from $2^k - 1$ to $2k$. Note that the analysis in this section utilizes the mean field equations of Sec. III B, and so we make the mean field assumption that all nodes are neighbors, and ignore fluctuations. We consider the thermodynamic limit, corresponding to an infinite number of nodes.

A. Zealots of only one type

The first case we consider is that in which there are only zealots corresponding to a single opinion. Without loss of generality, we may assume that $p := p_1$ is non-zero and $p_i = 0$ for all $i \geq 2$.

In the subcritical regime, simulations show that if we only have zealots with opinion 1, there are k stable steady states. There is always one steady state in which all nodes hold opinion 1, and there are $k-1$ additional steady states in which one of the other opinions gains the largest number of followers, opinion 1 holds a lower number of followers, and all the remaining end up with no followers at all. In Fig. 2 we show this behavior occurring in simulation of the process. In this special case, there exists a critical value p_c beyond which all nodes will eventually have only opinion 1. That is, there exists only one stable steady state and in that state $x_{\{1\}} = 1$ and $x_{\{i\}} = 0$ for $i \neq 1$. In Fig. 3 we show this behavior occurring in simulation of the process. Moreover, Fig. 2 and Fig. 3 together show that the critical point occurs somewhere around $p_c = 0.1$. Although these figures only show the case of k = 4, we observe similar behavior for k = 3 and k = 5.

In order to estimate the value of p_c, the critical value of p_1 at which $x_{\{1\}} = 1$ is the only stable steady state, we assume that in steady state $x_{\{3\}} = x_{\{4\}} = \cdots = x_{\{k\}} = 0$. That is, we assume that if any opinion aside from opinion 1 "wins" and is non-zero in steady state, then it is opinion 2. Since all opinions aside from opinion 1 are interchangeable, this assumption loses no generality. Note that in this case it is clear that $z_S = 0$ if the opinion set S contains any opinion that is not 1 or 2. It follows

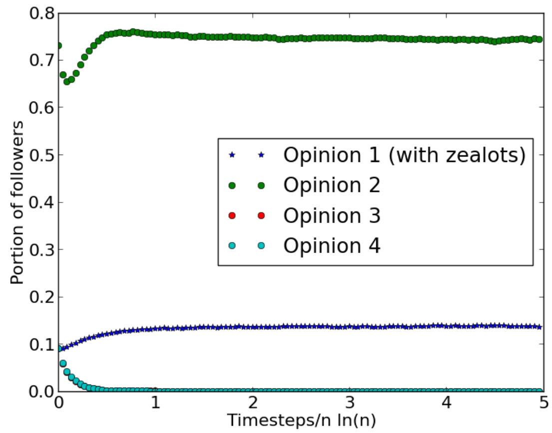

FIG. 2. The naming game with zealots of only one type for $k = 4$ and $p = 0.09$, with initial conditions that strongly favor an opinion without zealots. In steady state an opinion without zealots dominates due to the initial conditions which favor that opinion. Note that all but two opinions are eliminated, consistent with the assumption made in Sec. IV A in order to calculate critical points for all k.

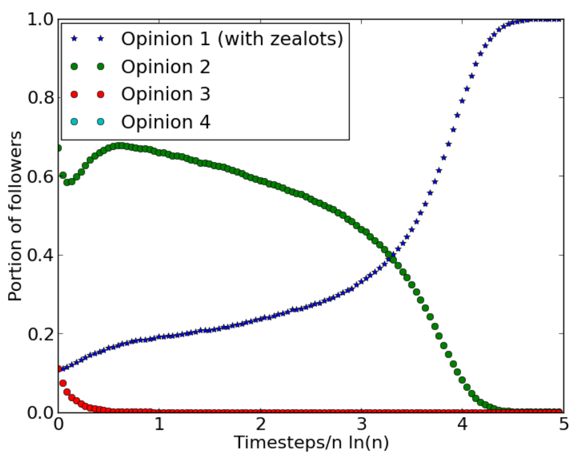

FIG. 3. The naming game with zealots of only one type for $k = 4$ and $p = 0.11$, with initial conditions that strongly favor an opinion without zealots. In steady state the opinion with zealots dominates despite initial conditions which are unfavorable for that opinion, and all others opinions are eliminated. This is consistent with the assumption made in Sec. IV A in order to calculate critical points for all k.

that the only non-zero variables in the system in steady state are $x_{\{1\}}$,$x_{\{2\}}$ and $z_{\{1,2\}}$. In other words, we are reduced to the case of two variables, and hence the value of p_c remains the same as k varies. To sum up, in the case where there are only zealots of one type, the fraction of zealots required to guarantee that the opinion *with zealots* wins out is approximately 0.1 regardless of the number of opinions k. Moreover, the possible steady states can be calculated by solving the equations in Sec. II, with the two decided opinions which do not die corresponding to the variables x and y, and the single undecided opinion which does not die out corresponds to the variable z. Alternatively, we may argue as follows.

