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Theory of the corrugation instability of a piston-driven shock wave

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We analyze the two-dimensional stability of a shock wave driven by a steadily-moving corrugated piston in an inviscid fluid with an arbitrary equation of state. For $h \leq -1$ or $h > h_c$, where h is the D'yakov parameter and h_c is the Kontorovich limit, we find that small perturbations on the shock front are unstable and grow — at first quadratically and later linearly — with time. Such instabilities are associated with non-equilibrium fluid states and imply a non-unique solution to the hydrodynamic equations. The above criteria are consistent with instability limits observed in shock-tube experiments involving ionizing and dissociating gases and may have important implications for driven shocks in laser-fusion, astrophysical and/or detonation studies.

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I. INTRODUCTION

A shock wave sustained by the steady motion of a planar piston in a fluid-filled channel is a classical problem of hydrodynamics. This system is also a familiar paradigm in other areas of physics for analyzing shocks driven by a variety of complex mechanisms, including solar flares [1], supernova ejecta [2], ablation surfaces in inertial-confinement-fusion targets [3], moving repulsive dipole beams in Bose-Einstein condensates [4] and flame fronts in combustible fluid mixtures [5]. Of central importance to the dynamics of this class of shock waves is the issue of stability. It is well known that a piston-driven shock in an ideal gas is unconditionally stable; even in the absence of viscous damping, small disturbances on the surface of the shock evanesce over time and the front eventually acquires a planar shape [6]. For a fluid characterized by a non-ideal equation of state (EOS), however, stability is not guaranteed and under certain circumstances perturbations to the zeroth-order flow conditions can amplify over time and give rise to the formation of turbulent-like states behind the shock [7]. Such unstable phenomena have been observed in a number of gases undergoing ionization and dissociation under shock compression [8–10] and are seen almost universally in detonations [11–19].

The determination of the precise conditions that an EOS must satisfy to admit unstable shock behavior has been the subject of numerous theoretical investigations over the last 60 years [20–25]. For the most part, those studies reach the same conclusion regarding shock stability and do so by adopting a common simplifying approximation known as the "isolated wave model" in which a steadily-propagating shock is treated as a *standalone* discontinuity and conservation of mass and momentum across the front constitute the sole boundary conditions in the problem [26]. According to this approach, linear corrugations on the shock front are unstable and grow exponentially in time when either of the criteria

$$h < -1 \qquad \text{or} \qquad h > 1 + 2M_1 \tag{1}$$

is satisfied [27, 28]. Here, the parameter $M_1 = (U - V)/a_1$ is a downstream Mach number satisfying $0 < M_1 < 1$ and the symbols U, V and a_1 represent the shock, particle and compressed-fluid sound speeds, respectively, in the laboratory frame. The dimensionless quantity h that appears in Ineq. (1) is known as the D'yakov parameter [20] and is a measure of the inverse slope of the Hugoniot curve in the plane of pressure P and mass density ρ; it is defined as $h = -(U^2/\eta^2)\,(d\rho/dP)_H$, where $\eta = \rho_1/\rho_0 > 1$ is the compression ratio across the front and subscripts "0" and "1" denote upstream and downstream states, respectively. In addition to the corrugation instability described above, the isolated wave model also predicts a special category of unstable shock behavior (known as the D'yakov-Kontorovich instability) for values of h that lie in the range

$$h_c < h < 1 + 2M_1, \tag{2}$$

where $h_c = (1 - M_1^2 - \eta\, M_1^2)/(1 - M_1^2 + \eta\, M_1^2)$ is the so-called Kontorovich limit [21]. For isolated shocks satisfying Ineq. (2), linear perturbations on the front remain stationary over time and emit sound and entropy-vortex waves in the downstream direction [27, 28].

