Frequency adjustment and synchrony in networks of delayed pulse-coupled oscillators

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Frequency adjustment and synchrony in networks of delayed pulse coupled oscillators

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We introduce a system of pulse coupled oscillators that can change both their phases and frequencies; and prove that when there is a separation of time scales between phase and frequency adjustment the system converges to exact synchrony on strongly connected graphs with time delays. The analysis involves decomposing the network into a forest of tree-like structures that capture causality. These results provide a robust method of sensor net synchronization as well as demonstrate a new avenue of possible pulse coupled oscillator research.

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I. INTRODUCTION

Pulse coupled oscillators (PCOs) have proven themselves an incredibly successful model of temporal coordination. Whether in biological, engineering or physical systems, the mix of discrete and continuous elements in PCO models allow for a study of synchronization in a surprisingly parsimonious and well motivated system [1].

One measure of the success of pulse coupled oscillator synchronization is its adoption for a family of wireless sensor network synchronization protocols [2–5]. However, while traditional PCO models provide an excellent tool to study synchronization in idealized settings or with specified network topologies, its application to wireless sensor networks has revealed that when such idealized PCOs are generalized to more realistic settings, they typically have great difficulty synchronizing. In particular, traditional PCO models are especially challenged by the combination of complex network topologies and signal delay [6–9]; this has naturally led to a number of design questions relevant to both those interested in superior wireless sensor network synchronization protocols and those interested in the theoretical limits of the PCO framework.

The design challenge posed by complex network topology and delays has been addressed by specialized PCO models which augment oscillators with: mixtures of inhibition and excitation [6–11], stochasticity [6], single bits of additional memory [12, 13] or other modifications [14]. These recent PCO models represent a surprisingly large break from traditional PCO and from Dynamical Systems more broadly—requiring new analytical techniques, new theoretical goals and new considerations for novelty.

However, these new models have been unable to address one of the more interesting traditional oscillator questions: can oscillators with heterogeneous frequencies synchronize? Of the PCO models able to synchronize on a complex network with delays, there is at best numerical evidence that they approximate synchrony when oscillator frequencies are heterogeneous. In such complicated settings, there is little understanding of how to design PCO systems to handle heterogeneous frequencies—and under reasonable assumptions, exact synchrony is clearly impossible. Yet, given that frequency alteration is common in Hebbian Learning [15] and the recent advances in continuous oscillators [16], a PCO model which allows individual oscillators to adjust their frequencies is not only well motivated, but promising.

As such, the main contribution of this paper is the introduction of a system of phase-frequency pulse coupled oscillators and proof that this system attains exact synchrony even in the presence of uniform time delays, complex connected networks, and oscillator frequency heterogeneity. Not only does this suggest an entirely new avenue of PCO model design, this definitively answers the question of whether exact synchrony is possible in such systems.

In particular, we show that when there is a separation in time scales between phase and frequency adjustments there is an invariant cascading regime of phase-frequency space and a corresponding phase-locked fixed point. To build the machinery for this result, we analyze both a single oscillator subjected to periodic forcing and pairs of oscillators. These simplified results are then conglomerated via an analysis of the emergent tree-like dependencies in the system, yielding the main convergence result for connected networks. Subsequently, at this fast-time fixed point, the slow-time frequency responses drive the system towards exact synchrony. Next, we give a lower bound on the probability of convergence based on an analysis of the networks' degree sequence. Finally, to bolster these analytic results, we provide numerical simulation demonstrating robustness.

While the primary aim of this paper is to develop theoretical results and understanding of a particular oscillator systems, we conclude with several suggested modifications of this system for those seeking to design oscillator systems for engineering applications.

A. System Description

Similar to [8, 9, 17, 18], consider a PCO model on an undirected graph $G = \{V, E\}$, where each oscillator $i \in V$ has phase $\phi_i(t) \in [0, 1]$, speed $\omega_i \in [1, 2)$ and $\frac{d\phi_i}{dt} = \omega_i$. When i reaches its terminal phase, $\phi_i(t) = 1$, it emits a signal and its phase is reset to 0. The signal from i takes time $\tau < \frac{1}{8}$ to reach i's neighbors in G: $N(i)$. Let t_i^n denote the time of the n'th firing of oscillator i. We limit the rate that oscillators can respond to incoming signals by introducing a 'quiescent' period, where after an oscillator j processes an incoming signal, it then ignores future signals for the next $q > 2\tau$ time. Otherwise, when a signal from i arrives at non-quiescent oscillator j, j adjusts both its phase and frequency according to its phase resetting curve, f and frequency response curve g. Namely: $\phi_j(t_i^n+\tau) \leftarrow f(\phi_j(t_i^n+\tau))$ and $\omega_j(t_i^n+\tau) \leftarrow \omega_j(t_i^n+\tau)[1+\epsilon g(\phi_j(t_i^n+\tau))]$ for small $\epsilon > 0$.

We consider phase resetting curves and frequency response curves of the form:

$$f(\phi) = \begin{cases} (1-\alpha)\phi & : \phi < B \\ 1 & : \phi \geq B \end{cases} \quad \Bigg| \quad g(\phi) = \begin{cases} = 0 & : \phi \in [0,\tau) \\ < 0 & : \phi \in [\tau, B) \\ \geq 0 & : \phi \geq B \end{cases}$$

with parameters $\frac{1}{2} < \alpha < 1$ and $4\alpha\tau < B \leq \frac{1}{2} - 2\tau$. Notice that the phase resetting curve $f(\phi)$ inhibits oscillators by a fixed proportion α when $\phi < B$ and excites oscillators to fire when $\phi > B$. The discontinuity in f could be relaxed, but at the expense of added complexity and a smaller basin of attraction (as supported by the arguments in [8]). While the slower time scale of frequency response implies that much of the details of $g(\phi)$ are unimportant, Fig. 1 displays a choice of ϵ and $g(\phi)$ that works well in numerical trials.

This model expands on those used in [8, 9, 17, 18] in two important ways. The first difference is that it allows oscillators to adjust their frequency via a frequency response curve—allowing oscillators to overcome heterogeneous frequencies. The second modification is the introduction of the quiescent period. The quiescent period operates analogously to, but is different than, the well studied refractory period [6, 19]. Whereas the refractory period prevents the processing of signals immediately after an oscillator fires, the quiescent period does so after the reception of a signal. Conceptually, the quiescent period represents oscillators whose receptors are overloaded by processing an incoming signal, such that they only process the first in a series of closely timed signals.

While other large oscillator systems can have extremely complicated patterns of interactions, the quiescent period tends to mute this possibility in our system. The main effect that the quiescent period has on the analysis of the system is that its presence eventually creates predictable tree-like dependencies, where oscillators only process signals from one of their neighbors. It is through analyzing these tree-like structures that convergence results will be derived from the analysis of small systems of oscillators.