We wish to show that the critical point does not change as the number of opinions is increased. The critical point p_c will be the value of p at which the variable $x_{\{1\}}$ is guaranteed to achieve full consensus. That is, in the supercritical regime we are guaranteed to achieve the absorbing steady state in which $x_{\{1\}} = 1$ and all other variables are equal to 0. Our approach is to consider the "worst case" initial conditions most likely to achieve a non-absorbing steady state in which consensus is *not* achieved. More specifically, since we only need one variable other than $x_{\{1\}}$ to be non-zero in order to be in the subcritical regime, we will try to find the initial conditions most likely to result in a non-zero $x_{\{2\}}$. This loses no generality, because:

1. There is symmetry amongst the variables $x_{\{2\}}, x_{\{3\}}, \ldots, x_{\{k\}}$ which represent decideds and have no zealots ,

2. If a variable representing undecideds is non-zero in steady state, one of the variables $x_{\{2\}}, x_{\{3\}}, ..., x_{\{k\}}$ must be non-zero in steady state as well.

To see that 2. holds, suppose $x_{\{i_1,i_2,\ldots i_j\}} \neq 0$ and $x_{\{i_1\}} = 0$. It is straightforward to verify from the mean field equations in Sec. III B that $x'_{\{i_1\}} > 0$, contradicting the assumption that we are in steady state. Therefore, if any variable representing undecideds in non-zero in steady state, there must be non-zero variables representing decideds as well. So it suffices to find the worst case conditions which make it least likely that $x_{\{2\}} = 0$ in steady state. It is intuitively clear that these initial conditions are $x_{\{1\}} = p$, $x_{\{2\}} = 1 - p$ and all other variables equal to 0. But with these initial conditions only the variables $x_{\{1\}}$, $x_{\{2\}}$ and $x_{\{1,2\}}$ will ever be non-zero, so it proceeds exactly as in the two-opinion case. If we were to find the value of p at which we first observe consensus with these "worst case" initial conditions, we would therefore find the same critical point as in the two-opinion case. This shows that as the number of opinions increases, the critical point remains stationary.

B. Equal zealots for all opinions

Next we consider the case in which there are an equal fraction of zealots representing all opinions. That is, we take $p = p_1 = p_2 = p_3 = \cdots = p_k$. Simulations show that in steady state, the variables describing decideds take on at most two distinct values. There will be one variable, say x_1, which has a higher value, and the remaining variables will all have the same value which is lower. See Fig. 4 for a plot of a particular simulation with random initial conditions and $k = 5$. We observe similar behavior for k = 3 and k = 4. Also note that in the supercritical regime,

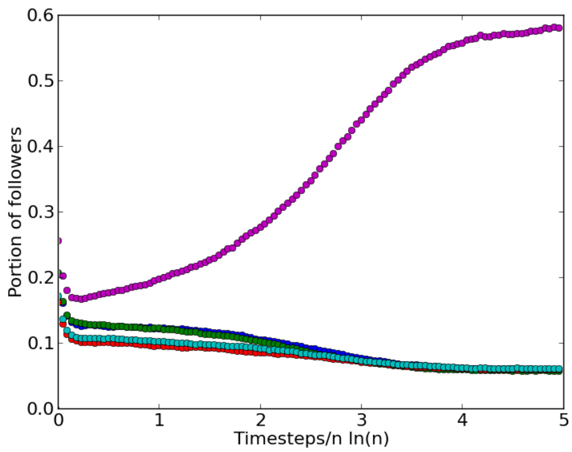

FIG. 4. The naming game with equal zealots of all opinions, random initial conditions, $k = 5$, and $p = 0.05$. In steady state it appears that the decideds for a single opinion dominates while the remaining decideds tie at a lower value, consistent with the assumption made in Sec. IV B in order to calculate critical points for all k.