In reality, of course, steady shocks are never truly isolated waveforms; the sustainment of the front requires the existence of a uniform driving agent such as a piston behind it — the removal of which leads to a decay of the shock into acoustic waves [11]. It is perhaps not surprising, then, that instability limits derived using the isolated wave model do not closely match results from driven-shock experiments [8, 29] and moreover, imply the existence of multivalued solutions to projectile impact problems [30, 31]. As suggested by Fowles and Swan, these inconsistencies are almost certainly a consequence of the failure of the theory to account for the influence of the piston on the shock dynamics [32]. In this paper, we perform for the first time a linear stability analysis of a two-dimensional shock that takes into account its acoustic interaction with the piston sustaining it. Perturbations are introduced through small corrugations on the face of the piston, which is assumed to move impulsively from rest into a stationary inviscid fluid and thereafter maintain a constant speed V. The principal conclusion of this study is that driven planar shocks obey instability criteria that are somewhat different from the widely-known conditions appearing in Ineqs. (1) and (2).

The majority of our analysis has already been presented in great detail in a previous publication [33]. In that work, it was shown that a shock driven by a corrugated piston is stable provided that the condition $-1 < h < h_c$ holds. For values of h within that range, linear perturbations on the front attenuate as $t^{-3/2}$ asymptotically (or as $t^{-1/2}$ as $h \to h_c$ from below), where t denotes time. Here, we derive the solution for all other values of h. For such cases, we find that unstable shock behavior results from linear perturbation amplitudes that grow *algebraically* in time (in contrast to the exponential dependence predicted with the isolated wave model), which occurs if

$$h \leq -1 \qquad \text{or} \qquad h > h_c \,. \tag{3}$$

Comparison of these expressions with Ineq. (1) shows that the second inequality above is a less stringent condition for the occurrence of the corrugation instability than that derived using the isolated wave model. This finding is supported by experimental observations reported in Ref. [8], where the instability condition $h > h_c$ was postulated for strong driven shocks in argon and carbon dioxide gases. It should also be noted that in the present theory, the range of h-values in Ineq. (2) corresponds to a region of absolute growth so that the D'yakov-Kontorovich instability evidently does not occur for driven planar shocks perturbed at the piston boundary.

These results may have have important consequences for driven shock fronts in fluids that are subject to extreme thermodynamic conditions and thus not adequately described by ideal-gas constitutive relations. Such conditions occur, for example, in laser-fusion targets where perturbed strong shocks driven by non-uniform ablation processes seed deleterious hydrodynamic instabilities [34]. An understanding of how perturbed shock fronts evolve in these targets — and the conditions for which those fronts behave stably— are crucial considerations for achieving high fuel compression and significant fusion-energy gain. The theory developed in this paper may also provide insight into certain magnetohydrodynamic and astrophysical phenomena in which the corrugation instability of shock fronts is thought to play a key role; these include the generation of strong magnetic fields in relativistically hot plasmas [35], "noise" effects in the accretion of compact stellar objects [36] and the acceleration of high-energy particles in strongly-magnetized white-dwarf stars [37].

A third potentially-important application of this work is in the investigation of the deflagration-to-detonation transition in combustible fluid mixtures [38–43]. In that process, a flame front — which is analogous to the piston in the present study— suddenly accelerates and drives a strong shock that may result in detonation of the fuel [44]. Although this phenomenon has yet to be fully elucidated, there is evidence to suggest that it is linked to the development of turbulent states in the downstream flow [39, 45] and the occurrence of the corrugation instability of shocks [15, 31]. Thus, by providing a hydrodynamic framework for modeling the flame-shock system, the analysis presented in this paper may help to clarify the role that shock instabilities play in triggering detonations.

