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A. Kolakowska and M. A. Novotny

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Nonuniversal Effects in Mixing Correlated-Growth Processes with Randomness: Interplay Between Bulk Morphology and Surface Roughening

A. Kolakowska*
Department of Physics, The University of Memphis, Memphis, TN 38152

M. A. Novotny
Department of Physics and Astronomy, and Center for Computational Sciences, P.O. Box 5167, Mississippi State, MS 39762-5167

To construct continuum stochastic growth equations for competitive non-equilibrium surface-growth processes of the type RD+X that mixes random deposition (RD) with a correlated-growth process X, we use a simplex decomposition of the height field. A distinction between growth processes X that do and do not create voids in the bulk leads to the definition of the *effective probability* $p_{\rm eff}$ of the process X that is a measurable property of the bulk morphology and depends on the *activation probability* p of X in the competitive process RD+X. The bulk morphology is reflected in the surface roughening via *nonuniversal* prefactors in the universal scaling of the surface width that scales in $p_{\rm eff}$. The equation and the resulting scaling are derived for X in either a Kardar-Parisi-Zhang or Edwards-Wilkinson universality class in (1+1) dimensions, and illustrated by an example of X being a ballistic deposition. We obtain full data collapse on its corresponding universal scaling function for all $p \in (0; 1]$. We outline the generalizations to $(1 + n)$ dimensions and to many-component competitive growth processes.

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I. INTRODUCTION

Many dynamical complex physical systems in nature are studied by their mapping onto a suitable nonequilibrium surface-growth problems. The dynamics of correlation buildup in these physical systems, and their other properties, can then be explored with the use of surface-growth methodologies. Numerous examples of such studies, experimental as well as theoretical and computational, come from a variety of fields such as tumor-growth processes [1] in cancer research; growth of cell colonies [2] in biophysics; roughening of lipid bilayers [3] in soft-matter bio-materials; dynamics of combustion fronts [5]; film-growth processes [6]; time-series and market price analyses in econo-physics [7]; and, scalability and synchronization of parallel-computing system [8, 9] in computer science, to give representative examples.

Large-scale properties are described within a continuum model by universal stochastic growth equations and tested with simulation models. On the theory side, the trouble is that simple discrete models, such as SOS, are often not adequate to reproduce the complex physics of observed surface phenomena, as they assume only one universal process alone being responsible for surface formation. Such an idealization does not reflect actual experimental settings where the observed surface phenomena may involve contributions from several universal processes. The continuum description of such a multicomponent growth process has not yet been developed. Such mixed-growth systems display many nontrivial properties [10–35].

A representative example comes from an applied model in computer science [9] when the asynchronous dynamics of conservative updates in a system of parallel processors is modeled as a virtual-time surface that represents nonequilibrium processes in this system. When the load per processor is minimal, this dynamics belongs to the Kardar-Parisi-Zhang (KPZ) universality class [37]. However, when the load is increased to reflect real operations, the realistic dynamics is a competitive growth process that combines a universal KPZ process with a random deposition (RD) process, i.e., is of the type RD+KPZ [22]. Consequently, in order to fully understand the statistics of the updates and make quantitative predictions, it is important to know how the *nonuniversal* properties of the multi-component processes affect the universal scaling of a RD+KPZ process. This is a still unsolved problem of nonequilibrium surface growth science. In applied modeling, even if not explicitly assumed, competitive dynamics naturally arises. Studying these systems should also contribute to the understanding of differences between the expected and the actual scaling of rough interfaces, often encountered both in simulations and in experiments.

By a competitive-growth process Y+X — alternatively called a two-component system or a mixed-growth process — we understand a dynamical process where process Y alternates with process X in accordance with the rule of the exclusive alternative: "*either* process Y (active with probability q) *or* process X (active with probability p)," is active. Here $q + p = 1$, and Y belongs to a different universality class than X. It is understood that in competitive surface-growth processes an event on the surface is triggered by only one process at a time, even if

*Electronic address: alice@kolakowska.us

both mechanisms X and Y are simultaneously present.

In this article we introduce a method by which a continuum stochastic growth equation can be constructed for competitive growth processes. We investigate a connection between surface roughening and the bulk morphology formed during the deposition in the competitive growth process RD+X, where X is a correlated growth process of universal dynamics different from RD. This connection has been already established in simulations of competitive growth models [10] and of binary growth of thin films [14], and for diffusion-limited-aggregation models [36]. A new aspect of our study is to provide a direct theoretical link between nonuniversal properties of process X, as read from the bulk, and the continuum equation that underlies the observed universal scaling laws for the competitive RD+X processes. In this work, we derive from first principles a continuum equation to show that its model-dependent coefficients do reflect the bulk structure. This will lead to a distinction between void-producing and simple desorption and adsorption processes. As discussed later, this division into subclasses is a *necessary* first step towards a theory of many-component processes. In particular, it explains variations in scale dilatations observed in RD+X models [17, 18, 21, 22, 28, 30–35]. In our analysis we use as an example the universal RD+KPZ growth process in $(1+1)$ dimensions, and generalize our approach to other processes in $(1+n)$ dimensions.

This article is organized as follows. Scaling of the interface width in competitive RD+X models is outlined in Sec. II, where we show that the full data collapse scaling can be obtained in a geometric scaling given by Eqs. (2)-(3). This type of scaling was heuristically proposed in Ref. [33] and its explicit form was derived in Ref. [34]. Geometric scaling confirms that the RD+X systems are in the universality class of process X [30], but, such data collapse is not a dynamic finite-size scaling. In the remaining part of Sec. II we focus on dynamic scaling that provides a connection with stochastic dynamics as described by a continuum-growth equation. In Sec. III, where we define the adsorption-bulk-compact and the dense-or-lace-bulk processes, we use a concept of simplectic decomposition to derive from first principles the stochastic growth equation for simple RD+X processes. Hence, we find a connection between the bulk morphology and the surface roughening for these processes. Results of Sec. III are discussed in Sec. IV, where we demonstrate by examples that in RD+X processes the nonuniversal prefactors in Family-Vicsek universal scaling function are nontrivial and have connection with the bulk morphology. In Sec. IV we also give the extension of the approach introduced in Sec. III to $(1+n)$ dimensional models of two-component processes, outline a possible generalization to many-component competitive processes, and discuss further developments. Conclusions are summarized in Sec. V.