II. FAST CONVERGENCE OF PHASES

To begin, we examine the fast time subsystem, where phases change but not frequencies. Since in this subsystem oscillators have different frequencies and there is time delay, it is clearly the case that exact synchronization is impossible. However, as we will show, the system is able converge to a solution that approaches synchrony as frequency differences approach zero.

A. Forcing and Pairs

Of the possibly relevant scenarios, the two simplest are an oscillator subjected to periodic forcing and a pair of oscillators. Indeed, it would seem reasonable that if these oscillators are eventually to arrive at a solution resembling synchrony, they had better be able to be, entrained by a periodic forcing term, and have a pair of them synchronize.

Suppose an otherwise isolated oscillator i receives a single signal at time t_*, how does this signal affect the next time i will fire, t_i^{n+1}? If the signal arrives when $\phi_i(t_*) < B$, i will be inhibited by $\alpha\phi_i(t_*)$ and will take an additional $\frac{\alpha}{\omega_i}\phi_i(t_*)$ time to fire. Since $\phi_i(t_*) = \omega_i(t_* - t_i^n)$, then relative to the last time i fired:

$$t_i^{n+1} = t_i^n + \frac{1}{\omega_i} + \alpha(t_*^n - t_i^n) \tag{1}$$

Alternatively, if i receives a signal when $\phi_i(t_*) \geq B$ then i will be excited to firing and $t_i^{n+1} = t_*$.

Thus, depending on the time that t_* arrives, two very different outcomes can result, one based upon inhibition and a lengthening of period, the other based upon excitation and the relative independence on t_i^n. Despite this complication, the following lemma clearly demonstrates that these oscillators can be entrained to a periodic signal.

Lemma 1. *If an isolated oscillator i receives a signal at times $t_*^{n+1} = t_*^n + \frac{1}{\omega_*}$ where $\frac{\omega_i}{B} \geq \omega_* \geq \frac{\omega_i}{1+2\alpha\tau\omega_i}$ then $t_i^n - t_*^n \to \min\{0, \frac{\omega_* - \omega_i}{\alpha\omega_*\omega_i}\}$.*

Proof. We consider two cases.

Case 1: $\omega_i > \omega_*$. If the nth forcing signal arrives when $\phi_i \geq B$ then i will be forced to fire and $t_i^n = t_*^n$. This gives that when the next forcing signal arrives i will have already fired by itself at time $t_*^n + \frac{1}{\omega_i} \leq t_*^{n+1}$. Thus, $\phi_i(t_*^{n+1}) = \omega_i(\frac{1}{\omega_*} - \frac{1}{\omega_i}) \leq 2\alpha\tau\omega_i < B$, which prompts the next step.

If the nth forcing signal arrives when $\phi_i < B$ then manipulating Eq. (1) yields,

$$t_i^{n+1} - t_*^{n+1} = t_i^n - t_*^n + \frac{1}{\omega_i} - \frac{1}{\omega_*} - \alpha(t_i^n - t_*^n). \tag{2}$$

Since, $\alpha \in (\frac{1}{2}, 1)$ this difference equation converges to $t_i^n - t_*^n = \frac{\omega_* - \omega_i}{\alpha\omega_*\omega_i}$.

Case 2: $\omega_* > \omega_i$. If the nth forcing signal arrives when $\phi_i < B$ then Δt_i^{n+1} is given by Eqn. 2. However, since the unique stable point of Eqn. 2 is positive if $\omega_* > \omega_i$, i will eventually receive a signal before it fires, and thus, when $\phi_i \geq B$.

If the nth forcing signal arrives when $\phi_i \geq B$ then $t_i^n = t_*^n$. The next signal arrives time $\frac{1}{\omega_*}$ later when

$\phi_i(t_*^{n+1}) = \frac{\omega_i}{\omega_*} \geq B$. Thus $t_i^{n+1} = t_*^{n+1}$, and i is entrained to the periodic signal and $t_i^n - t_*^n \to 0$. □

If the periodic signals in lemma 1 originate from another oscillator k, (uniform delay τ) then $t_i^n - t_k^n \to \min\{\tau, \tau + \frac{\omega_* - \omega_i}{\alpha\omega_*\omega_i}\}$.

More generally, if i receives multiple signals from other oscillators, but processes only a single signal between firings, then either:

$$t_i^{n+1} = t_i^n + \frac{1}{\omega_i} + \alpha(\tau - \Delta t_i^n) \tag{3}$$

where $\Delta t_i^n = t_i^n - \min_{j \in N(i)} t_j^n$, or i gets excited to firing by some oscillator k, leading to $t_i^{n+1} = t_k^{n+1} + \tau$.

Next, consider a pair of oscillators (i, j) with initial phases within τ of each other.

Lemma 2. *For a system of two oscillators, $\{i, j\}$ with $\omega_i \leq \omega_j \leq \frac{\omega_i}{B}$, let $\Delta t^n = t_i^n - t_j^n$. If $|\Delta t^n| \leq \tau$, then Δt converges to $\min\{\tau, \frac{\omega_j - \omega_i}{2\alpha\omega_i\omega_j}\}$.*

Proof. Similar to the proof of lemma 1, either j excites i to firing giving $\Delta t^{n+1} = \tau$ or

$$t_i^{n+1} = t_i^n + \frac{1}{\omega_i} + \alpha(\tau - \Delta t_i^n)$$

$$t_j^{n+1} = t_j^n + \frac{1}{\omega_j} + \alpha(\tau + \Delta t_i^n)$$

and, $\Delta t^{n+1} = \Delta t(1 - 2\alpha) + \frac{\omega_j - \omega_i}{\omega_i\omega_j}$. These conditions on Δt^{n+1} give that the system converges to stable fixed point: $\min\{\tau, \frac{\omega_j - \omega_i}{2\alpha\omega_i\omega_j}\}$. □

Notice that Eq. (3) also gives that the average period for a pair (i, j) is $\min\{\frac{1}{2}(\frac{1}{\omega_i} + \frac{1}{\omega_j}) + \alpha\tau, \frac{1}{\omega_j} + 2\alpha\tau\}$.

As we will show, the simple behavior of a pair of oscillators and an oscillator subjected to periodic forcing can be aggregated to describe the behavior of the overall fast system with static, but heterogeneous frequencies.