there is one distinct value among the variables describing decideds by definition. Hence in the supercritical regime it is trivially the case that there are at most two distinct values amongst the variables describing decideds. Furthermore, note that a small number of distinct values amongst the variables describing decideds in steady state implies a small number of distinct values amongst the remaining variables. For example, if $x_{\{2\}} = x_{\{3\}}$ in steady state, it must be the case that $x_{\{1,2\}} = x_{\{1,3\}}$ in steady state. By applying this logic across all variables, it is straightforward to determine that there are at most $2n$ distinct values amongst all variables in steady state. We can make use of this fact by "glueing" together any variables which appear to have the same value in steady state in order to reduce the total number of variables which must be considered. We refer to this glueing process as "identifying" two variables, that is, giving two variables the same identity. For example, we may identify the two variables x and y as a single variable z.

For the sake of analysis, we hypothesize that the observation that the variables $x_{\{1\}}, x_{\{2\}}, \ldots, x_{\{k\}}$ only take on two distinct values in any stable steady state is true. It then becomes possible to quickly and accurately calculate the stable steady states observed in simulations and estimate the critical value p_c for any fixed k. We refer to the variable which attains the higher value as x and identify all other variables which attain the lower value as $\bar{x}$. For example, if $x_{\{1\}}$ attains the higher value we would have $x \coloneqq x_{\{1\}}$ and $\bar{x} \coloneqq x_{\{2\}} = x_{\{3\}} = \cdots = x_{\{k\}}$. We also identify z_S and $z_{\hat{S}}$ as $z_{|S|}$ if sets S and $\hat{S}$ are the same size and both have the same status as to whether or not they contain x. That is, $Z_{\{1,3,4\}}$ and $Z_{\{1,2,3\}}$ are identified as $Z_{\{3\}}$ since both contain the dominant variable 1 and are of size 3.

In symbols, z_S and $z_{\hat{S}}$ are identified with each other if $|S| = |\hat{S}|$ and $\delta_{x\in S} = \delta_{x\in S'}$ where $\delta_A = 1$ if A is true and $\delta_A = 0$ otherwise. This results in the following system of $2k$ variables. Note that by definition $z_1 = x$ and $\bar{z}_1 = \bar{x}$.

$$
\begin{aligned}
x' &= 2(T_x - x)(C_x - x) + (T_x - x)x + x(C_x - x) \\
&\quad - (1 - T_x)(x - p) \\
y' &= 2(T_{\bar{x}} - \bar{x})(C_{\bar{x}} - \bar{x}) + (T_{\bar{x}} - \bar{x})\bar{x} + \bar{x}(C_{\bar{x}} \\
&\quad - \bar{x}) - (1 - T_{\bar{x}})(\bar{x} - p) \\
z_i' &= T_x \bar{z}_{i-1} + (i-1)T_{\bar{x}} z_{i-1} - (1 - z_i)z_i - 2z_i^2 \\
&\quad - z_i(\frac{1}{i}(C_x + (i-1))C_{\bar{x}}), \text{ for } i \geq 2 \\
\bar{z}_i' &= iT_{\bar{x}}\bar{z}_{i-1} - (1 - \bar{z}_i)\bar{z}_i - 2\bar{z}_i^2 - \bar{z}_i C_{\bar{x}}, \text{ for } i \geq 2 \\
T_x &= x_1 + \sum_{i=2}^{k} \frac{1}{i}\binom{k-1}{i-1} z_i \\
C_x &= x_1 + \sum_{i=2}^{k} \binom{k-1}{i-1} z_i \\
T_{\bar{x}} &= y + \sum_{i=2}^{k} \frac{1}{i}(\binom{k-2}{i-1}\bar{z}_i + \binom{k-2}{i-2} z_i) \\
C_{\bar{x}} &= y + \sum_{i=2}^{k} \binom{k-2}{i-1}\bar{z}_i + \binom{k-2}{i-2} z_i
\end{aligned}
$$

