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Critical Slowing Down in Networks Generating Temporal Complexity

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We study a nonlinear Langevin equation describing the dynamic variable $X(t)$, the mean field (order parameter) of a finite size complex network at criticality. The conditions under which the autocorrelation function of X shows any direct connection with criticality are discussed. We find that if the network is prepared in a state far from equilibrium, $X(0) = 1$, the autocorrelation function is characterized by evident signs of critical slowing down as well as by significant aging effects, while the preparation $X(0) = 0$ does not generate evident signs of criticality on $X(t)$, in spite of the fact that the same initial state makes the fluctuating variable $\eta(t) \equiv \mathrm{sgn}(X(t))$ yield significant aging effects. These latter effects arise because the dynamics of $\eta(t)$ are directly dependent on crucial events, namely the re-crossings of the origin, which undergo a significant aging process with the preparation $X(0) = 0$. The time scale dominated by temporal complexity, aging and ergodicity breakdown of $\eta(t)$ is properly evaluated by adopting the method of stochastic linearization which is used to explain the exponential-like behavior of the equilibrium autocorrelation function of $X(t)$.

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I. INTRODUCTION

Critical phenomena are of increasing interest in a broad range of disciplines ranging from the traditional field of physical phase transitions [1] to brain dynamics [2], epidemics [3, 4], internet traffic [5], climate change [6], perception theory [7] and opinion formation [8]. One measure of criticality is *critical slowing down*, namely, the divergence of the relaxation time in the neighborhood of the critical point [9, 10] and is a well known property of physical phase transitions. To date much less attention has been devoted to *temporal complexity* [11] although this, as explained herein, is an important manifestation of the realistic condition of a phase transition occurring very far from the ideal condition of the thermodynamic limit.

The Decision Making Model (DMM) [11] of complex dynamic networks generates an Ising-like phase transition and a mean field $X(t)$ that fluctuates around the origin, $X = 0$. In the case where the cooperating units are the nodes of a regular two-dimensional network the intensity of cooperation necessary to generate this phase transition is higher than in the case where each unit interacts with all the other units. This latter condition is denoted as all-to-all (ATA) coupling. In the case of a two-dimensional regular lattice, each unit is coupled only to four nearest neighbors. However, in both the lattice and ATA cases, the mean field fluctuates around the origin as a consequence of the finite number of units, N.

At criticality the distribution of the time intervals between consecutive re-crossings of the origin is described by an inverse power law, with index $\mu = 1.5$. The scale-free nature of the distribution of time intervals suggested the name *temporal complexity* [11] in analogy to topological complexity used to describe the scale-free nature of the connectivity of the network. Moreover, criticality was demonstrated to be the most efficient way to transport information from one unit to the other units in such a network [12]. This form of information transport is especially important, for example to explain the collective intelligence of a flock of birds postulated by Couzin [13]. Most of the birds in a flock are blind to the world outside the flock, while a few, the lookout birds, are sensitive to the environment and are therefore able to perceive the arrival of a predator. It is this arrival that constitutes the information transmitted to the other birds in the flock.

Although the transmission of information from the lookout birds to the blind ones suggests the action of an information wave, recent work [14] has shown that this misleading impression is the consequence of ignoring the role of crucial events. In the specific case of a flock, the relevant variable is not the mean field X, but it is rather $\eta(t) \equiv \mathrm{sgn}(X(t))$, which corresponds, so to speak, to the decision of flying to the right, $\eta(t) = 1$, or to the left, $\eta(1) = -1$. Therefore the re-crossing of the origin is the crucial event, called the "free will" condition [14], that makes it possible for the small bias exerted by a few lookout birds to direct the flock either to maintain or to change direction.