B. Finite Cascading Regime

As we consider larger systems of oscillators our analysis will focus on an invariant regime of phase space. We define the finite cascading regime as the regime where the nth time each oscillator fires, it does so within τ of its neighbors, i.e. $|t_i^n - t_j^n| \leq \tau$ for all $(i, j) \in E$. Inside the finite cascading regime, an oscillator i must receive signals at times in $[t_i^n, t_i^n + 2\tau]$, and since the quiescent period is larger than τ, i can only process the first of these signals. Furthermore, i cannot receive another signal until $t_i^{n+1} \geq t_i^n + \frac{1}{2} - \tau$. Thus, the finite cascading regime breaks the continuous time system into discrete 'rounds of firing', during which each oscillator fires for the n'th time and processes exactly one incoming signal. Since for a graph with diameter d the differences in phases inside

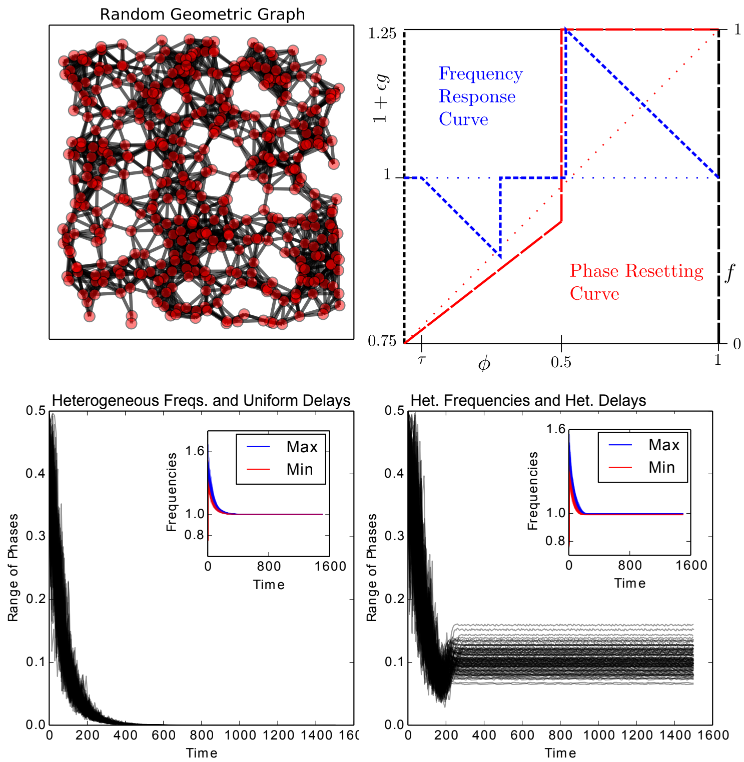

FIG. 1. (color online) Numerical simulations demonstrate robustness (bottom left) to large frequency response over 100 trials on a random geometric graph (top left) with phase resetting curves ($B = 0.5$) and frequency response curves (top right), $\tau = 0.05$, initial frequency range $[1, 1.2]$ and $q = 0.5$. The system remains robust (bottom right) to the addition of distance based delays $\tau_{ij} \in [0.5, 0.1]$.

the finite cascading regime are never more $d\tau$, if τ is very small then the finite cascading regime is very similar to synchrony. In the special case where $\tau = 0$, the following argument for entry into the finite cascading regime suffices as an argument for synchrony and the remaining theorems are unnecessary.

Moreover, when $\tau = 0$, each time an oscillator i fires, all of it's neighbors $j \in N(i)$ immediately receive it's signal and those with $\phi_j > B$ fire starting a cascade of firings where any oscillator k in that cascade of firings has $\phi_k = \phi_i = 0$. This logic can be extended, as in section section IV, to give a closed form lower bound on the probability that random initial conditions will enter the finite cascading regime for $\tau \geq 0$.

It is worth noting that on some large graphs there can exist infinite cascading regimes, where each oscillator fires within τ of their neighbors, but any attempted indexing of the firings would have some nth firings occurring within τ of a neighboring $(n+1)$th firing. For example, many such infinite cascading regimes exist on very large cycle graphs, and such firing patterns create a topologically different firing pattern. However, only the finite cascading regime is invariant on any connected undirected graph.

Lemma 3 (Invariance of the Finite Cascading Regime)**.**

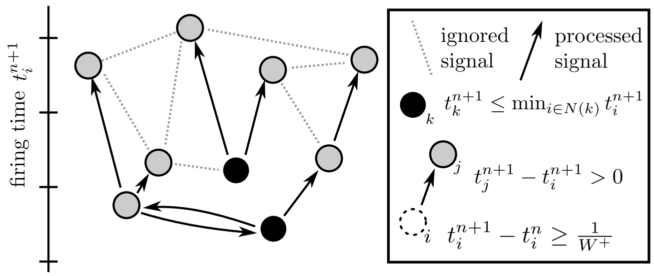

FIG. 2. The proof of lemma 3 involves an induction on nodes in an order consistent with the signal reception (black arrows). The result of the induction establishes that all nodes fit into two different classes, represented here as black and gray nodes.

If $\omega_i \in [W^-, W^+]$ for all i, $\tau < \frac{1}{2}$, $2\tau W^+ < B < \frac{W^-}{W^+}$ and $|t_i^n - t_j^n| \leq \tau$ for all $(i,j) \in E$ then $|t_i^{n+1} - t_j^{n+1}| \leq \tau$ for all $(i,j) \in E$.

Proof. Since $|t_i^n - t_j^n| \leq \tau$ for all $(i,j) \in E$ then any oscillator i receives its last signal associated with the nth set of firings before $t_i + 2\tau$. Since $2\tau\omega_i \leq 2\tau W^+ < B$ then these signals associated with the nth firings will not excite oscillators towards the $(n+1)$th firing.

Now, we will show two statements which will allow us to perform an induction on the graph structure, as depicted in Figure 2. First, consider any oscillator k for which $t_k^{n+1} \leq \min_{i\in N(k)} t_i^{n+1}$. Since k fires before its neighbors, and can't be excited by the nth round of firing then $t_k^{n+1} - t_k^n \geq \frac{1}{\omega_k} \geq \frac{1}{W^+}$.

Second, consider any oscillator j with neighbor $i = \text{argmin}_{p\in N(j)} t_p^{n+1}$ such that $t_j^{n+1} - t_i^{n+1} > 0$ and $t_i^{n+1} - t_i^n \geq \frac{1}{W^+}$. We will show that $t_j^{n+1} - t_j^n \leq \frac{1}{W^+}$.

Since in round n, j received a signal from i at time $t_i^n + \tau$, then it must be that $\phi_j((t_i^n + \tau)^+) \geq 0$, or else this or a previous signal would have excited j to firing. Thus, by $t_i^{n+1} + \tau$, either j will have already fired, or $\phi_j(t_i^{n+1} + \tau) \geq \omega_j(t_i^{n+1} - t_i^n)\frac{W^-}{W^+} > B$ and thus be excited to firing. In the former case, j fired of its own volition, giving that $t_j^{n+1} - t_j^n \geq \frac{1}{\omega_j} \geq \frac{1}{W^+}$ and $t_j^{n+1} - t_i^{n+1} \in [0,\tau)$, while in the later it fired τ time after i, giving that $t_j^{n+1} - t_i^{n+1} = \tau$ and $t_j^{n+1} - t_j^n \geq \frac{1}{\omega_i}$.