II. LINEAR STABILITY ANALYSIS

Our approach to analyzing the stability of a shock driven by a corrugated piston consists of first solving an auxiliary problem — that of a shock created by the motion of a wedge-shaped piston [see Fig. 1(a)] — whose solution can then be used to infer the result for a rippled driving surface [6]. We assume that the wedge angle δ is small and choose a co-moving system of coordinates (X, Y) in which the origin is at the vertex of the piston. In front of the piston, a slightly-bowed shock wave forms that propagates into a fluid with mass density ρ_0 and pressure P_0. At large distances on either side of the origin, the front is straight and moves with con-

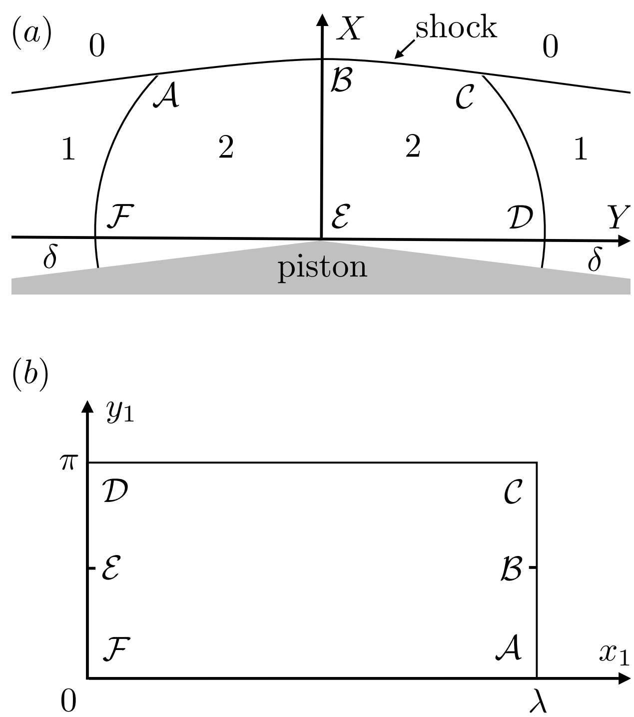

FIG. 1: The wedge-shaped piston with wedge angle δ (assumed to be small here). Region 2 in (a) becomes the rectangle (to zeroth-order) in (b) with $\lambda = \frac{1}{2}\ln[(1+M_1)/(1-M_1)]$ after a change to canonical field variables, a Busemann transformation and a conformal map. In the linear approximation, the solution to this problem can be used to determine the evolution of a shock front driven by a piston with a corrugated surface.

stant speed $U-V$ in a direction perpendicular to the piston's surface. Behind the shock, a circular acoustic wave with speed a_1 emanates from the vertex and separates the downstream fluid into two regions. Region 1 is the area of compressed fluid ahead of the acoustic wave with uniform mass density ρ_1 and pressure P_1. This region is bordered on each side of the X-axis by the straight section of the shock across which mass and momentum conservation give the relations $V = U(1-\rho_0/\rho_1)$ and $P_1 = P_0 + \rho_0 U^2(1-\rho_0/\rho_1)$. To first-order in δ, the fluid has a velocity given by $(0, \pm V\delta)$, where the plus and minus sign correspond to $Y > 0$ and $Y < 0$, respectively.

Region 2 is defined as the area behind the acoustic wave that is bounded by the piston and the shock front. In this region, the perturbed fluid-velocity vector can be expressed as $\mathbf{q}_2 = V(u, v)$, where u and v are dimensionless and of order δ. By linearizing the Euler equations, one can then show that P_2 satisfies the wave equation. In terms of the conical-field variables $x = X/(a_1 t)$ and $y = Y/(a_1 t)$, that equation becomes

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 1\right)\left(x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y}\right) = \nabla^2 p, \tag{4}$$

where we have introduced the dimensionless quantity $p = (P_2 - P_1)/(a_1 V \rho_1)$. We now look for a solution to Eq. (4) in region 2 subject to the following boundary conditions. On the surface of the piston, where $x = 0$ approximately, the normal fluid velocity must be the same as that of the piston itself; using the linearized momentum equation, this implies $\partial p/\partial x = 0$ when $x = 0$. On the boundary with region 1, the pressure must be continuous, so that $P_2 = P_1$ and $p = 0$ for $x^2 + y^2 = 1$. The other condition to be specified is that on the part of the shock bounded by the circular acoustic waves, which can be described by the equation $X_S = (U-V)\,t + \xi$, where $\xi = a_1 t\, f(y)$ is the deflection of the front from plane and f is of order δ. The condition that p satisfies there (*i.e.*, $x = M_1$, approximately) is [33]

$$\left(1 - M_1^2\right)\frac{\partial p}{\partial x} = \left(yM_1 + y\beta - \Gamma\, M_1^2\, y^{-1}\right)\frac{\partial p}{\partial y}\,, \tag{5}$$

where

$$\beta = \frac{1-h}{2M_1} \qquad \text{and} \qquad \Gamma = \frac{(1+h)\eta}{2M_1}. \tag{6}$$