A. Equations of interest

A central purpose of this paper is to establish a clear link between critical slowing down and temporal complexity. We note that a nonlinear Langevin-like equation of motion for the network dynamics expressed in terms of the mean field can be written, the form of which depends on whether the control parameter of the DMM is

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below, at or above criticality. The Langevin equation for the mean field away from the criticality condition can be expanded and to lowest order in the mean field is given by the ordinary linear Langevin equation [16]

$$\dot{X}(t) = -\gamma X(t) + f(t), \tag{1}$$

where $X(t)$ is either the time-dependent mean field in the subcritical condition, or the departure of the mean field from the equilibrium value in the supercritical condition. At criticality the linear term in the expansion of the equation of motion vanishes identically and the ordinary linear Langevin equation of Eq. (1) is replaced by the nonlinear Langevin equation

$$\dot{X}(t) = -\gamma X(t)^3 + f(t). \tag{2}$$

In both Eq. (1) and Eq. (2) the symbol f denotes a white noise of intensity proportional to $1/\sqrt{N}$ that in the absence of friction would generate free diffusion with diffusion coefficient

$$D \propto \frac{1}{N}. \tag{3}$$

In the case where the intensity of the random noise vanishes, the time evolution of $X(t)$, moving from the initial out-of-equilibrium condition $X(0) \neq 0$, is

$$X(t) = \frac{X(0)}{\sqrt{1 + 2\gamma X(0)^2 t}}, \tag{4}$$

namely, a relaxation process with an infinitely long lifetime, which is a typical manifestation of critical slowing down [9, 10]. In both the supercritical and subcritical case the regression to equilibrium is, on the contrary, characterized by exponential decay with decay rate γ. In conclusion we see that critical slowing down is a property of the network in the thermodynamic limit, where the stochastic force $f(t)$ vanishes.

Temporal complexity [11], as mentioned previously, is a property of the single trajectories of $X(t)$ of Eq. (2) in the presence of the stochastic force $f(t)$. The time interval τ between two consecutive re-crossings of the origin $X = 0$ was shown [11] to be described by the waiting-time distribution density $\psi(\tau)$

$$\psi(\tau) \propto \frac{1}{\tau^{\mu}}, \tag{5}$$

with $\mu = 1.5$ for times $\tau < T_{eq}$. The equilibrium time, T_{eq}, is given by [16]

$$T_{eq} \approx \frac{1}{\sqrt{\gamma D}}, \tag{6}$$

and $\psi(\tau)$ decays exponentially with the lifetime T_{eq} for times $\tau > T_{eq}$. For times $t < T_{eq}$ the dichotomous representation $\eta(t) = \mathrm{sgn}(X(t))$ shares the ergodicity breakdown of blinking quantum dots [15] and the authors of [16] conjecture that a connection must exist between the autocorrelation function of the fluctuating variable $X(t)$ and the temporary ergodicity breakdown of $\eta(t)$. Here we notice that the time region $t < T_{eq}$ can be made arbitrarily long by decreasing the noise intensity D. The cubic structure of Eq. (2) affords another way to explain the extended time regime of temporal complexity. We shall discuss this in more details in Section IV. Here we limit ourselves to noticing that the fluctuations f generate small values of X that become even weaker, by raising them to the third power, thereby producing a condition of classical diffusion.

It is important to notice that the equilibrium autocorrelation function of $X(t)$ has an exponential structure therefore conveying the impression that the fluctuations $f(t)$ may completely erase any memory of critical slowing down. This article sheds light onto the connection between critical slowing down and temporal complexity by: (i) revisiting the stochastic linearization assumption adopted in Ref. [16], and (ii) addressing the problem of analytically evaluating the non-stationary autocorrelation function

$$\Phi_x(\tau, t_a) = \frac{\langle x(t_a + \tau) x(t_a) \rangle}{\langle x(t_a)^2 \rangle}, \tag{7}$$

with t_a being the aging time. The autocorrelation function for the the preparation $X(0) = 1$ will be compared with that for the preparation $X(0) = 0$. Note that $\langle ... \rangle$ denotes averaging over an ensemble of realizations of the network dynamics. We shall see that the out-of-equilibrium preparation establishes a clear connection with the critical slowing down of Eq. (4), thereby entailing the study of the extended time of transition to equilibrium, namely, the extended time region responsible for the synchronization between two complex networks [14]. The preparation condition responsible for synchronization, and consequently for temporal complexity, is $X(0) = 0$, which makes complexity virtually invisible, in the sense that it provides no apparent connection with critical slowing down if it is monitored through the autocorrelation function of $X(t)$.