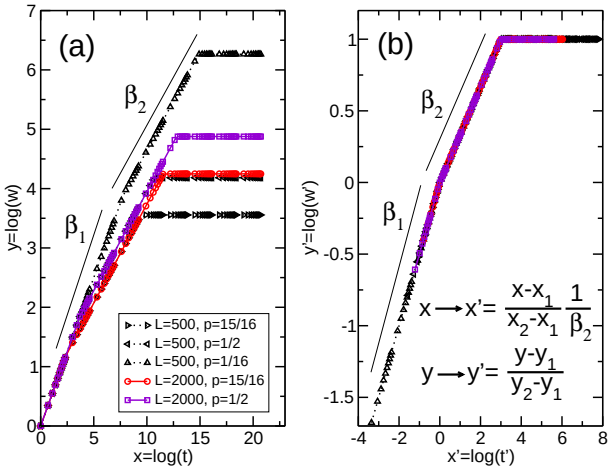

FIG. 1: (Color on line) Typical time evolution of the interface width $w(t)$ in a competitive process RD+X for selected system sizes L and activation probabilities p of process X. (a) Synthetic data from Eq. (1) when process X is in the KPZ universality class. (b) The collapse of the data in (a) that is produced by the geometric scaling $(x, y) \to (x', y')$ given by Eqs. (2)-(3) (shown in the figure). Here, $\beta_1 = 1/2$ and $\beta_2 = 1/3$.

II. SCALING OF THE INTERFACE WIDTH

Time-evolution of the correlation length is reflected in the interface width $w(t)$ of the growing surface. Both the correlation length and $w(t)$ have the same scaling properties. In SOS models of surfaces growing on a substrate of L sites, $w(t)$ is measured as: $\langle w^2(t)\rangle = \langle L^{-1}\sum_{k=1}^{k=L}[h_k(t) - \bar{h}(t)]^2\rangle$, where $h_k(t)$ is the column height at site k at time t, and $\bar{h}(t)$ is its mean over L sites. The time t is measured as a number of deposited monolayers. The angular brackets denote configurational averages. For brevity of notation, we set $w \equiv \sqrt{\langle w^2\rangle}$.

In competitive growth processes that mix correlated-growth process X with randomness, i.e., of the type RD+X, results of simulations with a flat-substrate initial condition at $t = 0$ can be summarized by the following two-parameter family of curves [22]:

$$w(t; L, p) = \begin{cases} c_1\sqrt{t} & , \quad t \in [0; t_1(p)] \\ c_2 t^{\beta_2} & , \quad t \in (t_1(p); t_2(L, p)) \\ c_3 L^{\alpha_2} & , \quad t \in [t_2(L, p); +\infty], \end{cases} \qquad (1)$$

where p and L are parameters; and, c_1, c_2, and c_3 are constants. The effect of the parameter p on the time-evolution of $w(t; L, p{=}1)$ is a nonuniversal dilatation of time and length scales, as discussed in Ref. [30]. Because of these dilatations the times $t_1(p)$, when the initial RD transients terminate, and the times $t_2(p)$, when the saturation phases begin, have different values for different curves $w(t; L, p)$. In Eq. (1) β_2 and α_2 are universal scaling exponents (the growth and the roughness exponents, respectively) characteristic of the universality class of process X. For processes RD+KPZ (when

$\alpha = \alpha_2 = 1/2$ and $\beta = \beta_2 = 1/3$) the family given by Eq. (1) is illustrated in Fig. 1a, where $y = \log w$ is plotted versus $x = \log t$. In the (x, y)-plane, the family in Eq. (1) can be collapsed onto one curve shown in Fig. 1b by the means of the shift-and-scale operator $\hat{G} : (x, y) \longrightarrow (x', y')$ [34],

$$x \ \rightarrow \ x' = \frac{x - x_1}{x_2 - x_1} \frac{1}{\beta_2}, \tag{2}$$

$$y \ \rightarrow \ y' = \frac{y - y_1}{y_2 - y_1}, \tag{3}$$

where $x' = \log t'$ and $y' = \log w'$. In Eqs. (2)-(3) the pairs of numbers $x_1 = \log t_1(p)$ and $y_1 = \log w_1(p)$, and, $x_2 = \log t_2(L, p)$ and $y_2 = \log w_2(L, p)$, have different values for each curve $w(t; L, p)$. Explicitly, $w_1 = w(t_1(p))$ and $w_2 = w(t_2(L, p))$ by Eq. (1). Operator $\hat{G}$ shifts all curves in Eq. (1) to one position where all crossover points $(t_\times, w_{\rm sat})$ to saturation are mapped onto one point $(1/\beta_2, 1)$. Subsequently, $\hat{G}$ scales the length $(x_2 - x_1)\sqrt{1 + 1/\beta_2^2}$ of the correlated-growth phase in (x, y)-plane for each curve to the length of $\sqrt{1 + 1/\beta_2^2}$. The full data collapse obtained by $\hat{G}$ is possible because each curve in Eq. (1) has one universal footprint, where the initial RD transient is followed by a specific universal correlation phase. Such a collapse of the data in the (x, y)-plane by geometric scaling is an illustration of the previously proven fact [30] that competitive-growth processes RD+X are in the universality class of process X. It must be stressed, however, that this geometric scaling expressed by $\hat{G}$ does not give the universal dynamic scaling function that would explain the universal shape of the curve in Fig. 1b in terms of finite-size scaling of the corresponding stochastic dynamics as described by the continuum model and, possibly, nonuniversal corrections to scaling when $p \neq 1$. Manipulation of Eqs. (2)-(3) to obtain explicitly $w(t)$ leads back to Eq. (1); thus, Eqs. (2)-(3) do not contain any new physical information in addition to that already present in Eq. (1). In summary, to this point the geometric scaling lacks physical meaning and does not connect with the Family-Vicsek dynamic scaling.