To establish the bounds for all nodes in the graph we now simply combine the previous two statements and perform induction. In particular, for each oscillator iteratively construct a sequence (or path) in the following way: for any oscillator j move to the oscillator $i = \text{argmin}_{p\in N(j)} t_p^{n+1}$ if $t_j^{n+1} - t_i^{n+1} > 0$ otherwise terminate at a local minimum. Notice that following this path in reverse allows for the immediate use of the previous two statements, the first at the local minimum, and the second at the remaining nodes.

This gives that for each node j either t_j^{n+1} is a local minimum or $t_j^{n+1} - \min_{i\in N(j)} t_i^{n+1} \leq \tau$, which, since all edges are undirected, is enough to conclude the desired result that $t_j^{n+1} - t_i^{n+1} \leq \tau$ for all $(i,j) \in E$. □

Notice, $|t_i^{n+1} - t_j^{n+1}| \leq \tau$ implies that for graph diameter d, $\max_{i\in V, j\in V} |t_i^{n+1} - t_j^{n+1}| < d\tau$. Since timing differences in each round are bounded, then in the finite cascading regime all oscillators have the same asymptotic frequency.

The remaining lemmas and theorems regard the behavior of the system inside the finite cascading regime.

C. Edge Based Analysis

In order to describe the structure of firing times inside the finite cascading regime, we will need to begin to look at the differences in oscillator frequencies. Indeed, the signature of heterogeneous oscillator frequencies is not in the asymptotic frequencies, but in the relative phase differences inside each round of firing. Ideally, it would be the case that the system dealt with these heterogeneous phases in an assortative way, where faster oscillators fire before slower ones: $t_i^n \leq t_j^n$ iff $\omega_i \geq \omega_j$. Interestingly, this is not the case.

Instead, we now show that pairs of oscillators can be well understood. First, denote the average time a pair of oscillators fires during the nth round as:

$$v_{i,j}^n = \frac{1}{2}(t_i^n + t_j^n) \tag{4}$$

for $(i,j) \in E$. Furthermore, for $(i,j) \in E$ define the innate pair period as $p_{i,j} = \min\{\frac{1}{2}(\frac{1}{\omega_i} + \frac{1}{\omega_j}) + \alpha\tau, \frac{1}{\omega_i} + 2\alpha\tau, \frac{1}{\omega_j} + 2\alpha\tau\}$ (notice: $p_{i,j}$ would be the average period of (i,j) if they were isolated). We will show that the order in which oscillators fire is assortative, not on the values of ω_i, but on the values $p_{i,j}$, where pairs with shorter innate $p_{i,j}$ tend to fire first.

D. The Role of Excitation

Excitation, though important for establishing the invariant cascading regime, is otherwise somewhat uninteresting for the resulting theorems. In particular, the situation where an oscillator is excited to firing, is indistinguishable from the situation where that same oscillator simply had a fast enough frequency to fire at the moment the first incoming signal arrived. Thus, rather than explicitly deal with excitation, we will instead analyze systems where in each round an oscillator i has firing-round specific frequency ω_i^n. When i isn't excited to firing $\omega_i^n = \omega_i$, but when i would be excited to firing, ω_i^n is set so that i fires at the exact moment the incoming excitation would have caused it to fire. Since replacing excitation with this contrived firing-round specific frequency preserves oscillators firing times, if one system converges so must the other. Before continuing though, we show that while in the finite cascading regime these contrived ω_i^n are restricted in their possible values, and in particular, can't be faster than the fastest oscillator.

Explicitly, if i were to be excited by oscillator k then ω_i^n must satisfy: $t_k^{n+1} + \tau = t_i^n + 1/\omega_i^n + \alpha(\tau - \Delta t_i^n)$. Similarly,

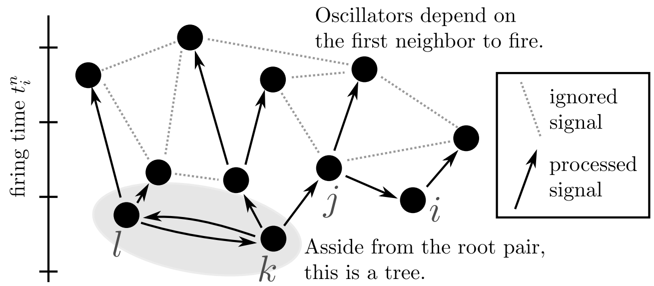

FIG. 3. The relationship between oscillator firing times and network topology reveals key structure. Since a oscillator processes only a single signal each round, a round of firing induces a tree-like set of dependencies, in this case with $l \to^n k \to^n j \to^n i$, and $k \to^n l$.

for oscillator k define induced period $p_k^n = t_k^{n+1} - t_k^n$, and notice:

$$\begin{aligned} 1/\omega_i^n &= t_k^{n+1} - t_i^n + \alpha(\Delta t_i^n) + (1-\alpha)\tau \\ &= (t_k^{n+1} - t_k^n) + (1-\alpha)(t_k^n - t_i^n) + (1-\alpha)\tau \\ &\geq t_k^{n+1} - t_k^n \\ &= p_k^n. \end{aligned}$$

With this reasonable bound on the new period $1/\omega_i^n$ we now show that we can continue to discuss comparisons of a round varying pair period $p_{i,j}^n$ in a meaningful way. First, let $p_{i,j}^n = \frac{1}{2}(\frac{1}{\omega_i^n} + \frac{1}{\omega_j^n}) + \alpha\tau$.

The following lemma shows that while oscillators can be excited to have faster periods, if the system obeys an assortativity property, oscillators can not have a faster pair period than the oscillator which excited them.

Lemma 4 (Frequency Exchange)**.** *For k that excites i to firing in the $(n+1)$th round, [and l that excites j], if $p_{i,j} \geq p_k^n$, [and $p_{i,j} \geq p_l^n$] then $p_{i,j}^n \geq p_k^n$, [$p_{i,j}^n \geq \min\{p_l^n, p_k^n\}$].*

Proof. First we consider the case that k excites i to firing and j is not excited to firing ($\omega_j^n = \omega_j$):

$p_{i,j}^n = \frac{1}{2}(\frac{1}{\omega_i{}^n} + \frac{1}{\omega_j^n}) + \alpha\tau$. Since, $p_{i,j} \geq p_k^n$ then by definition of $p_{i,j}$, $\frac{1}{\omega_j} + 2\alpha\tau \geq p_k^n$ giving:

$$\begin{aligned} p_{i,j}^n &= \frac{1}{2}\left(\frac{1}{\omega_i^n} + \frac{1}{\omega_j^n}\right) + \alpha\tau \\ &\geq \frac{1}{2}(p_k^n + p_k^n - 2\alpha\tau) + \alpha\tau \\ &= p_k^n \end{aligned}$$

When both i and j are excited to firing the situation is even simpler. In particular, $p_{i,j}^n = \frac{1}{2}(\frac{1}{\omega_i{}^n} + \frac{1}{\omega_j{}^n}) + \alpha\tau \geq \min p_l^n, p_k^n$. □

Thus, our new round dependent pair periods obey an analogous result to that for individual oscillators.