Since there is only a linear number of variables, it is possible to quickly solve these equations through numerical integration. In this special case there exists a critical point p_c. If $p > p_c$ a stalemate is achieved among all k opinions, and if $p < p_c$ then a single opinion dominates. To calculate the critical point it is necessary to find the lowest possible value of p for which any set of initial conditions will result in a stalemate: a"k-way tie" between all k opinions. In this context, a k-way tie is equivalent to $x = \bar{x}$. For any given p, if there exist any initial conditions for which $x \neq \bar{x}$ in steady state, then p is a lower bound on p_c. Note that the initial condition which maximizes x will not result in $x = \bar{x}$ in steady state so long as $p < p_c$. Therefore, to find an upper bound on the critical point we begin with the initial condition $x = 1 - (k-1)p$ and $\bar{x} = p$. This is the initial condition that maximizes x. If in steady state $x = \bar{x}$, then a stalemate has occurred and hence $p_c < p$. We may use these upper and lower bounds to efficiently approximate p_c for each value of k to an arbitrary level of accuracy. In Fig. 5 these critical points are plotted against the number of opinions k, along with a fitted lognormal curve, for $1 \leq k \leq 20$. Recall that a lognormal distribution is a 2-parameter distribution defined as $f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{\frac{\ln x - \mu}{2\sigma^2}}$, where μ is the mean and σ the standard deviation.

In Fig. 6, we show a similar plot up to $k = 100$. The parameters for the lognormal curve in Fig. 5 are $\mu = 6.05$ and $\sigma = 3.73$, and the parameters for the lognormal curve in Fig. 6 are the same as in Fig. 5. In both cases the

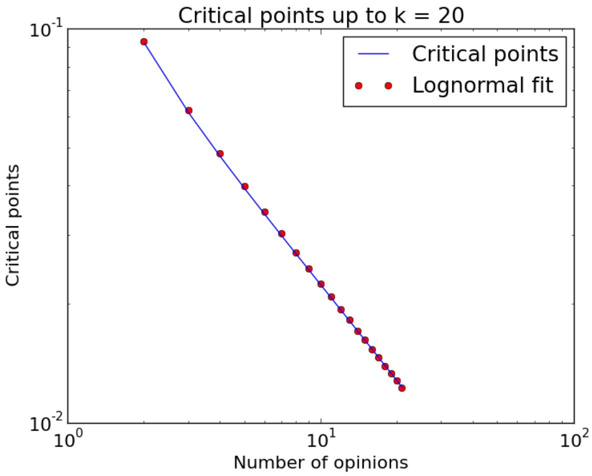

FIG. 5. Critical value p_c of p above which a stalemate is guaranteed, plotted against a fitted lognormal curve up to $k = 20$. The residual sum of squares is 2.05×10^{-6}.

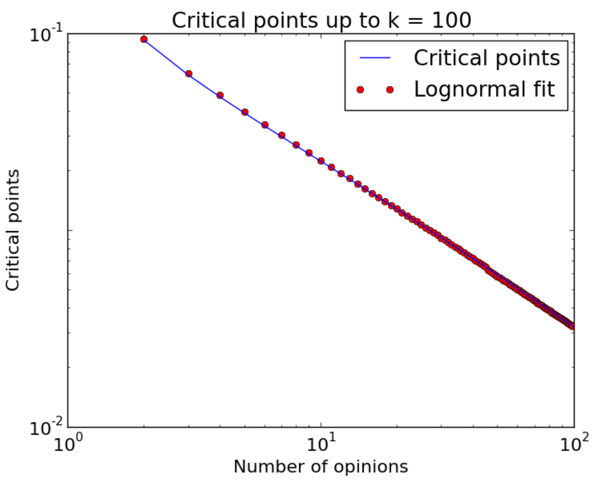

FIG. 6. Critical points plotted against a fitted lognormal curve up to $k = 100$. Note that the lognormal curve plotted here has the same parameters and is normalized as the lognormal curve in Fig. 5. The residual sum of squares is 2.34×10^{-6}.

residual sum of squares is less than 3×10^{-6}. This is a strong indication that the decay of the critical points is truly lognormal.

C. Verification of Critical Points

In order to calculate the critical point in Fig. 6 it was necessary to assume without proof that many of the variables would be equal in steady state. Here we will verify that the critical points are correct in lower dimensional cases. Given the calculated critical point p_c, we verify that $p_c + \epsilon$ is greater than the true critical point and $p_c - \epsilon$ is less than the true critical point, where $\epsilon = 0.001$. In order to verify that $p_c + \epsilon$ is greater than the true critical point, we calculate the steady state with $p = p_c + \epsilon$ and observe that the variables describing decideds all differ by less than 10^{-9}. In order to verify that $p_c - \epsilon$ is less than the true critical point, we calculate the steady state with $p = p_c - \epsilon$ and observe that the variables describing decideds can differ by more than 0.1. We performed this verification process for $k = 2, 3, 4, 5, 6$ and 7 .