A dynamic scaling hypothesis for competitive RD+X processes [30] states that if a correlated growth X occurs with a constant probability p, its continuum equation must be invariant under the scaling

$$x \rightarrow x \, , \ h \rightarrow h/g(p) \, , \ t \rightarrow t/f(p) \, , \tag{4}$$

where $g(p)$ and $f(p)$ are arbitrary suitable functions of $p \in (0; 1]$. This invariance implies that $f(p) = g^2(p)$. When X=KPZ, the dynamic scaling hypothesis leads to the KPZ equation [38] for the RD+KPZ mix [30]:

$$h_t = \nu_0 f(p) h_{xx} + (\lambda_0/2) f^{3/2}(p) h_x^2 + \eta(x, t) \, , \tag{5}$$

where $h \equiv h(x, t)$ is the height field; x and t are the spacial and time coordinates, respectively; subscripts denote partial derivatives; $\eta(x, t)$ is the white noise; and, ν_0 and λ_0 are constants. When $\lambda_0 = 0$, Eq. (5) is the Edwards-Wilkinson (EW) equation [39] when X=EW. When $\nu_0 = \lambda_0 = 0$, Eq. (5) defines universal RD dynamics.

Many simulation models of RD+EW and RD+KPZ growth processes [21, 30–32] suggest $g(p) = p^\delta$ in Eq. (4), which leads to the Family-Vicsek universal scaling [40] of the average surface width $w(p, t)$ [30]:

$$w(p, t) = \frac{L^\alpha}{p^\delta} F\left(p^{2\delta} \frac{t}{L^z} \right) . \tag{6}$$

For substrates of size L, $F(y)$ describes two limit-regimes of evolution: $F(y) \sim y^{\alpha/z}$ if $y \ll 1$ (growth); and, $F(y) \sim$ const if $y \gg 1$ (saturation). In Eq. (6), α and z are the universal roughness and dynamic exponents, respectively, of the universality class of the correlated-growth process X. The scale-dilatation exponent δ in the scaling prefactors in Eq. (6), however, is *nonuniversal*. It has been observed that in some models $\delta \approx 1$ across universality classes, and in some other models $0 < \delta \lessapprox 1$ within a single universality class [22, 28, 30]. Also, there are some models where the prefactors in Eq. (6) do not at all obey a power law in p [32, 41]. In the next section, we shall establish that this variation is not accidental in flux-conserving models, but rather reflects the properties of the bulk of the deposited material.

III. *AB INITIO* CONTINUUM EQUATION BY SIMPLECTIC DECOMPOSITION

Consider aggregations where identical particles fall onto a substrate of L sites. On the substrate, the incoming particles may be accepted in accordance with a rule that generates correlations among the sites, i.e., in accordance to process X. It is understood here that the flux of the incoming particles is uniform and time independent, i.e., the average rate at which the particles arrive at the substrate does not vary with time and does not vary with the position along the substrate. The correlated-growth X occurs with probability p, and competes with RD growth that occurs with probability $q = 1 - p$. It is understood here that the probability p remains constant for the entire duration of the process RD+X, i.e., the average frequency of process X does not change with time and does not depend on the position of the site on the substrate. When a particle is accepted at a site, the site increases its height by Δh. If, e.g., component *1* is RD, and component *2* is a correlated-growth in the KPZ universality class, their corresponding growth equations are

$$h_{1,t} \ = \ \eta_1(x, t) \, , \tag{7}$$

$$h_{2,t} \ = \ \nu_0 h_{2,xx} + (\lambda_0/2) h_{2,x}^2 + \eta_2(x, t) \, , \tag{8}$$

where $h_n(x, t)$, $n = 1, 2$, is the column height at x after time t when the component n acts alone. Assume for simplicity that the noise terms are of the same strength, i.e.,

$\eta \equiv \eta_1 = \eta_2$. In two-component growth, when both components act simultaneously together, the column height $h(x,t)$ is incremented due to either of the components with their corresponding probabilities $\tilde{p}$ and $\tilde{q}$, $\tilde{p}+\tilde{q}=1$:

$$\Delta h(x,t) = \tilde{p}\Delta h_2(x,t) + \tilde{q}\Delta h_1(x,t)\,. \qquad (9)$$

Here, probability $\tilde{p}$ (or $\tilde{q}$) is the fraction of contributions to h from component *2* (or *1*). For some processes this fraction is identical to a fraction of times when $h(x)$ is incremented due to component *2* (or *1*) for the times from 0 to t. However, as explained later, this is not so for all processes. In Eq. (9), Δh_n is understood as "being incremented due to the process n," $n = 1, 2$. In this statistical sense, Eq. (9) expresses a simplectic decomposition of $\Delta h(x,t)$ into its vertex-components $\Delta h_n(x,t)$. Dividing Eq. (9) by Δt, and taking the limit $\Delta t \to 0$, gives the equation for time rates, $h_t = \tilde{p}h_{2,t} + \tilde{q}h_{1,t}$, to which we substitute Eqs. (7)-(8) to obtain:

$$h_t = \nu_0\tilde{p}h_{2,xx} + (\lambda_0/2)\tilde{p}h_{2,x}^2 + (\tilde{p}+\tilde{q})\eta(x,t)\,. \qquad (10)$$

In Eq. (10), $h(x,t)$ is the column height that rises at x as the result of two processes acting simultaneously from the beginning to time t. Here, $h_2(x,t)$ is the part of $h(x,t)$ that was created by the component *2* in this time. The other part was created by component *1*. In other words, $h_2(x,t)$ is so far an unknown fraction of $h(x,t)$. To find a relation between h and h_2, one must consider nonuniversal properties of aggregation processes.