B. Critical slowing down and temporal complexity

In an interesting recent paper [17] devoted to illustrate the importance of criticality for biological processes, Mora and Bialek wrote: "Another and possibly adverse consequence of criticality is the phenomenon of critical slowing down-the system takes more and more time to relax as it approaches its critical point. For birds, this may seem like a very detrimental effect." To explain how critical slowing down may become compatible with the biological system to promptly react to an external stimulus, we have to understand the difference between critical slowing down and temporal complexity. In the subcritical regime where the equilibrium state corresponds to the vanishing value $X = 0$, if the variable X is assigned an initial value slightly departing from equilibrium, $X(0) \neq 0$, its regression to equilibrium is obtained by expressing its time derivative in the form

$$\dot{X} = -\sum_{k=1}^{\infty} \gamma_k X^k. \tag{8}$$

At criticality, the coefficient of the linear term vanishes. As a consequence, Eq. (8) becomes

$$\dot{X} = -\gamma_r X^r, \tag{9}$$

where γ_r is the lowest non-vanishing expansion coefficient, with

$$r > 1. \tag{10}$$

The higher order contributions can be neglected as a consequence of the assumption that $X(0)$ is very close to equilibrium. The solution of Eq. (9) is

$$X(t) = \frac{X(0)}{[1 + \gamma_r(r-1)tX(0)^{r-1}]^{\frac{1}{(r-1)}}}. \tag{11}$$

We see that the fastest regression to equilibrium is realized when $r = 2$. In a cooperative case that is outside the scope of this article, this fastest condition applies. Yet, in this case as well the relaxation time is infinite. In the specific case of DMM, $r = 3$, thereby making Eq. (11) identical to Eq. (4), with $\gamma = \gamma_3$, and the regression to equilibrium proportional to $1/t^{0.5}$. This is the important phenomenon of critical slowing down that the authors of Ref. [17] consider to be potentially detrimental for the free will of a swarm of birds.

Temporal complexity is realized when $N < \infty$. In this case, as explained in Ref. [16], the regression to equilibrium of $X(t)$ is affected by internal fluctuations and Eq. (8), as previously mentioned, is corrected by the addition on its right hand side of a random fluctuation $f(t)$ of intensity proportional to $1/\sqrt{N}$. In the subcritical regime, the stochastic force $f(t)$ must compete with the linear term $-\gamma_1 X(t)$. For slight deviations from equilibrium the process would be described by Eq. (1), with $\gamma = \gamma_1$. Let us rescale time so as to replace γ with 1. If we assign to $X(t)$ values of the order of $f(t)$, the friction term $-\gamma X(t)$ is of the order of $f(t)$, and the free-diffusion regime does not emerge. Thus, the time distance between two consecutive crossings of $X(0) = 0$ is described by an exponential waiting time distribution $\psi(t)$.

Criticality has the important effect of making a diffusional regime emerge and persist for an extended time. In the specific case of DMM, when Eq. (2) applies, the friction containment of the order of f^3 can be neglected compared to f and for an extended time of the order of T_{eq} of Eq. (6) the dynamics of the fluctuating variable $X(t)$ are virtually generated by free diffusion and the waiting-time distribution density achieves the inverse power law structure of Eq. (5) with $\mu = 1.5$.

It is important to stress that there may be asymmetric cases where the mean field $X(t)$ is not allowed to be negative, the fluctuation $f(t)$ may have an external as well as an internal origin, where this analysis cannot be applied, and μ can turn out to be different from 1.5 (see for example the case of Ref. [12] where $\mu \approx 1.35$). The discussion of these cases is outside the scope of this article. Herein, we limit our discussion to the temporal complexity emerging from Eq. (2). Temporal complexity requires a non vanishing fluctuation $f(t)$, and consequently $N < \infty$, but this condition generates both temporal complexity with $\mu = 1.5$ with the associate renewal aging process and eventually an ordinary equilibrium process. Eq. (6) with $D \propto 1/N$ implies T_{eq} proportional to $N^{1/2}$. Consequently the diffusional regime can be made as extended as we wish, but, of course the fluctuation intensity responsible for temporal complexity becomes weaker and weaker.

C. A look ahead

In Section II we refine the key approximation of stochastic linearization. The connection between critical slowing down and temporal complexity is studied in Section III with the help of an analytical expression interpolating the short-time regime, where critical slowing down emerges, with the long-time regime where stochastic linearization satisfactorily applies. We devote Section IV to the comparison between theoretical and numerical results. Finally, in Section V we compare the aging process of the non-stationary correlation function of Eq. (7) to the renewal aging associated to temporal complexity [14], and we make some concluding remarks.