V. A LOW DIMENSIONAL OPINION DYNAMICS MODEL SIMILAR TO THE NAMING GAME WITH ZEALOTS

Although we have studied the naming game model with zealots in two special cases, the methods used do not extend to the general case. However, simulations show that variables describing undecideds seem to "converge" more quickly than variables describing decideds in the sense that their derivatives quickly become small. One way to greatly simplify the naming game model is to assume that the variables describing undecideds converge instantanously at each timestep. That is, for each variable z_S where $|S| > 1$ we define z_S as the solution to $z'_S = 0$, which makes z_S a deterministic function of the random variables described by the decideds. Another way of stating this assumption is that the variables describing undecides are on a different time scale, having sufficient time to converge to steady state before any change occurs in the variables describing decideds. We begin with the case $k = 2$ for intuition and then proceed to the general case.

Note that the model described in Sec. II is two-dimensional due to the the restriction $x + y + z = 1$. By setting $y = 1 - x - z$, the system depends only on the variables x and z. We define a function $f(x)$ as the unique positive solution to $z' = 0$. The uniqueness of the positive solution to $z' = 0$ can be easily shown as follows:

$$
\begin{aligned}
z' &= z^2 + z[2x + \frac{-p+q+2}{2}] + 2x^2 \\
&\quad + x(-2 - p + q) + p \\
&= 0 \\
f(x) &= -\frac{1}{2}L(x) + \frac{1}{2}\sqrt{L(x)^2 - 4Q(x)}
\end{aligned}
$$

where L and Q are defined as follows:

$$
\begin{aligned}
L(x) &= 2x + \frac{-p+q+2}{2} \\
Q(x) &= 2x^2 + x(-2 - p + q) + p
\end{aligned}
$$

It is straightforward to show that $-\frac{1}{2}L(x) - \frac{1}{2}\sqrt{L(x)^2 - 4Q(x)}$ is non-positive. Since a negative solution to the equation $z' = 0$ is inadmissible, we need focus only on the positive root, which we denote by $f(x)$. That

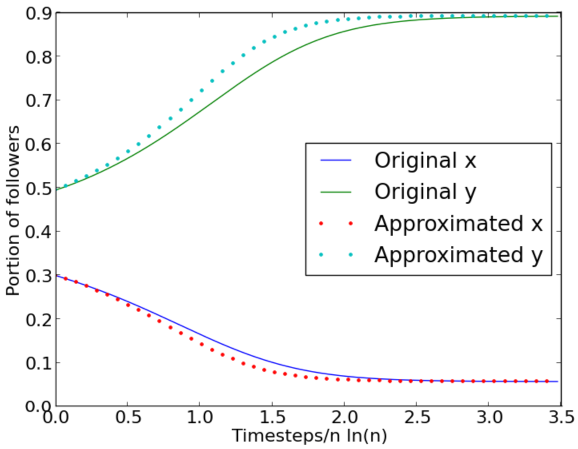

FIG. 7. Simplified versus original process with $k = 2$, $p = q = 0.05$ and initial condition $x = 0.3$ compared to the original process with $p = q = 0.05$ and initial conditions $x = 0.3$ and $z = f(x)$.

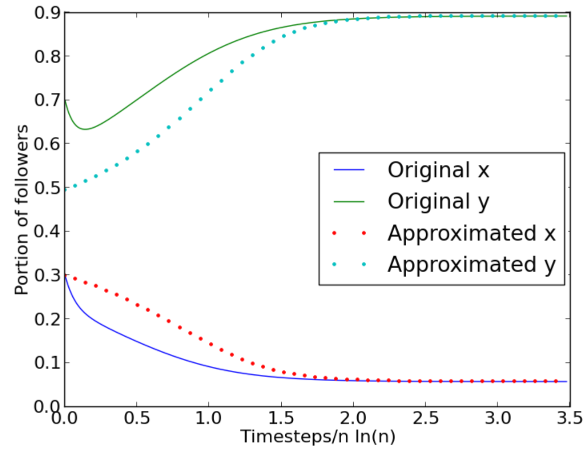

FIG. 8. Simplified versus original process with $k = 2$, $p = q = 0.05$ and initial condition $x = 0.3$ compared to the original process with $p = q = 0.05$ and initial conditions $x = 0.3$ and $z = 0.0$.

is, $f(x)$ is that function of x having the property that if z took the value $f(x)$, then z' would equal 0. Since at steady state, z' is indeed 0, a reasonable approximation it to set $z = f(x)$, and examine the resulting lower dimensional approximation of the true model.