We distinguish between the following two groups of surface growths processes. In one group we place all simple adsorption processes with conserved flux that do not create voids in the bulk of the deposited material. We call this group *adsorption-bulk-compact* (ABC) growths. For example, a simple random deposition or random deposition with surface relaxation fall into the ABC category. The other group, which we call *dense-or-lace-bulk* (DOLB) growths, contains processes that are not ABC-type. The DOLB group includes desorption processes that may lead to a dense bulk as well as adsorptions that lead to the formation of voids. The only type of desorption processes studied here are ones due to local spontaneous desorption at the surface, not desorption processes where an incoming particle strikes the surface and causes desorption. The latter type of desorption process would have shadowing effects, and hence would be extremely dependent on the direction of the incoming particles. Note, the DOLB category contains flux-conserving as well as flux-non-conserving processes. For example, ballistic deposition and deposition to local interface minima are both in the DOLB group. Note that all RD universal processes are ABC-growth processes. As we show in the next paragraph, when component *2* is of the ABC-type, probabilities $\tilde{p}$ and $\tilde{q}$ in Eq. (9) express fractional contributions to h in terms of times, and then $h_2(x,t) = ph(x,t)$. This is not true when component *2* is a DOLB growth.

Consider a discrete representation of events at coordinate x. Suppose, there are t deposition events in total, with t_1 events due to component *1*, and t_2 events due to component *2*, $t = t_1 + t_2$. In ABC growth, after t events, the total column height is $h = t\Delta h$, where contributions from components *1* and *2* are, respectively, $h_1 = t_1\Delta h$ and $h_2 = t_2\Delta h$. Thus, $h_1/h = t_1/t = q$ and $h_2/h = t_2/t = p$. Therefore, in ABC growth $h_2 = ph$, and in Eqs. (9)-(10) we can identify $\tilde{p} = p$ and $\tilde{q} = q$.

Next consider that the component *2* is a DOLB growth that creates voids. Now, an individual deposition event due to component *2* not only increases h by Δh, but

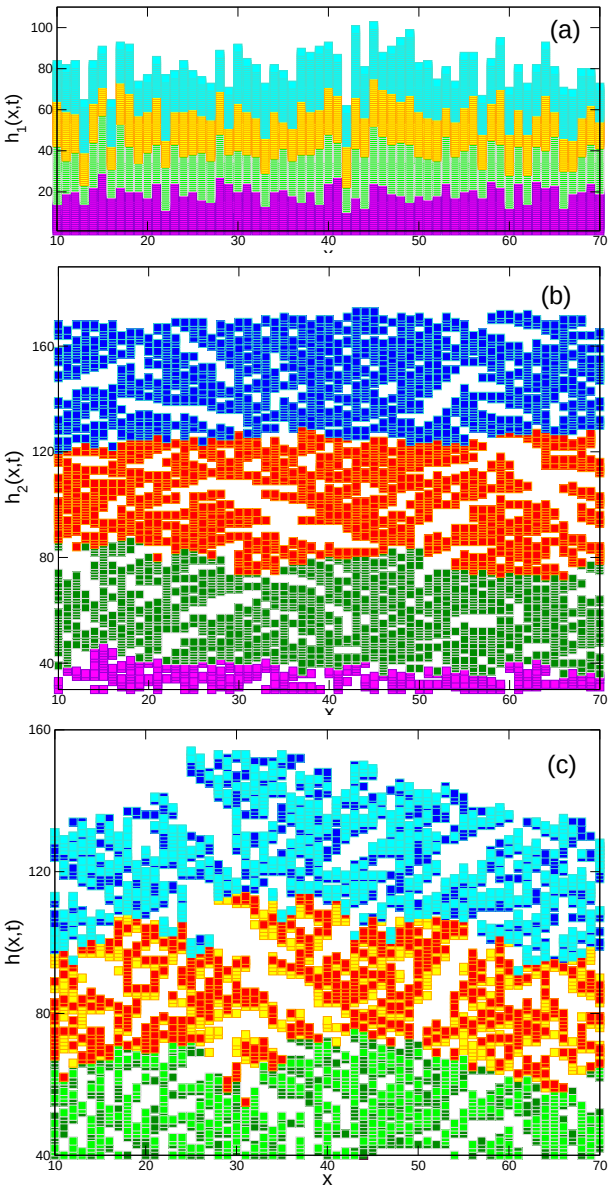

FIG. 2: (Color on line) Bulk sections obtained in Monte Carlo simulations of surfaces, generated by: (a) random deposition (RD); (b) ballistic deposition (BD) that creates the bulk compactness $c = 0.468$; and, (c) the competitive RD+BD process when $p = q = 1/2$. Coloring indicates time intervals. In (c), within a time interval, coloring is used to differentiate between deposits created by RD and those created by BD.

may also result in the creation of voids (as illustrated in Fig. 2*b*, an example of ballistic deposition). The net effect is as though component *2* deposited Δh *and* the voids. Therefore, in t_2 events, its contribution to the column height is $h_2 = (t_2 + m)\Delta h$, where $m\Delta h$ reflects the increase in the field height due to the presence of voids. The component *1* is RD, i.e., ABC-type, and $h_1 = t_1\Delta h$. After t events, the net column height is $h = h_1 + h_2 = (t + m)\Delta h$. Thus, $h_1/h = t_1/(t + m) < t_1/t = q$ and $h_2/h = (t_2 + m)/(t + m) > t_2/t = p$. The explicit form of these mutually complementary fractions, h_n/h for $n = 1, 2$, allows them to be directly measured from the bulk. They are, in fact, the *effective probabilities*

$$q_{\rm eff} \equiv h_1/h \quad \text{and} \quad p_{\rm eff} \equiv h_2/h \tag{11}$$

of deposition events due to components *1* and *2*, respectively, as they would result from measuring the column height.

For some types of two-component growth with RD, the probability $p_{\rm eff}$ can be expressed approximately as the power law $p_{\rm eff} = p^\delta$ [41], where the 'best' exponent δ can be estimated heuristically. For DOLB-type growth processes that produce voids, the exponent is $\delta < 1$ because $p_{\rm eff} > p$. When the component *2* is a DOLB growth with desorption, in the above reasoning one should change $m \to -m$. This will give $q_{\rm eff} > q$ and $p_{\rm eff} < p$, and $p_{\rm eff} = p^\delta$ with $\delta > 1$. The value of δ depends on nonuniversal particulars of the deposition rule of the component process *2*. Therefore, in a DOLB growth $h_2 = p_{\rm eff} h$, and in Eqs. (9)-(10) $\tilde{p} = p_{\rm eff}$ and $\tilde{q} = q_{\rm eff}$.

