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Successive Phase Transitions and Kink Solutions in ϕ^8, ϕ^{10} and ϕ^{12} Field Theories

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We obtain exact solutions for kinks in ϕ^8, ϕ^{10} and ϕ^{12} field theories with degenerate minima, which can describe a second-order phase transition followed by a first-order one, a succession of two first-order phase transitions and a second-order phase transition followed by two first-order phase transitions, respectively. Such phase transitions are known to occur in ferroelastic and ferroelectric crystals and in meson physics. In particular, we find that the higher-order field theories have kink solutions with algebraically-decaying tails and also asymmetric cases with mixed exponential-algebraic tail decay, unlike the lower-order ϕ^4 and ϕ^6 theories. Additionally, we construct distinct kinks with equal energies in all three field theories considered, and we show the co-existence of up to three distinct kinks (for a ϕ^{12} potential with six degenerate minima). We also summarize phonon dispersion relations for these systems, showing that the higher-order field theories have specific cases in which only nonlinear phonons are allowed. For the ϕ^{10} field theory, which is a quasi-exactly solvable (QES) model akin to ϕ^6, we are also able to obtain three analytical solutions for the classical free energy as well as the probability distribution function in the thermodynamic limit.

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I. INTRODUCTION

First- and second-order phase transitions are usually modeled by ϕ^6 and ϕ^4 field theories, respectively [1]. An asymmetric double well in ϕ^4 field theory can also describe first-order transitions [2]. However, if one has to capture *all* symmetry-allowed phases in a low-dimensional phase transition [3] or describe a succession of phase transitions, then one has to consider either multi-component field theories [4] or higher-than-sixth-order single-component field theories [5, 6]. For example, it is well known [3, 7] that while ϕ^8 field theory can describe a second-order phase transition followed by a first-order phase transition, one has to go to ϕ^{10} field theory to describe a succession of two first-order phase transitions. Indeed, there are examples of crystals undergoing two successive (ferroelastic and ferroelectric) first-order phase transitions [8]. The ϕ^8 field theory has also been used to model massless mesons with long-range interactions [5] as well as isostructural phase transitions [9]. Similarly, the ϕ^{10} field theory has been used in the study of crystallization of chiral proteins [10]. Meanwhile, the ϕ^{12} field theory has been invoked to describe the phenomenology of phase transitions in highly piezoelectric perovskite materials [11, 12].

The study of kinks (also known as *topological solitons* [13]) and domain walls in classical and quantum field theories [14, 15], in theories of gravity and cosmology [16, 17] and even in the nonlinear field theories of fluid mechanics [18] remains a topic of active research. Similarly, Ginzburg–Landau theories [19, 20] have been very successful in explaining superconducting, superfluid and many other transitions as well as in modeling topological defects (e.g., vortices and domain walls) in a variety of functional materials, through the inclusion of the gradient of the relevant order parameter in the free energy.

In this context, solitary wave solutions of some special octic potentials have been presented before [21, 22]. Similarly, generic properties of kink solutions of certain field theories with polynomial self-interaction have been studied previously [23–25]. However, to the best of our knowledge, the various kink solutions of the ϕ^8, ϕ^{10} and ϕ^{12} field theories with degenerate minima have not been studied systematically (and neither has the corresponding statistical mechanics). The purpose of this work is to provide such solutions. In addition, we show that as in ϕ^6 field theory (but unlike ϕ^4 field theory), it is possible to obtain an exact expression for the classical free energy and probability distribution function (PDF) [26, 27] at a given temperature in the thermodynamic limit of the ϕ^{10} field theory. This is related to the fact that the Schrödinger equation with a ϕ^{4n} (e.g., ϕ^8, ϕ^{12}) potential is not analytically solvable, whereas with a ϕ^{4n+2} (e.g., ϕ^6, ϕ^{10}) potential it is quasi-exactly solvable (QES) [28].

II. ϕ^8 FIELD THEORY

Throughout this paper, we refer to $\phi = \phi_e$ as an *equilibrium* value if $V(\phi_e) = 0$, by *degenerate extremum* we mean $\phi = \phi_e$ such that $V(\phi_e) = V'(\phi_e) = 0$, all potentials are assumed symmetric, i.e., $V(-\phi) = V(\phi)$, and we use Planck units ($m = c = \hbar = 1$) to simplify the notation.