II. STOCHASTIC LINEARIZATION

The autocorrelation function of the mean field involves an average over an ensemble of realizations of trajectories generated by the cubic Langevin equation. Equivalently, the average can be calculated using the probability density function (PDF) with $P(x,t)dx$ the probability that the dynamic variable $X(t)$ lies in the phase space interval $(x, x+dx)$ at time t. The PDF is obtained by solving the Fokker-Planck equation (FPE) associated with Eq. (2):

$$\frac{\partial}{\partial t} P(x,t) = \mathcal{L}_{FP} P(x,t), \tag{12}$$

where the Fokker-Planck operator $\mathcal{L}_{FP}$ is defined by

$$\mathcal{L}_{FP} \equiv \left(\gamma \frac{\partial}{\partial x} x^3 + D \frac{\partial^2}{\partial x^2} \right). \tag{13}$$

The equilibrium PDF is given by

$$P_{eq}(x) = \frac{e^{-\frac{\gamma x^4}{4D}}}{\int_{-\infty}^{\infty} e^{-\frac{\gamma x^4}{4D}} dx}. \tag{14}$$

To prepare the ground for the evaluation of the non-equilibrium autocorrelation function of Eq. (7) it is convenient to evaluate the equilibrium second moment

$$\left\langle x^2 \right\rangle_{eq} = \frac{\int_{-\infty}^{\infty} x^2 e^{-\frac{\gamma x^4}{4D}} dx}{\int_{-\infty}^{\infty} e^{-\frac{\gamma x^4}{4D}} dx}. \quad (15)$$

To establish the dependence of the second moment on γ and D it is convenient to use the dimensionless integration variable

$$z = \left(\frac{\gamma}{4D}\right)^{1/4} x, \quad (16)$$

which, after some algebra yields

$$\left\langle x^2 \right\rangle_{eq} = \alpha \left(\frac{D}{\gamma}\right)^{1/2}. \quad (17)$$

The new parameter α is expected to be a number on the order of unity and is obtained by direct integration to be

$$\alpha = \frac{2 \int_{-\infty}^{+\infty} dz z^2 e^{-z^4}}{\int_{-\infty}^{+\infty} dz e^{-z^4}} \approx 0.69. \quad (18)$$

This new parameter provides the information necessary to determine the magnitude of Γ.

It is important to notice that due to the cubic nature of the Langevin equation given by Eq. (2) the second moment departs drastically from the well known Einstein fluctuation-dissipation relation [19]. In fact, the linear Langevin equation

$$\dot{X}(t) = -\Gamma X(t) + f(t), \quad (19)$$

yields a Gaussian distribution for the solution to the equivalent FPE. In the latter case the second moment of the mean field satisfies the Einstein relation

$$\left\langle x^2 \right\rangle_{eq} = \frac{D}{\Gamma}, \quad (20)$$

which depends on the ratio of D to Γ rather than on its square root. We conclude that the effective linear Langevin equation generates results as close as possible to those of the cubic Langevin equation of Eq. (2) when the constraint

$$\frac{D}{\Gamma} = \alpha \left(\frac{D}{\gamma}\right)^{1/2} \quad (21)$$

is satisfied. This constraint is established by assuming that the second moment of Eq. (17) is equal to that of Eq. (20) leading to the value for the effective dissipation rate

$$\Gamma = \frac{1}{\alpha} (\gamma D)^{1/2} = 1.49 (\gamma D)^{1/2}. \quad (22)$$

This value of Γ is very close to the prediction of Ref. [16] and, even more remarkable, close to the eigenvalue of the first state of a Schrödinger-like equation that the authors of Ref. [20] propose to study the nonlinear stochastic process discussed herein. It is important to stress that using the assumption that Γ is this eigenvalue leads us to $1/\alpha = 1.4$, namely $\alpha = 0.71$ rather than $\alpha = 0.69$. Adopting intuitive arguments, we can explain stochastic linearization as a consequence of the fact that $xP_{eq}(x)$, due to symmetry, mainly overlaps only one of the eigenstates of the operator $\mathcal{L}$, the first excited state [20].