In general, relations $h_2(x, t) = p_{\rm eff} h(x, t)$ and $\tilde{p} \equiv p_{\rm eff}$ hold for all processes. When the correlation component is an ABC growth, its effective probability is identical with its frequency: $p_{\rm eff} = p$, provided that column-height increments are identical for the both processes 1 and 2. Thus, when process X is in the KPZ universality class, Eq. (10) gives the *exact* stochastic dynamics for the competitive RD+X processes:

$$h_t = \nu_0 p_{\rm eff}^2 h_{xx} + (\lambda_0/2) p_{\rm eff}^3 h_x^2 + \eta(x, t)\,. \tag{12}$$

When the correlation component X is a DOLB growth, and when the effective probability is well approximated by a power law p^δ, the above result can be summarized as $p_{\rm eff} = p^\delta$, where $\delta = 1$ for ABC growths, and $\delta \neq 1$ for DOLB growths. This result is combined with Eq. (12) to give the *approximate* continuum equation for the RD+KPZ mix:

$$h_t = \nu_0 p^{2\delta} h_{xx} + (\lambda_0/2) p^{3\delta} h_x^2 + \eta(x, t)\,. \tag{13}$$

When in Eq. (8) $\lambda_0 \equiv 0$, the analogous reasoning gives the *exact* result for RD+EW dynamics:

$$h_t = \nu_0 p_{\rm eff}^2 h_{xx} + \eta(x, t)\,. \tag{14}$$

In Eq. (14), we can explicitly set $p_{\rm eff} = p$ because all processes in the Edwards-Wilkinson universality class are ABC-type processes. When the flux particles are identical, the exponent $\delta = 1$ is exact.

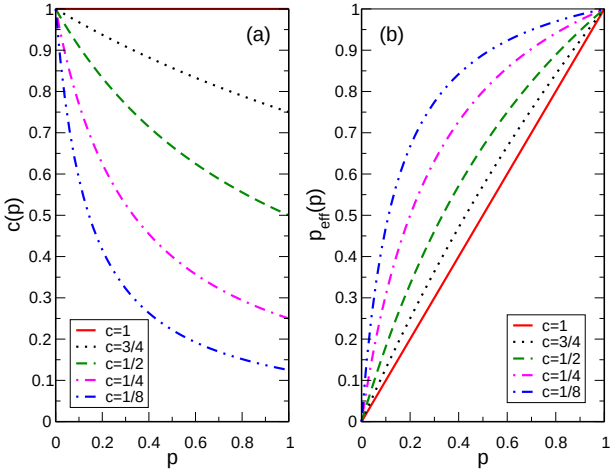

FIG. 3: (Color on line) Properties of the bulk formed in competitive process RD+X, where X is an adsorption process that creates voids. When X acts alone, i.e., in the absence of RD, it produces a bulk with compactness c. (a) Compactness of the bulk $c(p)$ as a function of the activation probability (i.e., frequency) p of X, plotted for selected values of c. (b) The effective probability $p_{\rm eff}(p)$ of X for selected values of c.

IV. DISCUSSION

Both results, Eqs. (12)-(13) and Eq. (14), are in accord with our former derivation that lead to Eq. (5) [30]. Matching Eq. (13) with Eq. (5) gives $f(p) = p^{2\delta}$, which form of $f(p)$ was used formerly to derive the approximate prefactors in Eq. (6). The inverse of the scaling (4) when applied to Eqs. (13)-(14) transforms them to continuum equations for a "pure" correlated processes of $p = 1$. Explicitly, it collapses *all* evolution curves $w(p, t)$ (for all L and p) either onto $w(1, t)$ or onto a neighborhood of $w(1, t)$ [30], following Eq. (6), provided the effective probabilities $p_{\rm eff}$ can be well approximated by the power-law p^δ. When such a fit is not possible, Eq. (6) is still obeyed but then the scaling prefactors must be expressed directly in terms of effective probabilities. This is because the factor p^δ in the coefficients of Eq. (13) is only a fit to the effective probability $p_{\rm eff}$.

Effective probabilities, defined by Eq. (11), are functions of the activation probability p of process X, i.e., the average frequency of process X in the mix RD+X. Properties of the bulk morphology created by RD+X can be equivalently expressed either in terms of $p_{\rm eff}(p)$ or in terms of $q_{\rm eff}(q)$, because of the identities $p_{\rm eff} + q_{\rm eff} = 1$ and $p + q = 1$. Effective probability $p_{\rm eff}$ of X in the mix RD+X can be expressed in various equivalent functional forms that may involve either the average compactness $c(p)$ [or the number density of voids $v(p)$] of the bulk created by the mix RD+X or the average compactness c (or the number density v of voids) of the bulk created by the process X acting alone (i.e., in the absence of RD when

$p = 1$):

$$p_{\rm eff}(p) = \frac{h_2}{h} = \frac{m + t_2}{m + t} = v(p) + pc(p) = 1 - qc(p), \quad (15)$$

where $c(p) = t/(m + t)$, $v(p) = m/(m + t)$, and $c(p) + v(p) = 1$;

$$p_{\rm eff}(p) = \frac{h_2}{h} = \frac{m + t_2}{m + t} = \frac{p}{c + pv} = \frac{p}{p + qc}, \quad (16)$$

where $c = t_2/h_2$, $v = m/h_2$, $p = t_2/t$, and $c + v = 1$. Combining Eq. (15) with Eq. (16) gives the compactness $c(p)$ of the RD+X bulk as a function of the activation probability p of process X:

$$c(p) = \frac{c}{c + pv}. \quad (17)$$

Equations (15), (16), and (17) show that $c(p)$ and $p_{\rm eff}(p)$ can be easily measured in experiment as well as in simulations: All that is needed is to measure the average density of voids in a sample cross section of the bulk when the correlated process X acts alone (i.e., in the absence of RD when $p = 1$). Equations (16) and (17), plotted for several values of c in Fig. 3, show that neither the compactness $c(p)$ nor the effective probability $p_{\rm eff}(p)$ follow a power law in p when $c < 1$.