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First, we will discuss the general picture, describing different possible phases for a generic potential V, then we will discuss the kink solutions in the various phases.

A. The Various Phases

The ϕ^8 potential (free energy) is given, generically, by

$$V(\phi) = \lambda^2(\phi^8 - \alpha_6\phi^6 + \alpha_4\phi^4 - \alpha_2\phi^2 + \alpha_0), \quad (1)$$

where, without loss of generality, the coefficient of the ϕ^8 term is set to $+1$ in units of λ^2. The coefficients of ϕ^6, ϕ^4 and ϕ^2 are, in general, arbitrary and there are eight different possibilities, depending on whether all three, two, one or none of the coefficients are positive. However, if one wants to consider a model describing a second-order transition followed by a first-order transition, then one must take $\alpha_{6,4,2} > 0$ in (1). Additionally, a particular choice of α_0 ensures that the minimum value of the potential is zero, i.e., $\min_\phi V(\phi) = 0$.

While the potential is determined by three parameters ($\alpha_{6,4,2}$), one can show, by using scaling arguments, that only two of them are independent. In a Landau-type theory, the coefficients $\alpha_{6,4,2}$ have some dependence on the thermodynamic temperature T. Thus, at the first-order phase transition point $T = T_c^I$, V has four degenerate minima, and the coefficients α_6, α_4 and $\alpha_2 = \alpha_2^c$ are related by

$$\alpha_4 = \frac{\alpha_6^2}{4} + \frac{2\alpha_2^c}{\alpha_6}. \quad (2)$$

In particular, if the four degenerate minima are at $\phi = \pm a, \pm b$, then the potential at $T = T_c^I$ has the factorized form

$$V(\phi) = \lambda^2(\phi^2 - a^2)^2(\phi^2 - b^2)^2. \quad (3)$$

Without loss of generality, we choose $b > a$ throughout this paper, unless otherwise specified. On comparing (1) and (3), and enforcing $\min_\phi V(\phi) = 0$, it is clear that the relationship between $\alpha_{6,4,2,0}$ and a, b is given by

$$\begin{aligned} \alpha_6 &= 2(b^2 + a^2), \\ \alpha_4 &= (b^4 + a^4 + 4a^2b^2), \\ \alpha_2 &= \alpha_2^c = 2a^2b^2(b^2 + a^2), \\ \alpha_0 &= a^4b^4. \end{aligned} \quad (4)$$

Clearly, $\alpha_{6,4,2,0} > 0$. In this case, one can also show that the potential has maxima at $\phi = 0, \pm\sqrt{(b^2 + a^2)/2}$. From (4), we also find that α_4/α_6^2 is constrained to satisfy the inequality

$$\frac{1}{4} < \frac{\alpha_4}{\alpha_6^2} < \frac{3}{8}. \quad (5)$$

Now, what happens as T is slowly increased from T_c^I? It is easily shown that, in this model, keeping α_6, α_4 fixed and decreasing α_2 (from its value α_2^c at $T = T_c^I$), T goes above T_c^I. As soon as T is slightly greater than T_c^I, the potential has two degenerate absolute minima at $\phi = \pm\hat{a}$, where $0 < \hat{a} < a$. Furthermore, there are now two local minima at $\phi = \pm\hat{b}$, where $0 < \hat{b} < b$, and there are three maxima including one at $\phi = 0$. As the temperature is further increased, the degenerate absolute minima at $\phi = \pm a$ persist until the onset of the second-order transition at $T = T_c^{II}$, which corresponds to $\alpha_2 = 0$. Beyond this point ($T > T_c^{II}$), the absolute minimum is now at $\phi = 0$, not at $\phi = \pm a$. Meanwhile, it can be shown that, as long as $1/4 < \alpha_4/\alpha_6^2 < 9/32$, there are local minima at $\phi = \pm\hat{b}$ even at $T = T_c^{II}$ (i.e., $\alpha_2 = 0$); if $\alpha_4/\alpha_6^2 = 9/32$, then for $\alpha_2 = 0$, there are inflection points at $\phi = \pm\hat{b}$. However, if $9/32 < \alpha_4/\alpha_6^2 < 3/8$, then, as the temperature is slowly increased from T_c^I, the local minima at $\phi = \pm\hat{b}$ disappear even before the second order transition point $T = T_c^{II}$ (i.e., $\alpha_2 = 0$) is reached.