It should be noted that the calculation in Ref. [33] also yields the conditions $u = \beta\, p$, $y\,(\partial v/\partial y) = M_1\Gamma(\partial p/\partial y)$ and $\partial\xi/\partial t = a_1 M_1 \Gamma\, p$ on the surface $x = M_1$.

Following Refs. [46, 47], the solution to Eq. (4) is obtained by enlisting the following procedure. First, a Busemann transformation [48] to the coordinates (r, θ) is made using the relations $x = 2r\cos\theta/(1+r^2)$ and $y = 2r\sin\theta/(1+r^2)$, which transforms Eq. (4) to Laplace's equation. Second, a conformal map in the form $z_1 = x_1 + i\,y_1 = \log\left[(1+\zeta)/(1-\zeta)\right] + i\,\pi/2$, where $\zeta = r\exp(i\theta)$, is employed. The result is that region 2 becomes, approximately, the rectangle shown in Fig. 1(b) with $0 \le x_1 \le \lambda$ and $0 \le y_1 \le \pi$, where $\lambda = \frac{1}{2}\ln[(1+M_1)/(1-M_1)]$. In the z_1-plane, the shock front $\mathcal{ABC}$ corresponds to the side of the rectangle given by $x_1 = \lambda$, $0 \le y_1 \le \pi$, the piston's surface $\mathcal{FED}$ becomes the segment $x_1 = 0$, $0 \le y_1 \le \pi$ and the arcs $\mathcal{DC}$ and $\mathcal{FA}$ map to the edges $0 \le x_1 \le \lambda$, $y_1 = \pi$ and $0 \le x_1 \le \lambda$, $y_1 = 0$, respectively. The forms of the boundary conditions on the latter three surfaces are not affected by the transformation to the new coordinate system so that $\partial\, p/\partial x_1 = 0$ on $\mathcal{FED}$ and $p = 0$ on $\mathcal{FA}$ and $\mathcal{DC}$. Furthermore, we have the additional condition that $\partial p/\partial y_1$ must vanish on the midline $\mathcal{EB}$ due to the symmetry of the problem.

The remaining boundary condition is that on $\mathcal{ABC}$, for which $y = -(1-M_1^2)^{1/2}\cos y_1$. The form of Eq. (5) suggests it is useful to introduce the complex function

$$w(z_1) = \frac{\partial p}{\partial x_1} - i\,\frac{\partial p}{\partial y_1} \tag{7}$$

and look for a solution in terms of this quantity rather than p directly. Note that w is imaginary on $\mathcal{EFA}$ and $\mathcal{EDC}$, and vanishes at $\mathcal{E}$. On $\mathcal{ABC}$, Eq. (5) implies [33, 47]

$$\arg w = \tan^{-1}\left(\mu_+ \tan y_1\right) + \tan^{-1}\left(\mu_- \tan y_1\right), \tag{8}$$

where

$$\mu_\pm = \frac{1 \pm \sqrt{1 - 4(\Gamma/\alpha^2)(\beta - \Gamma/\alpha^2)}}{2(\beta - \Gamma/\alpha^2)} \tag{9}$$

and we have introduced the additional EOS parameter

$$\alpha^2 = \frac{1-M_1^2}{M_1^2}. \tag{10}$$

For $-1 < h < h_c$, the coefficients $\mu_\pm$ are either both real and positive, or both complex with positive real parts (see Fig. 2). The solutions for those cases have already been given in Ref. [33] and correspond to stable shock behavior. For $h < -1$ or $h > h_c$, the coefficients $\mu_\pm$ are purely real, but with *opposite* signs and this leads to a different class of solutions, as we now demonstrate.