The dynamic scaling hypothesis for RD+X processes, Eq. (4), can be reinstated by expressing $f(p)$ explicitly in terms of $p_{\rm eff}$: $f(p_{\rm eff}) = g^2(p_{\rm eff})$, since Eq. (16) can be inverted to give $p(p_{\rm eff})$. Repeating the steps outlined in Ref. [30] gives the following generalization of Family-Vicsek scaling for the surface roughness:

$$w^2(p, t) = \frac{L^{2\alpha}}{f(p_{\rm eff})} F_{\rm RD+X}\left(f(p_{\rm eff})\frac{t}{L^z}\right), \quad (18)$$

where $F_{\rm RD+X}(\cdot)$ describes the three regimes of the evolution seen in Fig. 1b. The effect of the nonuniversal prefactors $f(p)$ in Eq. (18) is a dilatation of length and time scales, as discussed in Ref. [30]. The physical meaning of p is that of a noise-tuning parameter.

In Fig. 4 we give an example of the exact scaling where nonuniversal prefactors in Eq. (6) are directly expressed by $p_{\rm eff}$ via the substitution $p^\delta \to p^\delta_{\rm eff}(p) = \sqrt{f(p)}$ for the RD+BD model when ballistic deposition (BD) is the NN sticking rule [42]. Here, the effective probability depends on both p and the mean compactness $c(p)$ of the bulk formed in the RD+BD process, given by Eq. (15) [41]. The excellent data collapse in the *full range* of $p \in (0; 1]$, seen in Fig. 4, can be contrasted with Fig. 5 of Ref. [30] that shows only an approximate data collapse for the same system with the best fit exponent $\delta \approx 0.41$ in Eq. (6). It needs to be said explicitly that the scaling where $\delta = 1/2$ in Eq. (6), proposed in Refs.[26, 31] for RD+BD models, does not produce data collapse at all. The RD+BD model when BD is the NNN sticking rule [42] provides an example where $p_{\rm eff}(p)$, and thus the nonuniversal prefactors $f(p)$ and $g(p)$ in Family-Vicsek universal scaling, cannot be expressed by a power-law p^δ [32]. In this system the surface roughening obeys power laws in effective probability that incorporates either the compactness or the void density of the bulk, resulting in excellent data collapse of $w(p, t)$, similar to that seen in Fig. 4 [41].

The approach introduced here by the example of a KPZ processes, can be applied to a broad range of stochastic growth models RD+X, where component *2* can be any isotropic growth in $(1 + n)$ dimensions:

$$h_{2,t}(\vec{x}, t) = \hat{D}(h_2) + \eta_2(\vec{x}, t), \quad (19)$$

where $\vec{x}$ is n dimensional, and the operator $\hat{D}$ represents only local interactions [42]. In the general case, Eq. (10) is written as $h_t = p_{\rm eff} h_{2,t} + q_{\rm eff} h_{1,t}$, and combined with Eqs. (7) and (19), to find for the competitive growth

$$h_t(\vec{x}, t) = p_{\rm eff}\hat{D}(p_{\rm eff} h) + \eta(\vec{x}, t), \quad (20)$$

where $\eta = (1 - p_{\rm eff})\eta_1 + p_{\rm eff}\eta_2$, and the noise strengths may be different. Eqs. (19)-(20) represent the same universality class since the multiplication by $p_{\rm eff}$ does not modify local interactions: $p_{\rm eff}$ affects the noise strength and the gradient of the height field, but does not generate any new terms other than those already given by operator $\hat{D}$. Hence, if a correlated growth belongs to a given universality class, its mix with RD will remain in the same universality class. Elementary calculations show that Eq. (20) is invariant under the scaling $g(p)h(\vec{x}, t) = h'(\vec{x}, t' = f(p)t)$. If $g(p) = p_{\rm eff}(p)$ and $f(p) = p^2_{\rm eff}(p)$, and if the noise strengths are the same, this scaling maps the universal dynamics (20) of RD+X onto the universal dynamics of X. In this case the invariance implies $g(p)w(p, t) = w'(f(p)t)$, where $w'(\cdot)$ has universal scaling properties of the process X. When X is either in the KPZ or in the EW universality class, and if additionally $p_{\rm eff} \approx p^\delta$, we recover Eq. (6).

When both the RD and the correlation component *2* have deposits of unit height, when $p_{\rm eff} \approx p^\delta$, we have

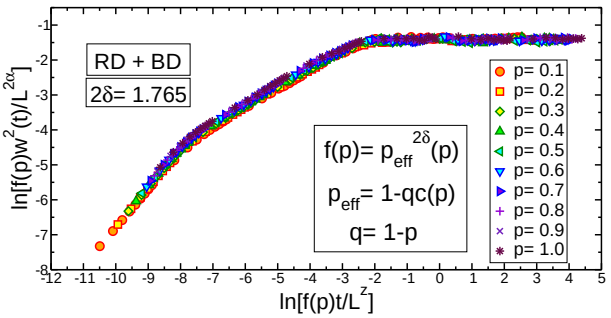

FIG. 4: (color on line) Scaled time-evolution $w^2(p, t)$ in the RD+BD model. In this example, the scaling function $f(p) = g^2(p)$ explicitly incorporates the compactness $c(p)$ of the bulk formed in the RD+BD process. Here, $L = 500$, $2\alpha = 1$, and averaging was performed over 400 surface configurations, i.e., independent simulations.

$\delta = 1$ if component *2* is of the ABC-type; and, $\delta \neq 1$ if it is of the DOLB-type. In the latter case, the value of the exponent δ is specific to component *2*. When $p_{\rm eff}$ incorporates explicitly bulk properties, the scaling is $g(p) = p_{\rm eff}^{\delta}(p)$, where the new scale-dilatation exponent δ is obtained from the slope of $\ln w^2(p)$ plotted vs $\ln p_{\rm eff}(p)$ at saturation. In DOLB growth with voids, $p_{\rm eff}$ can be determined by measuring the mean density of voids in the bulk (Figs. 2 and 4). Similarly, in DOLB growth with desorption, $p_{\rm eff}$ is connected to the mean fraction of the removed material (or flux) [41].