E. Causal Trees

We can now investigate the inner structure of causality inside a round of firing. Denote i processing a signal by j in the nth round of firings as $j \to^n i$. Notice, $j \to^n i$ is equivalent to $j \in \operatorname{argmin}_{k \in N(i)} t_k^n$ or $\Delta t_i^n = t_i^n - t_j^n$. The quiescent period gives that in a round of firing each oscillator responds to exactly one incoming signal, and thus each round of firings takes the full network of oscillators and creates a directed tree-like structure described by which oscillator depends on which other oscillator. Indeed, for each n decompose the graph G into a disjoint forest of directed tree-like structures $\{T_r^n\}$, where tree T_r^n with nodes i, j includes directed edge $j \to i$ if $j \to^n i$ (in instances where $j \to^n i$ and $k \to^n i$ because $t_k^n = t_j^n$, arbitrarily break the tie).

In these tree-like structures, loops are disallowed with one notable exception: the root of each tree must still process exactly one signal from one of its neighbors, and thus, as seen in Fig. 3, the root is effectively a pair of nodes. Denote the vertex set of an nth firing tree with root pair r as T_r^n. As we see in the next lemma, the period of a root pair is transparent.

Lemma 5. *For (i,j) a root pair in the nth round of firing: $v_{i,j}^{n+1} - v_{i,j}^n = p_{i,j}^n$.*

Proof. Since (i,j) is a root pair $i \to^n j$ and $j \to^n i$ implying that $\Delta t_i + \Delta t_j = 0$. Thus, via Eqn. 3,

$$t_i^{n+1} + t_j^{n+1} = t_i^n + t_j^n + 1/\omega_i^n + 1/\omega_j^n + \alpha(2\tau) \qquad (5)$$

which immediately gives the result.

□

Corollary 6. *For any $(i,j) \in E$, $v_{ij}^{n+1} - v_{ij}^n \leq p_{ij}$.*

Proof. Since for neighbors $\Delta t_i + \Delta t_j \geq 0$, a similar manipulation of Eqn. 3 in lemma 5 gives that $v_{ij}^{n+1} - v_{ij}^n \leq p_{ij}^n \leq p_{ij}$. □

We can similarly relate firing times to pair periods for edges in any tree T_r^n

Lemma 7. *For oscillators i, j and k, (see Fig. 3) with dependencies on the nth round of firing $l \to^n k$, $k \to^n j$, and $j \to^n i$ for some oscillator l (possibly $l = j$),*

$$v_{i,j}^{n+1} - v_{j,k}^{n+1} \geq p_{i,j}^n - p_{j,k}^n,$$

with equality only if $t_i^n = t_k^n$ and $t_j^n = t_l^n$.

Proof. Manipulating Eqn. 3 and Eqn. 4 yields that:

$$v_{i,j}^{n+1} = v_{i,j}^n + p_{i,j}^n - \frac{\alpha}{2}(\Delta t_i^n + \Delta t_j^n).$$

Since, $j \to^n i$ and $k \to^n j$ then $\delta t_i + \delta t_j = t_i^n - t_k^n$. Thus:

$$\begin{aligned} v_{i,j}^{n+1} &= v_{i,j}^n + p_{i,j}^n - \frac{\alpha}{2}(t_i^n - t_k^n) \\ v_{j,k}^{n+1} &= v_{j,k}^n + p_{j,k}^n - \frac{\alpha}{2}(t_j^n - t_l^n). \end{aligned}$$

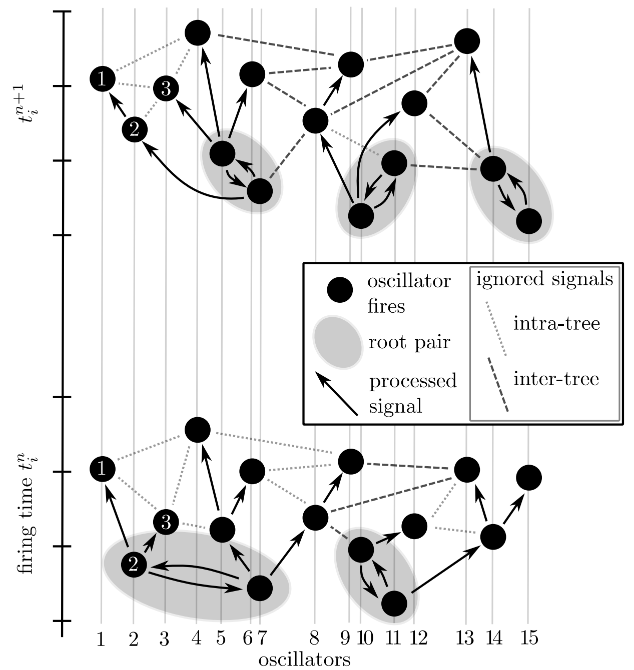

FIG. 4. From round to round firing trees may change. However, lemmas 8 and 9 establish bounds for pair differences across any of the three types of edges. For instance, in round n, lemma 9 can apply to edge $(8, 10)$ with dependencies on oscillators 2, 7, and 11, as well as edge (9,13) with dependencies on oscillators 7, 8, 11, and 14.

Subtracting these two, yields:

$$v_{i,j}^{n+1} - v_{j,k}^{n+1} = \frac{1}{2}((1-\alpha)(t_i^n - t_k^n) + \alpha(t_j^n - t_l^n)) + p_{i,j}^n - p_{j,k}^n. \tag{6}$$

Notice that since $i \to^n j \to^n k$, then $t_i^n - t_k^n \geq 0$ and since $j \to^n k \to^n l$ then $t_j^n - t_l^n \geq 0$ we are left with the desired inequality, and equality when $t_i^n = t_k^n$ and $t_j^n = t_l^n$.

□

Corollary 8. *For any two pairs of non-excited oscillators (k, l) with a path to (i, j) in T_r^n: $v_{i,j}^{n+1} - v_{k,l}^{n+1} \geq p_{i,j}^n - p_{k,l}^n$.*

Proof. Summing lemma 7 along the path from (k, l) to (i, j) telescopes to the desired equation. □

Similarly, we can now relate oscillators timings for a pair that bridge different branches of the same or different firing trees.

Lemma 9. *For oscillators i and j with $(i, j) \in E$, dependencies on the nth round of firing $k \to^n j$, $l \to^n k$, $p \to^n i$, and $q \to^n p$ for some oscillators l, k, q and p (possibly $l = j$, $q = k$, $l = p$ and/or $k = i$),*

$$(v_{i,j}^{n+1} - v_{j,k}^{n+1}) + (v_{i,j}^{n+1} - v_{i,p}^{n+1}) \geq (p_{i,j}^n - p_{j,k}^n) + (p_{i,j}^n - p_{i,p}^n),$$

with equality only if $t_i^n = t_k^n$ and $t_j^n = t_l^n$.