Let us first consider the case $\mu_+ > 0$ and $\mu_- < 0$ (*i.e.*, $h < -1$). For these conditions, Eq. (8) can be written [49]

$$\arg w(\lambda + iy_1) = -\sum_{n=1}^{\infty}\left(b^{-n} - a^n\right) n^{-1}\sin 2ny_1\,, \tag{11}$$

where $a = (\mu_+ - 1)/(\mu_+ + 1)$ and $b = (\mu_- - 1)/(\mu_- + 1)$. We see that as $y_1 \to 0$ or π, the above expression tends to zero. On $\mathcal{FA}$ and $\mathcal{DC}$, however, the function w is purely imaginary. Thus, $\arg w$ is discontinuous at the points $\mathcal{A}$ and $\mathcal{C}$, which implies that w must have a zero or a pole at each of those points. In what follows, a solution will be found that is regular inside the rectangle in Fig. 1(b) with *simple* zeros at the points $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{E}$. It must be the only regular solution with zeros at these points, because if there were another with higher-order zeros, $\widetilde{w}(z_1)$ say, then the ratio $\widetilde{w}(z_1)/w(z_1)$ would be purely *real* on the

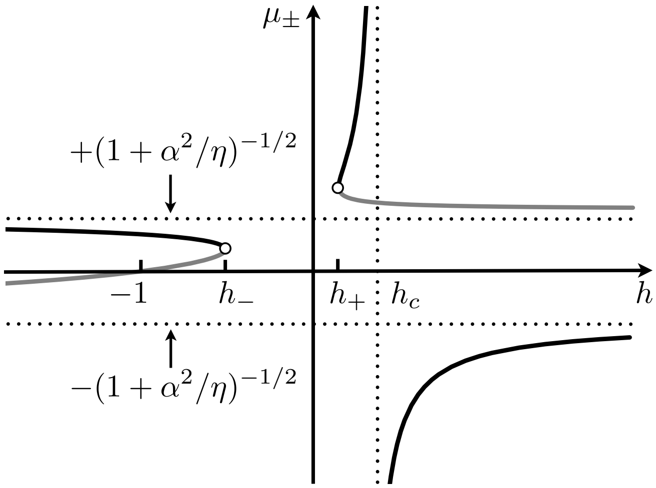

FIG. 2: Qualitative plots of μ_- (gray line) and μ_+ (black line) versus h. For $-1 < h < h_c$, the shock front is stable and $\mu_\pm$ are either both real and positive or they are complex conjugates of each other with positive real parts; see Ref. [33]. Outside of this range, the coefficients $\mu_\pm$ have opposite signs, which results in an unstable solution. The parameters $h_\pm$ in this figure denote the roots of the discriminant in Eq. (9).

boundary and thus equal to a constant [50], which, as shown below, can have only one determination in this problem. Note also that the assumption of zeros at $\mathcal{A}$ and $\mathcal{C}$ requires $\arg w$ to be $\pi/2$ on $\mathcal{EFA}$ and $3\pi/2$ on $\mathcal{EDC}$ [51].

An analytic function that satisfies all of the boundary conditions stated above is

$$\begin{aligned} w(z_1) &= iK\,\frac{\theta_2(-iz_1,q)\,\theta_4(-iz_1,q)}{\theta_3^2(-iz_1,q)} \\ &\quad\times\exp\left\{-\sum_{n=1}^{\infty}\frac{(b^{-n}-a^n)\cosh 2nz_1}{n\sinh 2n\lambda}\right\}, \end{aligned} \quad (12)$$

where θ_2, θ_3 and θ_4 denote Jacobi theta functions [52] with $q = \exp(-2\lambda)$. Note that this function is regular inside the rectangular boundary of Fig. 1(b) and has an integrable, second-order pole at $\mathcal{B}$. The constant K is determined by the fact that the change in velocity $V\!\int(\partial v/\partial y)\,dy$ along the shock is $2V\delta$, or equivalently