III. NON-STATIONARY CORRELATION FUNCTION OF X

In this section we discuss how to find a general analytical expression for the non-stationary correlation function of Eq. (7). We first study the case of a brand new correlation function, that being one with no aging time, $t_a = 0$, as well as the opposite limit of an infinitely old correlation function, $t_a = \infty$. Then we move to address the more general case of a correlation function of arbitrary age, $0 < t_a < \infty$.

A. Brand new correlation function

Since we adopt the preparation $X(0) = 1$, the brand new autocorrelation function is seen to coincide with the average mean field

$$\langle X(t) \rangle = \int_{-\infty}^{+\infty} x P(x,t) dx, \quad (23)$$

where $P(x,t)$ is the solution to the FPE of Eq. (12) with $P(x,0) = \delta(x-1)$. Taking the time derivative of Eq. (23) and using Eq. (12) we find

$$\frac{d}{dt} \langle X(t) \rangle = -\gamma \left\langle X^3(t) \right\rangle. \quad (24)$$

In principle we may get the exact time evolution of $\langle X(t) \rangle$, by using the FPE to determine the time derivative of $\left\langle X^3(t) \right\rangle$, which would be expressed as the linear combination of $\langle X(t) \rangle$ and $\left\langle X^5(t) \right\rangle$, and so on to build a hierarchy of moment rate equations. By applying this procedure N times, we would obtain a $N \times N$ matrix that could be diagonalized to yield an expression for $\langle X(t) \rangle$ that is expected to converge to the exact solution for $N \to \infty$. This is known as the Carleman embedding technique [18]. However, for our present purposes an approximated analytical expression in which the accuracy can be controlled by comparison with the numerical solution of the problem is preferred.

Thus, we adopt the following approximation

$$\left\langle X^3(t) \right\rangle = \left\langle x^2 \right\rangle_{eq} \langle X(t) \rangle + \langle X(t) \rangle^3. \quad (25)$$

This approximation bridges two limiting conditions, the short-time condition where $\left\langle X^3(t) \right\rangle = \langle X(t) \rangle^3$ is exact, and the condition close to equilibrium where, according to stochastic linearization

$$\frac{d\langle X(t)\rangle}{dt} = -\Gamma \langle X(t)\rangle . \quad (26)$$

Note that inserting Eq. (25) into Eq. (24) yields the nonlinear rate equation

$$\dot{y}(t) = -ay(t) - by(t)^3, \quad (27)$$

with $y(t) = \langle X(t)\rangle$, $a = \gamma$ and $b = \gamma \langle x^2\rangle_{eq}$. This equation can be easily solved to yield

$$y(t) = \frac{y(0)e^{-at}}{\left[1 + \frac{b}{a}y(0)^2\left(1 - e^{-2at}\right)\right]^{1/2}}. \quad (28)$$

In conclusion, setting the preparation condition $\langle X(0)\rangle = 1$, this expression reduces to the solution for the average mean field

$$\langle X(t)\rangle = \frac{e^{-\Gamma t}}{\left[1 + \frac{\gamma}{\Gamma}\left(1 - e^{-2\Gamma t}\right)\right]^{1/2}}, \quad (29)$$

with Γ given by Eq. (22).

B. Infinitely aged correlation function

The time-dependent solution to the FPE is taken up in [20] using a spectral decomposition method. It is sufficient for our purposes here to note that the contribution from the non-zero eigenvalues to the asymptotic PDF all vanish. In fact, to evaluate the autocorrelation function we adopt an average over a Gibbs ensemble of networks with the preparation $P(x, 0) = \delta(x - 1)$. This ensemble evolves in time and for $t_a \to \infty$ the corresponding PDF becomes equivalent to $P_{eq}(x)$ of Eq. (14). This allows us to write the autocorrelation function, using the operator notation $X(t) = e^{\mathcal{L}_{FP}t}X(0)$, as

$$\Phi_x(t, t_a = \infty) = \left\langle xe^{t\mathcal{L}_{FP}}x\right\rangle = \frac{\int_{-\infty}^{+\infty} xe^{t\mathcal{L}_{Fp}}xP_{eq}(x)dx}{\int_{-\infty}^{+\infty} x^2P_{eq}(x)dx}. \quad (30)$$

Using the quantum-like spectral decomposition arguments of Ref. [20] immediately leads to the exponential relaxation of the autocorrelation function

$$\Phi_x(\tau, t_a = \infty) = e^{-\Gamma t}, \quad (31)$$

with the effective dissipation parameter

$$\Gamma = 1.4(\gamma D)^{1/2}. \quad (32)$$

This result is based on the assumption that $xP_{eq}(x)$ overlaps only with the first eigenstate of the Fokker-Planck operator $\mathcal{L}_{Fp}$. This important symmetry based assumption is discussed in Ref. [20].