Let us now discuss what happens as temperature is lowered from T_c^I, i.e., α_2 is increased from its value α_2^c at T_c^I (while keeping α_6, α_4 fixed). As soon as T is slightly less than T_c^I, the potential has degenerate absolute minima at $\phi = \pm\hat{b}$, where $\hat{b} > b$, while there are degenerate local minima at $\phi = \pm\hat{a}$, where $\hat{a} > a$, and three maxima including one at $\phi = 0$. Finally, beyond a critical point, the local minima at $\phi = \pm\hat{a}$ disappear, and the potential only has absolute minima at $\phi = \pm\hat{b}$ and a maximum at $\phi = 0$. This picture persists no matter how much further the temperature is lowered (i.e., α_2 is increased). For example, it is easily shown that at $\alpha_2 = 2\alpha_2^c$ (for given α_6, α_4), the potential (1) can be written as

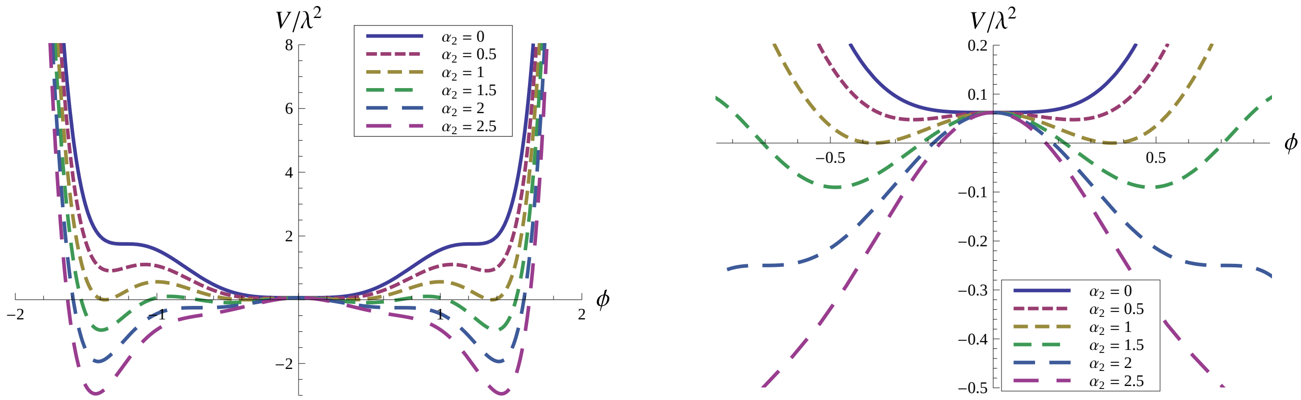
$$V(\phi) = \lambda^2[\phi^2 - (b^2 + a^2)]^2[\phi^4 + 2a^2b^2]. \quad (6)$$

Hence, at $\alpha_2 = 2\alpha_2^c$, V has absolute minima at $\phi = \pm\sqrt{b^2 + a^2}$, a maximum at $\phi = 0$ and *no* local minima as long as $(b^2 + a^2)^2 < 16a^2b^2$. Using (4), it follows that the local minima at $\phi = \pm a$ disappear for some value of $\alpha_2 < 2\alpha_2^c$ if $9/32 < \alpha_4/\alpha_6^2 < 3/8$. On the other hand, if $1/4 < \alpha_4/\alpha_6^2 < 9/32$, then V has local minima at $\phi = \pm\hat{a}$ with $a < \hat{a}$, while for $\alpha_4/\alpha_6^2 = 9/32$, V has inflection points at $\phi = \pm\sqrt{(b^2 + a^2)/2}$.

As an illustration, consider the potential

$$V(\phi) = \lambda^2[\phi^8 - 4\phi^6 + (9/2)\phi^4 - \alpha_2\phi^2 + (1/16)] \quad (7)$$