$$\frac{(b-a)\,K}{2\,\delta\,q^{1/4}} = \frac{\alpha}{\Gamma}\left[\int_0^{\pi}\tan^2 y_1\,G(y_1)\,dy_1\right]^{-1}, \quad (13)$$

where

$$\begin{aligned} G(y_1) = \prod_{m=0}^{\infty} &\frac{(1-2\,b^{-1}q^{2m+2}\cos 2y_1 + b^{-2}\,q^{4m+4})}{(1-2\,a\,q^{2m}\cos 2y_1 + a^2\,q^{4m})} \\ &\times\frac{(1-2\,q^{2m+2}\cos 2y_1 + q^{4m+4})}{(1+2\,q^{2m+2}\cos 2y_1 + q^{4m+4})^2} \\ &\times(1+2\,q^{2m+1}\cos 2y_1 + q^{4m+2})\,. \end{aligned} \quad (14)$$

Since $q < 1$, the infinite product above converges rapidly.

Following Refs. [6, 33], we can now deduce the shape of a shock launched by a corrugated piston with profile $\varepsilon\exp(i\omega Y)$, where ω is a spatial frequency and the amplitude ε is small compared to ω^{-1}. Consider a point on the piston $Y = Y'$ and another a distance dY' away. The change in slope between these two points is $-\varepsilon\,\omega^2\exp(i\omega Y')\,dY'$. Thus, if we replace 2δ by this quantity in the above analysis, we can write an expression for the contribution to $\partial p/\partial Y$ on the shock from an infinitesimal section of the piston:

$$\left(\frac{\partial p}{\partial Y}\right)_{Y'}^{Y'+dY'} = \frac{K_1\,\varepsilon\,\omega^2\,e^{i\omega Y'}\tan y_1\,G(y_1)}{a_1 t\,(1-M_1^2)^{1/2}}\,dY'\,, \quad (15)$$

where the constant K_1 is given by the right side of Eq. (13). Using $Y - Y' = -a_1 t\,(1-M_1^2)^{1/2}\cos y_1$, three integrations can be performed on Eq. (15). The first is over the infinite surface of the piston, the second is with respect to the variable Y and the third is over time [6, 33]; the resulting expression for the shock-ripple amplitude is

$$\frac{\xi}{\varepsilon e^{i\omega Y}} = \frac{\int_{-1}^{1} e^{-i\tau z}\,z^{-2}\left(1-z^2\right)^{1/2}\,F(z)\,dz}{\int_{-1}^{1} z^{-2}\left(1-z^2\right)^{1/2}\,F(z)\,dz}\,, \quad (16)$$

where

$$\begin{aligned} F(z) &= G[\cos^{-1}(-z)] \\ &= \prod_{m=0}^{\infty}\frac{(1-2\,b^{-1}q^{2m+2}(2z^2-1) + b^{-2}\,q^{4m+4})}{(1-2\,a\,q^{2m}(2z^2-1) + a^2\,q^{4m})} \\ &\qquad\times\frac{(1-2\,q^{2m+2}(2z^2-1) + q^{4m+4})}{(1+2\,q^{2m+2}(2z^2-1) + q^{4m+4})^2} \\ &\qquad\times(1+2\,q^{2m+1}(2z^2-1) + q^{4m+2}) \end{aligned} \quad (17)$$

and $\tau = \omega\,a_1 t\,(1-M_1^2)^{1/2}$ is a dimensionless time. At $\tau = 0$, we see that the shock has the shape of the piston, as it should. Moreover, one can show that in the limit $h \to -1$, Eq. (16) and the solution derived in Ref. [33] yield the same expression — as they must since μ_- changes sign continuously there; see Fig. 2. The solution for $\mu_+ < 0$ and $\mu_- > 0$ (*i.e.*, $h > h_c$) is obtained by interchanging the parameters a and b in the above formulae.