C. Correlation function of arbitrary age

Note that when the intensity of the stochastic force $f(t)$ vanishes, Eq. (29) is replaced by the exact solution for the order parameter and not its average value

$$X(t) = \frac{1}{(1 + 2\gamma t)^{1/2}}. \quad (33)$$

The prescription of Eq. (7) applied to this deterministic condition formally yields

$$\Phi_x(\tau, t_a) = \left[\frac{(1 + 2\gamma t_a)}{1 + 2\gamma(\tau + t_a)}\right]^{1/2}, \quad (34)$$

which we rewrite in the more attractive form

$$\Phi_x(\tau, t_a) = \frac{1}{(1 + 2\gamma(t_a)\tau)^{1/2}}, \quad (35)$$

with

$$\gamma(t_a) = \frac{\gamma}{1 + 2\gamma t_a}. \quad (36)$$

We make the assumption that this age-dependent friction has an unlimited range of validity, extending from the short time regime, where it is exact, to the long-time regime dominated by fluctuations, where it may significantly depart from the correct behavior. Thus, we obtain

$$\Phi_x(\tau, t_a) = \frac{e^{-\Gamma\tau}}{\sqrt{\left[1 + \frac{\gamma(t_a)}{\Gamma}\left(1 - e^{-2\Gamma\tau}\right)\right]}}. \quad (37)$$

It is interesting to notice that this analytical expression connects the brand new condition for $t_a = 0$ to the infinitely aged $t_a = \infty$ and shows that this form of aging process has a very extended time duration because of the inverse power law regression to the vanishing value of $\gamma(t_a)$. In the next section we show that this formula leads to a remarkably good agreement with the numerical results.

IV. NUMERICAL TREATMENT

Adopting the integration time step $\Delta t = 1$, the cubic Langevin equation becomes

$$X_{n+1} = X_n - \gamma X_n^3 + \sigma\xi_n, \quad (38)$$

where the fluctuation $\xi_n = \pm 1$ is generated according to a coin-tossing prescription. The second-order differential operator with respect to X of Eq. (13) is derived by using the central limit theorem and it is independent of the details of the fluctuations $f(t)$. It only depends on the second moment of this fluctuation that according to $f(n) = \sigma\xi_n$ is given by σ^2. Thus

$$D = \frac{\sigma^2}{2}. \quad (39)$$

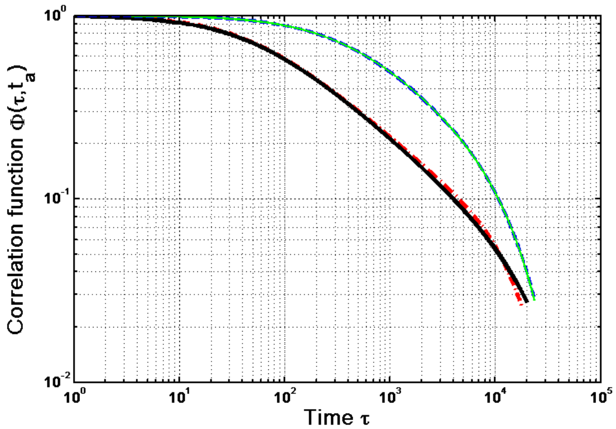

FIG. 1. Age-dependent correlation function with preparation $X(0) = 1$. The bottom doublet, solid black line (theoretical), dashed line (numerical), refers to $t_a = 0$. The top doublet, solid line (theoretical), dashed line (numerical) refers to $t_a = 100$. $\gamma = 0.01$, $\sigma = 0.001$