Proof. This proceeds similar to the proof of lemma 7. Notice that:

$$\begin{aligned} t_i^{n+1} &= t_i^n + 1/\omega_i^n + \alpha(\tau - t_i^n + t_p^n) \\ t_j^{n+1} &= t_j^n + 1/\omega_j^n + \alpha(\tau - t_j^n + t_k^n) \\ t_k^{n+1} &= t_k^n + 1/\omega_k^n + \alpha(\tau - t_k^n + t_l^n) \\ t_p^{n+1} &= t_p^n + 1/\omega_p^n + \alpha(\tau - t_p^n + t_q^n) \end{aligned}$$

Thus,

$$\begin{aligned} 2v_{i,j}^{n+1} - v_{j,k}^{n+1} - v_{i,p}^{n+1} \geq 2{p^n}_{i,j} - {p^n}_{j,k} - {p^n}_{i,p} + \alpha[(t_i^n - t_q^n) \\ + (t_j^n - t_l^n)] + (1-\alpha)[(t_i^n - t_k^n) + (t_j^n - t_p^n)]. \end{aligned}$$

Notice that the firing dependencies imply that $t_i^n - t_q^n \geq 0$, $t_j^n - t_l^n \geq 0$, $t_i^n - t_k^n \geq 0$, and $t_j^n - t_p^n \geq 0$, giving the desired result; with equality only if these timing differences are zero as well. □

With these preliminaries we now proceed to show that eventually the fastest pair of oscillators is the first to fire. Denote the $v_-^n = \min_{e \in E} v_e^n$ and $p_- = \min_{e \in E} p_e$.

Lemma 10. *There exists N_0 such that for all $n > N_0$, $v_{i,j}^n = v_-^n$ implies that $p_{i,j} = p_-$ and $v_{i,j}^{n+1} = v_-^{n+1}$*

Proof. Combining corollary 8 with lemma 9 reveals that for neighbors i and j, with $i \in T_r$ and $j \in T_s$:

$$(v_{i,j}^{n+1} - v_r^{n+1}) + (v_{i,j}^{n+1} - v_s^{n+1}) \geq ({p^n}_{i,j} - {p^n}_r) + ({p^n}_{i,j} - {p^n}_s). \tag{7}$$

Eqn. 7 gives that

$$2v_{i,j}^{n+1} \geq 2p_{i,j}^n + v_r^{n+1} - p_r^n + v_s^{n+1} - p_s^n.$$

Since both r and s are root pairs in round n, then by lemma 5, $v_r^{n+1} - v_r^n = p_r$ and $v_s^{n+1} - v_s^n = p_s$. Thus,

$$2v_{i,j}^{n+1} \geq 2p_{i,j}^n + v_r^n + v_s^n.$$

This gives that $v_{i,j}^{n+1} - v_-^n \geq p_{i,j}^n$.

Since this holds for any pair, it holds for all (l, k) such that $v_{l,k}^{n+1} = v_-^{n+1}$. Being the first to fire in round $n+1$ implies that (l, k) are roots in $n+1$, and thus they cannot have been excited to firing from neighbors, giving that $\omega_l^n = \omega_l$ and $\omega_k^n = \omega_k$, so $p_{l,k}^n = p_{l,k}$. Thus, $v_-^{n+1} - v_-^n \geq p_-$ and the system is prevented from firing faster than p_-.

Since by lemma 6 any fastest pair c, $p_c = p_-$ has $v_c^{n+1} - v_c^n \leq p_c$, then the long term average period must be less than or equal to p_- giving that there exists N_1 such that eventually $v_-^{n+1} - v_-^n \leq p_c$ for $n \geq N_1$.

Finally, now consider some i, j such that $v_{i,j}^n = v_-^n$ for some $n \geq N_1$. If $p_{i,j} = p_-$ then, $v_{i,j}^{n+1} \leq v_{i,j}^n + p_- = v_- = v_-^n + p_- = v_-^{n+1}$ and thus $v_{i,j}^{n+1} = v_-^{n+1}$. Moreover, for any i, j to have $v_{i,j}^{n+1} = v_-^{n+1}$ would require i, j to be roots in round $n+1$ and thus, $p_{i,j}^n = p_{i,j} = p_-$. Thus, for all $n \geq N_1 + 1 = N_0$ the desired claim must hold.

□

Assortativity on pair periods allows an understanding of the phase locking fixed point inside the finite cascading regime. While actual convergence occurs much faster than in the following argument, we can show that eventually it must come to be that the system converges to fixed phase differences.

Theorem 11 (Convergence of Phase Differences)**.** *For fixed frequencies ω, every oscillator i has some neighbor j such that the difference in oscillator firing times converges to: τ if $p_i \geq p_- + \tau$ and otherwise one of $\tau - \frac{p_j - p_-}{\alpha}$ or $\frac{\omega_j - \omega_i}{2\alpha\omega_i\omega_j}$ depending on oscillator frequencies.*

Proof. By lemma 10, there exists N_0 such that for all $n > N_0$, $v^n_{i,j} = v^n_-$ implies that $p_{i,j} = p_-$ and $v^{n+1}_{i,j} = v^{n+1}_-$, and $v^{n+1}_{i,j} = v^{n+1}_-$ implies that $v^n_{i,j} = v^n_-$ and $p_{i,j} = p_-$.

Our argument will proceed iteratively, beginning at the first pairs to fire, and sequentially determining oscillator timings outward. At any stage in the argument, let R be the set oscillators whose timings have been determined, and let δR and $\mathrm{int}(R)$ represent the boundary and interior of R in the network, respectively. We will further require that for all $i \in \mathrm{int}(R)$, $t^n_i \leq t^n_j$ for all $j \in \delta R$.

Take any i, j with $v^n_{i,j} = v^n_-$, with $t^n_i \leq t^n_j$. Since $v^n_{i,j} \leq v_{k,l}$ for all $k \in N(i)$ and $l \in N(k)$, then $t^n_i \leq t^n_l$, implying that $i \to^n k$ and all $k \in N(i)$ process the signal from i. Further, since i, j will be a root for all rounds past N, then it will necessarily be that $i \to^n k$ for all $n > N$. Notice, this gives that i, j only process signals from each other, and thus by lemma 2, i, j converge to a fixed phase difference, and each converges to $t^{n+1}_i - t^n_i = p_{i,j} = p_-$. As i, j converge, i begins to provide a periodic signal to it's neighbors, and thus by lemma 1, each oscillator $k \in N(i)$ converges to a fixed phase difference with i, and fires every p_- time. In this way, we have established that some oscillator i and all of its neighbors will converge to set of fixed phase differences, and will fire every p_- time. We now consider i and its neighbors to be in R and note that i is necessarily in the interior of R and the interior oscillators fire before the boundary.