Fixing $z = f(x)$ yields a 1-dimensional approximation of the original model. The one dimensional process has the following mean field equation:

$$\begin{aligned} x' &= z(x + z + \frac{p}{2}) - y(x - p) \\ &= f(x)(x + f(x) + \frac{p}{2}) - [1 - x - f(x)](x - p) \end{aligned}$$

We may compare this simple system to the larger system of two variables introduced in the previous section. We observe an excellent agreement in the trajectory of the variable x if both the original and reduced dimension system start with the initial condition $z = f(x)$, as shown in Fig. 7. This is true regardless of p, q and the initial choice of x. However, if initially $z \neq f(x)$, then the processes may be quite different, although they necessarily have the same steady states. This is demonstrated in Fig. 8 in which initially $z = 0.0$.

Based on the appearance of the curve, we conjecture that for a short time frame the z variable is converging to the deterministic function f, at which point both processes look similar. This intuition can be confirmed by delaying the start of the one dimensional process until some time t_0. As t_0 is increased, the curves look more and more similar as shown in Fig. 9 and Fig. 10.

The $k = 3$ opinion case yields a similar two dimensional approximation, by setting $z'_{xy} = z'_{xw} = z'_{wy} = z'_{xyw} = 0$. The k variables in this process correspond to the variables $x_{\{i\}}$, with all other variables z_S depending on them in a deterministic way. Specifically, we numerically determine

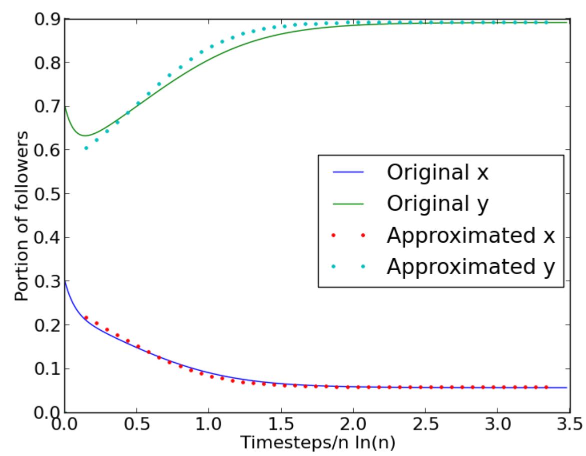

FIG. 9. The original process is run with $x = 0.3$, $y = 0.7$, and $z = 0.0$. The simplified process is run starting at time $0.15n\ln(n)$ with initial conditions equal to the values of the original process at that time.

the value of the variable z_S as the solution to $z'_S = 0$ according to equations (4)-(6). For convenience, we define $\overline{z_S}$ as the solution to $z'_S = 0$. In mean field analysis it does not matter how we define a process which corresponds to these equations, since it depends only on the system of derivatives. For concreteness a process may be defined as follows: at each timestep we choose a random node and a random neighbor, and transmit a random opinion of the first node to its neighbor. At this point we utilize the interaction rules of the naming game with zealots model as usual. However, after the rules have been applied to determine the net opinion sets of the chosen nodes, each node whose opinion set contains more than one opinion must randomly choose a new opinion set with the set S

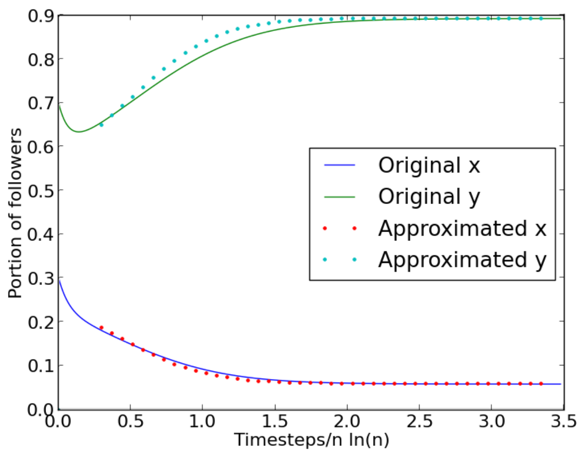

FIG. 10. The original process is run with $x = 0.3$, $y = 0.7$, and $z = 0.0$. The simplified process is run starting at time $0.3n \ln(n)$ with initial conditions equal to the values of the original process at that time.

chosen proportional to $\overline{z_S}$. Hence, in the mean field, the variables x_i are still updated in exactly the same manner as in the naming game with zealots model, but all other variables are deterministic functions of the x_i's.