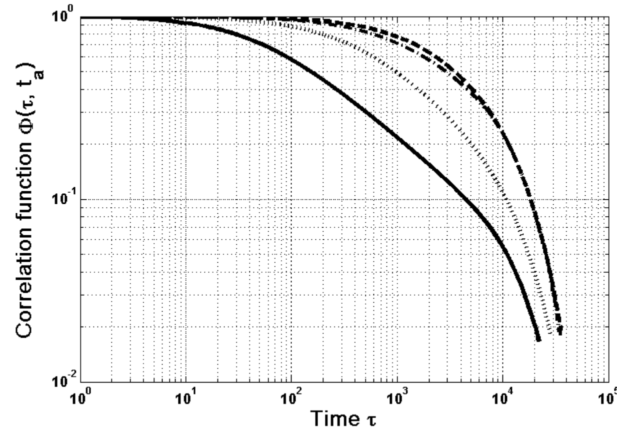

FIG. 2. The correlation function with preparation $X(0) = 1$ at different ages. The solid line denotes $t_a = 0$, the dotted line $t_a = 100$, and the bottom dashed line $t_a = 1000$. These three curves are numerical. The top dashed line is the theoretical equilibrium ($t_a = \infty$) correlation function, the exponential function of Eq. (31) with the parameter Γ given by Eq. (32). $\gamma = 0.01$, $\sigma = 0.001$

Note that Eq. (38) facilitates the understanding of the emergence of temporal complexity from criticality. In fact, as shown in discussion of the DMM made in Ref. [16] and as stressed in Section I B to make this article as self-contained as possible, in either the sub- or super-critical condition, a small departure from equilibrium is driven by a linear Langevin equation. On the other hand, the criticality condition annihilates the linear dependence of the Langevin equation on the mean field, thereby activating the nonlinear structure of Eq. (2) and its discrete-time counterpart Eq. (38). We see that, moving from $X = 0$, the variable X produces values on the order of $\sigma \ll 1$. Consequently the nonlinear friction $-\gamma X_n^3$ generates values on the order of $\gamma\sigma^3$, which are negligible compared to σ, thereby making Eq. (38) virtually identical to a free-diffusion process. Criticality plays the crucial role of extending the regime of validity of the free-diffusion process, and consequently of making evident, according to the Sparre-Andersen theorem [21] the regime of regression to the origin $X = 0$ with the inverse power law $1/t^{3/2}$.

Fig. 1 shows the remarkable agreement between the important theoretical prediction of Eq. (37) and the numerical treatment by means of two doublets of curves referring to $t_a = 0$ and $t_a = 100$, with the theoretical prediction virtually coincident with the numerical result. Fig. 2 shows that the numerical results tend to the equilibrium correlation function of Eq. (31), with the curve $t_a = 1000$ being very close to it.

Note that the aging process is much less evident if preparation is done at $X(0) = 0$, because for even very small t_a the non-stationary autocorrelation function is close to the equilibrium autocorrelation function. The synchronization between two complex systems at criticality [14] is determined by the influence that the dichotomous signal $\eta(t) = \mathrm{sgn}(X(t))$ of the perturbing network exerts on the $\eta(t) = \mathrm{sgn}(X(t))$ process of the perturbed network. In spite of the synchronization proved by the numerical work of Ref. [14], the corresponding X-autocorrelation functions do not reveal any significant sign of criticality. Fig. 3 shows that at $t_a = 10^4$ this age-dependent autocorrelation function virtually coincides with the autocorrelation function of Eq. (31) with Γ given by Eq. (32).

V. CONCLUDING REMARKS

We notice that stochastic linearization plays the important role of turning a difficult nonlinear stochastic process into an analytically tractable form of nonequilibrium dynamics. It sheds light on the intriguing issue of temporal complexity. In fact, the power index $\mu = 1.5$ is the well known result of the Sparre-Andersen theorem, which is based on ordinary diffusion, thereby casting doubts on temporal complexity that would depend on ordinary diffusion. It is important to notice that the correspondence between critical slowing down, with the power index 0.5 and the survival probability $\Psi(\tau)$ associated to the waiting time distribution density with $\psi(\tau) \propto 1/\tau^{1.5}$, and consequently with the same power index 0.5, is a coincidence. Nevertheless, temporal complexity remains a consequence of criticality. In fact, criticality generates the nonlinear Langevin equation of Eq. (2) and this equation, through stochastic linearization yields the lifetime Eq. (6), which defines the time region $t < T_{eq}$ where the variable $\eta = \mathrm{sgn}(X(t))$ has the effect of generating an extended regime of free diffusion. As a further way of illustrating the benefits of stochastic linearization, we notice that the stochastic linearization is