The analysis presented here explains scaling results of the following mixed-growth models in (1+1) dimensions. *Model A* [17, 21, 30]: component *2* is RD with surface relaxation. *Model B* [30]: component *2* simulates a deposition of a sticky non-granular material of variable droplet size. *Model C* [18, 21, 30]: component *2* is the NN sticking rule of BD. *Model D* [22, 30]: component *2* is a deposition of Poisson-random numbers to the local surface minima. *Models A* and *B* are ABC growths in the EW universality class, where $p_{\rm eff} = p$ and $\delta = 1$ (Fig. 5). *Models C* and *D* belong to the KPZ universality class. *Model C* is an example of DOLB growth with voids, with a 53.2% void density in the bulk when $p = 1$, and in this case $\delta \approx 0.41 < 1$. *Model D* is a DOLB-type growth that produces a compact bulk but component *2* is flux non-conserving, and here $\delta \approx 1$. Extensions of *Models A* and *C* to $(1 + n)$ dimensions [21], $n = 2, 3$, yield results that conform to our theoretical predictions of $p_{\rm eff} \approx p^{\delta}$ with $\delta \neq 1$ for mixing RD with DOLB processes, and $\delta = 1$ for mixing RD with ABC processes. Additional examples include cases [28] when component *2* is a restricted Kim-Kosterlitz solid-on-solid model [43] (where RD+X is in KPZ universality class), and when it simulates a conserved restricted SOS growth of Kim *et al* [44]. In the latter case the process RD+X is in the Villain-Lai-Das Sarma universality class [24, 45].

An interesting lattice simulation model has been recently considered by Banerjee *et al* [35] in an attempt to describe a realistic sedimentation. The Banerjee *et al* model is a competitive growth process that has three component processes: one RD process and two DOLB processes, where one DOLB process is BD with the NN sticking rule and the other DOLB process is BD with the NNN sticking rule. In the language of our study, the overall process is the RD+X process, where X=X1+X2 and, as the convex linear combination of two KPZ processes, X is in KPZ universality class. Accordingly, this system should obey the scaling law of Eq. (18) in the effective probability of the combined process X. Time-evolution plots of the surface roughness in Ref. [35] suggest such scaling.

The extension of the approach presented here to other competitive growth processes may provide a tool to understand the observed dynamics of surface growth. Realistic systems may involve many component-processes, some of which may be dominant. Within our formalism

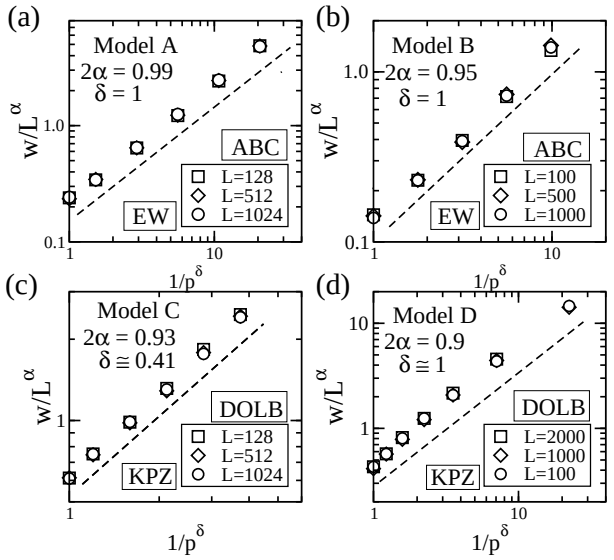

FIG. 5: Scaled widths at saturation w vs the parameter $1/p^{\delta}$: (a) and (b) are for *Models A* and *B*, respectively; (c) and (d) are for *Models C* and *D*, respectively. Reference lines have slope 1. Data are scaled with the α values shown.

a departure point may be a generalization of Eq. (9):

$$\Delta h(\vec{x}, t) = \sum_k p_{\rm eff}^{(k)} \Delta h^{(k)}(\vec{x}, t)\,, \qquad (21)$$

where the summation is over all contributing processes, and $\Delta h^{(k)}$ is the column-height increment due to the $k^{\rm th}$ process. In a first approximation, component processes are not explicitly correlated. Each process is encountered with the activation probability or frequency p_k, $\sum_k p_k = 1$, and contributes to the growth with an effective probability $p_{\rm eff}^{(k)}$, $\sum_k p_{\rm eff}^{(k)} = 1$. In the trivial case when all components are ABC-type models with the unit mean deposit height we have $p_{\rm eff}^{(k)} = p_k$. For a DOLB-type growth the $p_{\rm eff}^{(k)}$ will have to be determined. Depending on the model, $p_{\rm eff}^{(k)}$ can be estimated by analyzing the growth when process k acts alone, and measuring either the mean bulk density or the mean fraction of the detached material or both [41]. Simplectic decompositions like the one proposed in Eq. (21) have a long history of applications in many diverse fields and are the precursors of probability measures.

Stochastic theory of multi-component competitive far-from-equilibrium surface-growth processes is a newly emerging topic in statistical physics. During the recent two decades, in addition to model-specific simulation studies of two-component (either RD+EW or RD+KPZ lattice) growth models, a special case of RD+RD has been considered both in a theoretical mean-field approach and in simulations [46]. The absence of a consistent continuum theory for the RD+X mix hindered scale-invariance studies of more complex systems such as EW+EW or KPZ+KPZ or EW+KPZ, or more general three-component systems (such as, e.g., those in

Ref. [35]). Understanding the dynamics of real physical systems calls for more realistic models that would go beyond a one-process theory of kinetic roughening. For example, in realistic modeling of ion-bombardment experiment, where shadowing effects matter, the mechanism of ion desorption must be (at least) accompanied by ion diffusion along the substrate as well as by random deposition. The construction of a continuum stochastic growth equation, outlined in this article, will be helpful for future studies of two- and three-component competitive non-equilibrium growth systems.