VI. CONCLUSION

In this paper, we analyzed a generalized version of the binary naming game with zealots by considering an arbitrary number of opinions. We were able to numerically calculate critical points in two special cases: the case in which there are zealots of only one type, and the case in which there are an equal fraction of zealots of each type. The primary challenge was in the exponential number of variables in the system for general k, with $2^k - 1$ different variables corresponding to the non-empty subsets of $\{1, 2, 3, \ldots, k\}$. We observed in simulations that in these two special cases there were not very many distinct variables in steady state, with many variables being equal to each other in value. This led to the hypothesis that many variables could be "glued" together in order to obtain a smaller system.

Recall that in the first special case, in which there are only zealots of one type, we defined the critical point as the threshold value beyond which the opinion with zealots will always "win" by having more proponents than any other opinion regardless of initial conditions.We reduced the system from $2^k - 1$ variables to only 3. The critical point, therefore, does not depend on k and is always approximately 0.1. That is, it is necessary for 10% of the nodes in the system to be zealots in support of a single opinion in order to guarantee that this opinion is held by more nodes than any other in steady state.

In the second special case, in which there are an equal fraction of zealots for each opinion, the critical point is the threshold value beyond which a stalemate is guaranteed between all k opinions. We reduced the system of $2^k - 1$ variables to a system of $2k$ variables. This system can be solved numerically even for very large k, and we determined that the critical points appeared to decay precisely as a lognormal curve in k. This slow decay indicates that as the number of opinions increases, it still requires a massive fraction of zealots in order to achieve a stalemate, which shows that there will almost certainly be a dominant variable in any real-world situation which is well-described by naming game dynamics and in which there are an equal fraction of zealots of each type. We presented strong evidence that the critical points decay precisely as a lognormal curve in the special case in which there is an equal fraction of zealots of each type. To do this, we first found the best fit of a lognormal curve to the critical points from $k = 1$ to $k = 20$ with respect to the residual sum of squares error, and then plotted this curve against the larger collection of critical points from $k = 1$ to $k = 100$. We found that the residual sum of squares error remained on the order of 10^{-6}. It is not clear at present why the curve of critical points would show precisely lognormal decay. The lognormal distribution may arise as the limiting distribution of a product of random variables, which can often explain it's appearance as a probability distribution [26]. However, critical points are certainly not probability distributions arising from a process, so this cannot explain the appearance of lognormal decay. Moreover, although we presented strong evidence, we did not rigorously prove that the decay is precisely lognormal. It may be possible to prove analytically that the decay is precisely lognormal by first assuming that the critical points assume a general lognormal form and then solving for the undetermined parameters. We leave this for future work.

It is unlikely that in any real-world scenario the number of zealots corresponding to each opinion would be precisely equal. For this reason, it is desirable to show that the result we have given in this paper is stable. That is, if the fraction of zealots of each type is slightly perturbed, will there still be a critical point at which a stalemate (or near stalemate) is guaranteed and will the critical point be in nearly the same location? Note that since such a perturbation takes us out of the special case in which there is only a single parameter, p, to vary, it is not immediately clear how to extend the definition of "critical point." Moreover, a perfect stalemate will almost certainly no longer occur. A possible approach is to set some zealotry parameters to $p + \epsilon$ and others to $p - \epsilon$ so that there is still only one parameter, p, to vary. An additional benefit of this approach is that the methods presented in this paper may be applied to it. In fact, the general method of "glueing" together variables allows efficient steady state analysis in any case in which the zealotry parameters take on only a finite number of different values. Concretely, suppose there are two values p_1 and p_2 representing the fractions of zealots for different opinions, which could be considered an extension of the case in which the fraction of zealots for each opinion is p. By setting $p_1 = p - \epsilon$ and $p_2 = p + \epsilon$, we may study the effects of perturbation on steady states and still remain in a numerically tractable case.

Finally, we presented a model which behaves similarly to the naming game but has only k random variables. This model may be easier to study analytically, and makes the assumption that all variables corresponding to undecideds converge instantaneously at each timestep. In the context of this paper, this gives evidence that the naming game with zealots is fundamentally low dimensional in the sense that there are only k stable steady states, all of which are accounted for in our analysis.

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