The evolution of the shock-ripple amplitude ξ with time can be determined by integrating the right side of Eq. (16) by parts and then evaluating the resulting expressions numerically for particular values of the parameters η, M_1 and h. Figure 3 shows a plot of the real part of $\xi/(\varepsilon\,e^{i\omega Y})$ obtained by following this procedure for $\eta = 3$, $M_1 = 0.5$ and $h = -1.1$. From this figure, we see that in the linear approximation, the shock-ripple amplitude grows (approximately) quadratically with τ initially and then switches over to a linear dependence at later times. Note that this behavior is considerably different from the exponential growth predicted for unstable, *isolated* shock fronts [20–25]. For comparison, Fig. 3 also contains an example of a stable solution satisfying the condition $-1 < h < h_c$, which decays asymptotically as $\tau^{-3/2}$. That class of driven shock waves was discussed

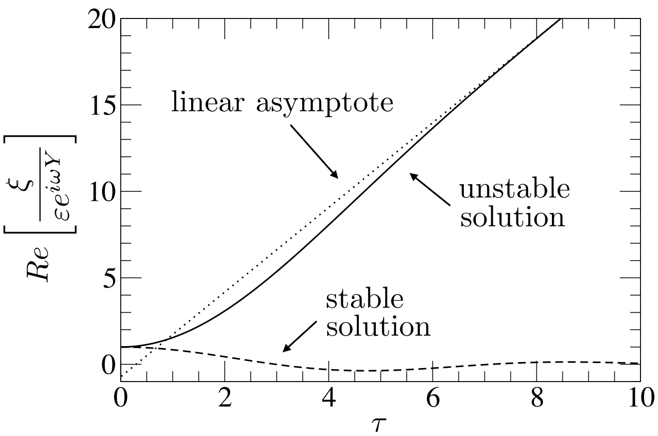

FIG. 3: Plot of the real part of the normalized shock-ripple amplitude in Eq. (16) versus dimensionless time τ for $\eta = 3$, $M_1 = 0.5$ and $h = -1.1$ (solid line). At late times, the growth is essentially linear with τ. A stable solution is also shown (dashed line) for $\eta = 2.197$, $M_1 = 0.4782$, $h = -0.1647$ and $h_c = 0.2111$; see Ref. [33] for the derivation of that class of solutions.

in detail in Ref. [33] and is not considered here, but in passing, we note that the stable branch of the solution has the form of Eq. (16) with $F(z)$ replaced by

$$\prod_{m=0}^{\infty} \frac{\left[1 - 2q^{2m+2}(2z^2-1) + q^{4m+4}\right]}{\left[1 - 2aq^{2m}(2z^2-1) + a^2q^{4m}\right]} \times \frac{\left[1 + 2q^{2m+1}(2z^2-1) + q^{4m+2}\right]}{\left[1 - 2bq^{2m}(2z^2-1) + b^2q^{4m}\right]}$$

and the factors of z^{-2} omitted from the integrands. One can show that at the upper limit of the stable regime (*i.e.*, as $h \to h_c$ from below), the asymptotic behavior changes to a $\tau^{-1/2}$ dependence due to the appearance of a term $(1 - z^2)$ in the denominator of the infinite product above [53].

III. THERMODYNAMIC CONSIDERATIONS

We now demonstrate that the instability criteria derived in this paper are associated with shocked fluid states that are not in thermodynamic equilibrium. Much of the basis for this conclusion has already been provided by Fowles [31], who analyzed the shock stability problem using a principle of irreversible thermodynamics that says the approach to equilibrium of two disparate systems is characterized by the conditions $dS \geq 0$ and $dE' \leq 0$, where S and E' denote entropy and reduced internal energy, respectively, with $dE' = dE - P_0\, d\rho/\rho^2$. (The quantity E here is the conventional internal energy of the fluid.) According to this principle, a shock transition from an initial state to a given final state is thermodynamically unstable if there exists an alternative point on the Hugoniot curve for which the entropy is larger, or the reduced internal energy is smaller, than that given state. Fowles showed that the former situation arises whenever $h < -1$, which is also a condition that ultimately leads to the splitting of the front into a double shock-wave structure with a greater net increase in entropy [54, 55].