V. CONCLUSION

In summary, we have derived continuum stochastic-growth equations and the resulting scaling for competitive RD+X growth processes. The RD+X growth processes show that model-dependent prefactors in universal scaling laws can be linked with the bulk morphology and determined from bulk structures. This necessitates the distinction between the adsorption-bulk-compact (ABC) and the dense-or-lace-bulk (DOLB) growth processes in dynamic-scaling analysis of competitive mixed-growth models. For competitive systems, the activation probability of process X, i.e., its frequency in the RD+X mix, alone does not provide sufficient information to correctly describe their dynamics. The essential physical non-universal parameter here is the effective probability, $p_{\rm eff}$, of X. The bulk morphology allows one to obtain $p_{\rm eff}$ for either experimental or computational studies. Furthermore, $p_{\rm eff}$ is parameterized by the activation probability.

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[1] A. Brú, S. Albertos, J. L. Subiza, J. L. Garcia-Asenjo, and I. A. Brú, Biophys. J. **85**, 2948 (2003).
[2] M. A. C. Huergo, N. E. Muzzio, M. A. Pasquale, P. H. González, A. E. Bolzán, and A. J. Arvia, Phys. Rev. E **90**, 022706 (2014).
[3] P. Verma, M. D. Mager, and N. A. Melosh, Phys. Rev. E **89**, 012404 (2014).
[4] J. Soriano, A. Mercier, R. Planet, A. Hernández-Machado, M. A. Rodriguez, and J. Ortin, Phys. Rev. Lett. **95**, 104501 (2005).
[5] A. S. Balankin and D. A. Matamoros, Phys. Rev. E **71**, 056102 (2005).
[6] A. C. Dürr, F. Schreiber, K. A. Ritley, V. Kruppa, J. Krug, H. Dosch, and B. Struth, Phys. Rev. Lett. **90**, 016104 (2003).
[7] A. S. Balankin, Phys. Rev. E **76**, 056120 (2007).
[8] G. Korniss, M. A. Novotny, H. Guclu, Z. Toroczkai, and P. A. Rikvold, Science **299**, 677 (2003).
[9] A. Kolakowska and M. A. Novotny, in *Artificial Intelligence and Computer Science* ed. by S. Shannon (Nova Science Publishers, New York, 2005) p.151.
[10] P. Pellegrini and R. Jullien, Phys. Rev. Lett. **64**, 1745 (1990).
[11] M. Ausloos, N. Vandewalle, and R. Cloots, Europhys. Lett. **24**, 629 (1993).
[12] W. Wang and H. A. Cerdeira, Phys. Rev. E **47**, 3367 (1993).
[13] M. Kortla, F. Slanina, and M. Predota, Phys. Rev. B **58**, 10003 (1998).
[14] B. Drossel and M. Kardar, Phys. Rev. Lett. **85**, 614 (2000).
[15] H. F. El-Nashar and H. A. Cerdeira, Phys. Rev. E **61**, 6149 (2000).
[16] T. J. Da Silva and J. G. Moreira, Phys. Rev. E **63**, 041601 (2001).
[17] C. M. Horowitz, R. A. Monetti, and E. V. Albano, Phys. Rev. E **63**, 066132 (2001).
[18] C. M. Horowitz and E. V. Albano, J. Phys. A: Math. Gen. **34**, 357 (2001).
[19] A. Chame and F. D. A. Aarão Reis, Phys. Rev. E **66**, 051104 (2002).
[20] F. D. A. Aarão Reis, Phys. Rev. E **66**, 027101 (2002).
[21] C. M. Horowitz and E. V. Albano, Eur. Phys. J. B **31**, 563 (2003).
[22] A. Kolakowska, M. A. Novotny, and P. S. Verma, Phys. Rev. E **70**, 051602 (2004).
[23] D. Muraca, L. A. Braustein, and R. C. Bruceta, Phys. Rev. E **69**, 065103(R) (2004).
[24] F. D. A. Aarão Reis, Phys. Rev E **70**, 031607 (2004).
[25] C.-C. Chien, N.-N. Pang, and W.-J. Tzeng, Phys. Rev. E **70**, 021602 (2004).
[26] L. A. Braunstein and C.-H. Lam, Phys. Rev E **72**, 026128 (2005).
[27] I. Irurzun, C. M. Horowitz, and E. V. Albano, Phys. Rev. E **72**, 036116 (2005).
[28] F. D. A. Aarão Reis, Phys. Rev. E **73**, 021605 (2006).
[29] T. J. Oliveira, K. Dechoum, J. A. Redinz, and F. D. A. Aarão Reis, Phys. Rev. E **74**, 011604 (2006).
[30] A. Kolakowska, M. A. Novotny, and P. S. Verma, Phys. Rev. E **73**, 011603 (2006).
[31] C. M. Horowitz and E. V. Albano, Phys. Rev. E **73**, 031111 (2006).
[32] A. Kolakowska and M. A. Novotny, Phys. Rev. E **81**, 033101 (2010).
[33] Y.-L Chou and M. Pleimling, Phys. Rev. E **79**, 051605 (2009).
[34] A. Kolakowska, Phys. Rev. E, in press (2014).
[35] K. Banerjee, J. Shamanna, and S. Ray, Phys. Rev. E **90**, 022111 (2014).
[36] J. Krug, in *Scale Invariance, Interfaces, and Non-*

Equilibrium Dynamics, ed. A. McKane, M. Droz, J. Vannimenus, and D. Wolf, NATO ASI Series **344**, 1-61 (Springer US, 1995).
[37] G. Korniss *et. al.*, Phys. Rev. Lett. **84**, 1351 (2000).
[38] M. Kardar, G.Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[39] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London, Ser. A **381**, 17 (1982).
[40] F. Family and T. Vicsek, J. Phys. A **18**, L75 (1985).
[41] Details will be given in a later publication.
[42] A.-L. Barabasi and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, 1995).
[43] J. M. Kim and J. M. Kosterlitz, Phys. Rev. Lett. **62**, 2289 (1989).
[44] Y. Kim, D. K. Park, and J. M. Kim, J. Phys. A: Math. Gen. **27**, L533 (1994).
[45] J. Villain, J. Phys. I **1**, 19 (1991); Z.-W. Lai and S. Das Sarma, Phys. Rev. Lett. **66**, 2348 (1991).
[46] M. A. Novotny and A. Kolakowska, Int. J. Mod. Phys. C **20**, 1377 (2009).