From the perspective of the rest of the network, as the oscillators in R converge to their established limits, there are ever decreasing effects to changing the properties of oscillators in R provided the oscillators in δR maintain the same limiting firing times. Specifically, if each of $i \in \mathrm{int}(R)$ and $k \in N(i)$ are replaced by new oscillators $\hat{i}$ and $\hat{k}$, where $\hat{i}$ is slightly slower than i then the following changes maintain the same limiting firing times for each of $k \in N(i)$.

$$t^n_{\hat{i}} = t^n_i + \frac{1}{\alpha}\left(\frac{1}{\omega_{\hat{i}}} - \frac{1}{\omega_i}\right)$$
$$\frac{1}{\omega_{\hat{k}}} = \frac{1}{\omega^n_k} + \frac{1}{\omega_i} - \frac{1}{\omega_{\hat{i}}}$$

The iterative argument now proceeds as follows.

Consider the continuous family of systems, created by taking each $i \in \mathrm{argmin}_{j\in \mathrm{int}(R)} t^n_j$ and replacing each i with an ever slower $\hat{i}$ in the method described above (consequently causing $t^n_{\hat{i}}$ to increase). Since this family of systems maintains firing times on δR, the remaining portion of the graph must evolve the same for all members of this family. Thus, as these $\hat{i}$ become ever slower, it's possible that some system has a pair $p, q \notin R$, $t^n_p \leq t^n_q$, with $v^n_{p,q} = v^n_-$. Alternatively, some system has slowed the $\hat{i}$ to the point where there exists $p \in \delta R$ such that $t^n_p = t^n_{\hat{i}}$. In either case, in this system there now exists an oscillator p, such that all of the neighbors of p must process its signal now and by lemma 10, for all future rounds. Following the earlier argument, we now add p and the neighbors of p to R. In this way R increases until it contains all nodes, giving that eventually all nodes must converge to phase differences determined above. □

Notice that the above argument determines a unique set of phase differences when the fastest pair is unique. When there are multiple pairs $r_1, r_2 ...$ such that $p_{r_i} = p_-$, there exist many possible arrangements of G into firing trees T_{r_i} with corresponding stable phase differences. However, in the simplest case, theorem 11 leads immediately to:

Corollary 12. *If oscillator frequencies are homogeneous, a system in the finite cascading regime converges to exact synchrony.*

Proof. Homogeneous frequencies give that oscillators converge to fixed phase differences that do not involve excitation, and thus by theorem 11, all phases differences are: $\frac{\omega_j - \omega_i}{2\alpha\omega_i\omega_j} = 0$, which is exact synchrony. □

In the next section we consider how with the above fixed phase differences lead frequencies to converge.

III. SLOW CONVERGENCE IN FREQUENCY

Next, we consider the slow time behavior of the system, when the system has phases as described by theorem 11 and frequencies change via the frequency response curve g.

Theorem 13 (Slow Convergence of Frequencies)**.** *Suppose initially, $\omega_i \in (1, M)$, and at the reception of a pulse $\omega_i(t^+) = \omega_i(t^-)(1 + \epsilon g(\phi_i(t^-))$, then in the singular limit $\epsilon \to 0$ the slow subsystem converges to $\omega_i = 1$ for all i.*

Proof. We show this by looking at each of the three separate cases. In the first case consider the oscillator pair at the root of their firing tree. As shown in theorem 11, these oscillators will have a phase difference of $t^n_i - t^n_j = \frac{\omega_j - \omega_i}{\alpha\omega_i\omega_j}$. Without loss of generality, suppose $\omega_j \geq \omega_i$. This fixed difference in firing times, gives that

j receives a signal from i when,

$$\phi_j(t_i^n+\tau) = \omega_j\left(\tau + \frac{\omega_j-\omega_i}{\alpha\omega_i\omega_j}\right) = \omega_j\tau + \frac{\omega_j-\omega_i}{\alpha\omega_i},$$

giving that $\phi_j(t_i^n+\tau) \geq \tau$, since $\omega_j \geq 1$. Since the oscillators are in the cascading regime, the latest j can receive a signal is when $\phi_j(t_j^n+2\tau) = 2\omega_j\tau \leq 2M\tau$ Since the frequency response curve reduces frequencies for $\phi \in [\tau, M\tau]$ it's the case that at the stable phase fixed point the very first oscillator to fire will have it's frequency decreased, unless $\omega_j = \omega_i = 1$. Similarly

As i and j are the fastest pair the only possible frequency fixed point in $\Omega \in [1, M]$ is $\Omega = \vec{1}$.

Next we show that the remaining oscillators cannot have their frequencies slowed below 1 or accelerated past the root node .

Now consider an oscillator i in a forced pair. At the stable phase fixed point it receives a signal at:

$$\phi_i(\hat{t}^n+\tau) = \frac{\omega_i-\hat{\omega}}{\alpha\hat{\omega}}$$

Because i is in the cascading regime, it must be the case that this is less than $2\omega_i\tau$. We are then interested in when this phase causes i to have a slower or an accelerated frequency. Notice, that for an oscillator to be slowed, it must be the case that $\phi_i(\hat{t}^n+\tau) > \tau$ which implies that $\omega_i > \hat{\omega}(1+\alpha\tau)$. As $\frac{1}{\hat{\omega}} = \frac{1}{2}(\frac{1}{\omega_j}+\frac{1}{\omega_k}) + \alpha\tau \leq 1+\alpha\tau$, with equality only when all oscillators have frequency 1, then for i to be slowed it must be that $\omega_i > 1$. As $\epsilon \to 0$ this gives that no oscillator's frequency can be slowed below 1.

Conversely, in order for an oscillator to be accelerated it must be i had not fired before the forcing signal arrives, giving that $\omega_i \leq \hat{\omega}$. Since $\hat{\omega}$ is slower than the root node, oscillator i can not accelerate past the root, and cannot be slowed below 1.

Since $\hat{\omega}$ must gradually decrease in step with the slowing root pair, and $\Omega \in [1, M] \to [1, M]$ it must be that the system converges to the only fixed point, that of $\Omega = \vec{1}$

□

IV. PROBABILITY OF CONVERGENCE

This raises the natural question: what is the probability random initial conditions put the system in the finite cascading regime? A straightforward application of Techniques in [8] yields the following lower bound on the probability that the system synchronizes. Namely: for uniform random initial phases and $\frac{\alpha B}{\omega_+} > 2\tau$ there exists some fixed $q < 1$ such that the probability a system on network G with degree sequence d_i converges to the finite cascading regime and therein synchronizes is:

$$P_{sync}(G) \geq 1 - \Sigma_i^n q^{d_i+1}. \quad (8)$$

Such bounds are known to produce surprisingly strong results [8]. For example, such a bound gives a lower bound for a phase transition (from no synchrony to synchrony) on an Erdős-Rényi random graph at only a constant multiple of the critical parameter for percolation. Similar bounds exists for random geometric graphs and in fact any random graph model which produces predictable degree distributions. Moreover, this bound demonstrates that for these phase-frequency oscillators, designing a network to synchronize can be simply reduced to increasing the minimum degree in the graph.