It is straightforward to see that satisfaction of the condition $h > h_c$ also implies the existence of a non-equilibrium fluid state from the equivalent inequality [56]

$$\left(\frac{\partial E'}{\partial S}\right)_{\mathcal{H}} = T\left[1 - M_1^2(\eta - 1)(1+\gamma)\right] < 0\,, \qquad (18)$$

where γ is the Grüneisen parameter and $\mathcal{H}$ and T denote enthalpy and temperature, respectively. Note that for a shock in an ideal gas, the left side of the above inequality reduces to $T\,P_0/P$, which is strictly positive. For non-ideal fluids, however, it is possible that $(\partial E'/\partial S)_{\mathcal{H}}$ is negative at sufficiently high shock-compression. The instability that develops in such a case effects a transition to an alternate (and likely turbulent) solution of the hydrodynamic equations in which the reduced internal energy and particle velocity of the compressed fluid are minimized [31]. We speculate that detonations are related to corrugation instabilities of this variety.

IV. SUMMARY AND CONCLUSIONS

In summary, we have solved for the first time the problem of a two-dimensional, planar shock front created, sustained and perturbed by a piston moving in a stationary inviscid fluid with an arbitrary EOS. Our theory predicts both stable and unstable behavior, depending on the value of h. For $-1 < h < h_c$, we find that linear perturbation amplitudes on the shock front attenuate asymptotically as $t^{-3/2}$ (or as $t^{-1/2}$ as $h \to h_c$ from below). Outside of this range, they grow — at first quadratically and later linearly — with time. It is important to notice that the upper stability limit found in this study is smaller than that derived using the isolated wave model and cited in standard fluid-dynamics textbooks (*e.g.*, see Ref. [27]), but agrees precisely with observations in driven-shock experiments involving ionizing and dissociating gases [8]. We should also remark that the D'yakov-Kontorvich instability of isolated shocks — which shares the same threshold condition $h > h_c$, but is characterized by a solution with *stationary* perturbations [57–61] — evidently does not occur for fronts sustained and perturbed by a moving piston. (Note that an earlier study by Wouchuk and Cavada [62] reached a different conclusion regarding the occurrence of the D'yakov-Kontorovich instability for shocks driven by a corrugated piston; this is perhaps not too surprising, though, since their analysis neglected the homogenous solution to the functional equation for the Laplace transform of the first-order pressure and moreover, only considered values of h in the limited range $h_c < h < 1 - 2M_1^2$.) We have further shown in this work that corrugation instabilities are associated with shock-compressed fluid states not in thermodynamic equilibrium — a condition that implies the existence of a non-unique solution to the hydrodynamic equations. In practice, the occurrence of such instabilities signals a transition to an alternate, post-shock fluid state in which the entropy is larger, or the reduced internal energy is smaller, than the original.

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[49] Note that the Fourier expansion $\tan^{-1}(c\tan x) = \sum_{n=1}^{\infty}(d^n - 1)\, n^{-1}\sin 2nx + \pi/2$ holds for $c > 0$ with $d = (c-1)/(c+1)$. For $c < 0$, the term in parentheses becomes $(1-d^{-n})$ and the factor $+\pi/2$ changes to $-\pi/2$.
[50] The real and imaginary parts of an analytic function are harmonic. Thus, if such a function is purely real on the domain boundary, then its imaginary part must be zero everywhere since Laplace's equation admits no internal extrema. Application of the Cauchy-Riemann equations then shows that the function must be a real constant.
[51] For example, the transition from $z_1 = \lambda - \epsilon$ to $z_1 = \lambda + i\epsilon$, where ϵ is small, real and positive, changes $\arg w(z_1)$ by $+\pi/2$ if a simple pole exists at $\mathcal{A}$ and by $-\pi/2$ if there is a simple zero at that point.
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