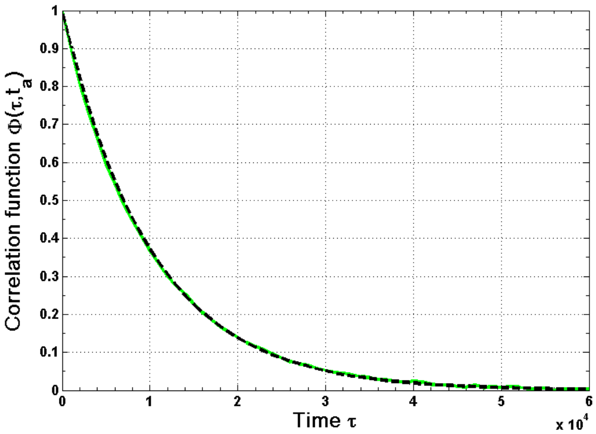

FIG. 3. The (solid) green curve is the numerical correlation function with preparation $X(0) = 1$ with $t_a = 10^4$ and the (dashed) black curve is the exponential function of Eq. (31) with the parameter Γ given by Eq. (32). $\gamma = 0.01$, $\sigma = 0.001$. Numerical work shows that preparation $X(0) = 0$ with $t_a = 10^4$ generates indistinguishable results.

equivalent to setting the condition

$$\Gamma \propto < x^2 >_{eq} \gamma, \quad (40)$$

which shows the friction becoming negligible, given the fact that at criticality $|x| \ll 1$.

The formal result of this paper, Eq. (37) turns out to be surprisingly accurate and serves very well the purpose of explaining the connection between critical slowing down and temporal complexity. Both processes show up in the time window $0 < t < T_{eq}$. Temporal complexity is a property of $\eta(t) = \mathrm{sgn}(X(t))$ and the preparation of the network at $X(0) = 0$ is compatible with the emergence of significant aging manifestations [12], if these events are observed. The aging process is connected to the well known fact that a process generating renewal events with a waiting-time distribution density $\psi(t) \propto 1/t^{\mu}$ with $\mu < 2$, prepared at time $t = 0$ generates a cascade of events the number of which per unit of time decreases as $1/t^{2-\mu}$ [22]. The response of a complex network under this non-equilibrium condition to external perturbations affecting the occurrence of these crucial events requires the adoption of a generalized form of Linear Response Theory (LRT) as explained by the authors of Ref. [23] who showed that this generalized LRT is consistent with experiment, affording clear evidence of the occurrence of this effect.

The autocorrelation function of X (see Fig. 2), on the contrary, does lead to significant aging effects, if the preparation $X(0) = 1$ is adopted. The aging effects become visible with the preparation $X(0) = 1$. The linear response studied in Ref. [16] corresponds to perturbing the variable X and to observing the response of the system during the regression to equilibrium of the autocorrelation function of the variable X. Although the autocorrelation function of the dichotomous signal $\eta(t)$ with preparation at $X(0) = 0$ generates a pronounced aging effect [12] , the aging of the autocorrelation function of X and not $\eta(t)$ is much less pronounced, thereby leading to weaker physical effects [16].

Although different preparations lead to different behavior, a significant aging effect with the preparation $X(0) = 1$, and very weak aging manifestation with the preparation $X(0) = 0$ [16], as expected, at $t_a = 10^4$ the dependence on the preparation is lost. In fact, Fig. 3 shows the preparation $X(0) = 1$ leads to a result virtually indistinguishable from the preparation $X(0) = 0$. In conclusion, physical processes depending on X rather than on $\eta(t) \equiv \mathrm{sgn}(X(t))$ require a strongly non-equilibrium preparation to produce significant physical manifestations of criticality. Alternatively, we can conclude that the manifestations of temporal complexity in the response to perturbation, as a form of response to perturbation in an out of equilibrium regime [23], become evident when the external stimulus perturbs the variable $\eta(t) \equiv \mathrm{sgn}(X(t))$, namely, the crucial events that generate temporal complexity.

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