V. NUMERICAL SIMULATION AND ENGINEERING MODIFICATIONS

While Eq. (8) is extremely effective in bounding the convergence probabilities of large random graphs, Fig. 1 displays numerical results, showing that the system converges to exact synchrony reliably for intermediate sized systems, and that the system is robust to the inclusion of heterogeneous delays as well as fast frequency response. In particular, notice that the frequency response curve in Fig. 1 differs significantly from a constant function with value 1, implying that in some situations the separation of time scales is less important. However, as the network size increases, and/or for networks which take longer to converge to fixed phase differences (such as lattices), the separation between time scales may become more important, and convergence may require smaller ϵ.

In comparison to many oscillator studies, the above numerical simulations already place these oscillators in a particularly difficult situation for synchronization, but they still include oscillators which suffer no error in their frequencies. Consider now a scenario where each time an oscillator fires it's frequency is slightly perturbed, so that $\omega_i(t_n^+) = (1+\varepsilon)\omega_i(t_n^-)$ where ε is a random variable drawn from a uniform distribution $\mathcal{U}(-a, a)$. Without frequency adjustment, frequency drift would cause the system's frequencies to diverge and thus preclude synchrony, or any approximately synchronous behavior. When a and the associated errors are relatively small the system first approaches synchrony but is unable to correct for small errors that decrease the minimum frequency, as in Fig. 5 (top left). Interestingly, when the random frequency errors are large enough, as in Fig. 5 (bottom left), the freq perturbations disrupt the structure in phases significantly, which has the perhaps unexpected effect of stabilizing a near synchronous solution. While understanding these phenomena is a worthwhile goal for later investigation, unconstrained drift of oscillator frequencies is a strong and somewhat unnatural assumption.

Indeed, a more reasonable modeling assumption is that oscillator frequencies undergo random drift about some innate frequency to which they regularly return. For instance, consider a system where upon firing an oscillator updates its frequency according, $\omega_i(t_n^+) = (1-\eta+\varepsilon)\omega_i(t_n^-) + \eta\omega_i(0)$, where parameter $\eta \in [0, 1)$ determines

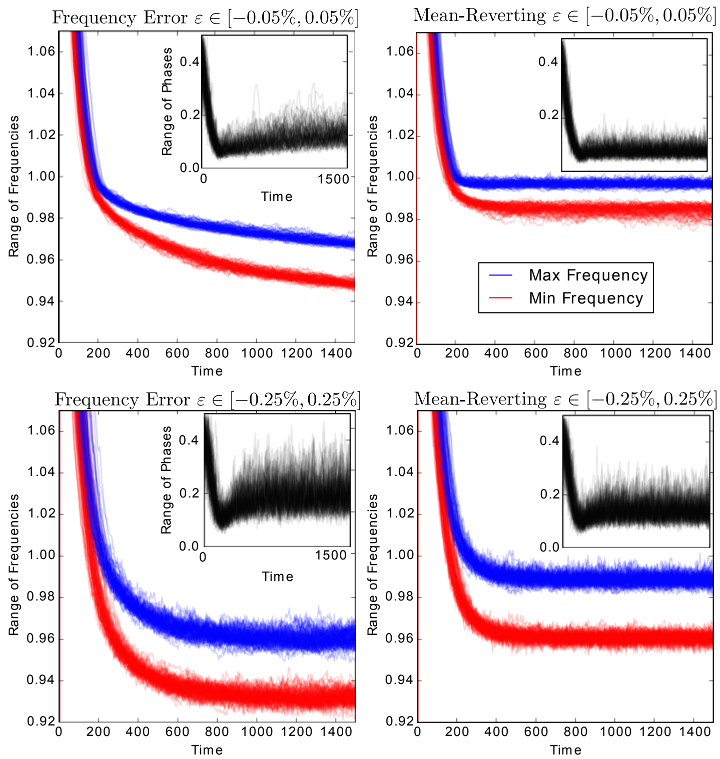

FIG. 5. (color online) Under the same parameters as in Fig. 1, but when oscillators are additionally subjected to random errors in frequencies, the system is still able to approximate synchrony (results from 100 simulations shown). For small error rates, the system can display a troubling continual decrease in frequency (top left), whereas for larger error rates (bottom) and for errors which are mean-reverting with $\eta = 0.1\%$ (right), the oscillator frequency range stabilizes.

the extent to which the oscillators are biased towards returning to their innate frequency. In such a setting, reasonable choices of η provide significant robustness, though at the cost of exact synchrony, Fig. 5.

VI. CONCLUSIONS

In this paper have shown that a particular system of pulse coupled oscillators with both phase and frequency response is able to synchronize even when there are delays, heterogeneous frequencies and a complex network topology, a task impossible for oscillators limited to modifying only their phases. The analysis builds upon a separation of time scales and an understanding of small sets of oscillators so as to describe an arbitrary connected undirected network as a forest of tree like structures. However, while the separation of time scales is vital to the analytic proof, numerical simulations suggest that the system's behavior isn't sensitive to changes in the relative time scales, suggesting stronger statements about synchrony may be possible.

In order to modify our results for implementation in engineered systems, a number of details will need to be addressed and some of the assumptions useful in the above analysis may need to be relaxed. For instance, the form of frequency updating allowed for exact synchrony, but also leads some networks and initial conditions to grow oscillators frequencies without bound. Instead, modifying the oscillators so that their frequencies drift towards their original innate frequency would preclude exact synchrony, but could provide an overall more robust system, and would further have the benefit of better addressing random errors in oscillator frequency. Additionally, in situations where oscillator to oscillator delays can be estimated, each oscillator could update their frequency response curves so that $g_{ij}(x) = 0$ for $x <= \tau_{ij}$.

The decentralized coordination of timing is a fundamentally and perhaps surprisingly, hard problem. Previous oscillator synchronization protocols can give almost global convergence properties only when frequencies are homogeneous, and exact synchrony isn't even a solution for these other protocols. In this light, our probabilistic lower bound on convergence that goes to 1 for large graphs with only slightly more edges than $n \ln(n)$ is arguable comparable to an analytic global result fragile to heterogeneous oscillators. Indeed, it is hoped that this work will improve the state of the art in decentralized synchronization and spur future inquiry into more robust pulse coupled oscillators. In particular, augmenting traditional coupled oscillators with frequency adaptation will almost certainly aid attempts to design pulse coupled oscillators capable of synchronizing networks of oscillators with heterogeneous delays, and heterogeneous frequencies.

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