for various values of the parameter α_2. For $\alpha_2 = \alpha_2^c = 1$ this potential has four degenerate minima with $b^2 + a^2 = 2$ and $b^2 - a^2 = \sqrt{3}$ [see (3) and (4)], hence this case corresponds to the first-order phase transition at $T = T_c^I$. Furthermore, for the potential in (7), $\alpha_4 = (9/32)\alpha_6^2$, hence at $T = T_c^{II}$ (i.e., $\alpha_2 = 0$), V has an absolute minimum at $\phi = 0$ and inflection points at $\phi = \pm\sqrt{(3/4)(b^2 + a^2)} = \pm\sqrt{3/2}$. Similarly, for $\alpha_2 = 2\alpha_2^c = 2$, there are absolute minima at $\phi = \pm\sqrt{2}$, a maximum at $\phi = 0$ and points of inflection at $\phi = \pm\sqrt{(b^2 + a^2)/2} = \pm 1$. Thus, for $0 < \alpha_2 < 1$, the potential (7) has absolute minima at $\phi = \pm\hat{a}$, local minima at $\phi = \pm\hat{b}$ and three maxima,



(a) (b)

FIG. 1: (Color online.) (a) Example potentials of the form (7) for various illustrative values of the coefficient of the quadratic term, α_2, showing the various phases and phase transitions in the ϕ^8 theory. (b) Zoom-in of (a) near the origin.

including one at $\phi = 0$. Similarly, for $1 < \alpha_2 < 2$, the potential has absolute minima at $\phi = \pm\hat{b}$, local minima at $\phi = \pm\hat{a}$ and three maxima, including one at $\phi = 0$. For $\alpha_2 < 0$, the potential has a single minimum at $\phi = 0$, while for $\alpha_2 > 2$, the potential has degenerate minima at $\phi = \pm b$, a maximum at $\phi = 0$ and no local minima. In Fig. 1, we show plots of the example potential (7) for $\alpha_2 = 0, 0.5, 1, 1.5, 2, 2.5$ (in units of λ^2) to illustrate its structure.

B. Four Degenerate Minima

1. $T = T_c^I$

At the first-order phase transition, i.e., $T = T_c^I$, the potential can always be written in the form (3) with a, b and $\alpha_{6,4,2}$ being related by (2). Since there are four degenerate minima, we expect two different kinds of kinks, one connecting a to b (or, equivalently, $-b$ to $-a$) and another connecting $-a$ to $+a$, as x goes from $-\infty$ to $+\infty$. In general, these kinks have different energies. Notice that if a kink goes from a to b, as x goes from $-\infty$ to $+\infty$, then the kink's energy is given by

$$E_k = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right] = \int_a^b d\phi\, \sqrt{2V(\phi)}, \tag{8}$$

where the last equality follows from the first integral of the equation of motion, i.e., $d\phi/dx = \sqrt{2V(\phi)}$ [5, 30]; again, without loss of generality, it is assumed that $\min_\phi V(\phi) = 0$, i.e., $V(\phi) \geq 0$ for all ϕ, which is always true when kink solutions exist.

From the first integral of the equation of motion [31], the shape of the kink can be found by quadrature:

$$\sqrt{2}\lambda x = \int \frac{d\phi}{\sqrt{(a^2-\phi^2)^2(b^2-\phi^2)^2}}. \tag{9}$$

As mentioned above, there are two kinds of kinks to be considered, which leads to two possible choices in the branch cut of the square root in (9). Let us consider the two cases separately.

a. Kink connecting $-a$ to $+a$ In this case, $|\phi| < a$ and $b > a$ by convention, hence (9) becomes

$$\sqrt{2}\lambda x = \int \frac{d\phi}{(a^2-\phi^2)(b^2-\phi^2)}. \tag{10}$$

The integral is evaluated using partial fractions to obtain the implicit solution, which was also found by Lohe [5, Eq. (63)]:

$$e^{\mu x} = \left(\frac{a+\phi}{a-\phi}\right)\left(\frac{b-\phi}{b+\phi}\right)^{a/b}, \tag{11}$$

where $\mu = 2\sqrt{2}\lambda a(b^2 - a^2)$. The approach to the asymptotes at $\phi = \pm a$ can be shown to be exponential from (11):

$$\phi(x) \simeq \begin{cases} -a + 2a\left(\frac{b-a}{b+a}\right)^{a/b} e^{\mu x}, & x \to -\infty, \\ +a - 2a\left(\frac{b-a}{b+a}\right)^{a/b} e^{-\mu x}, & x \to +\infty, \end{cases} \tag{12}$$

from which it follows that this kink is symmetric. The corresponding kink energy is obtained using (8):

$$E_k^{(1)} = \frac{4\sqrt{2}}{15}\lambda a^3(5b^2 - a^2). \tag{13}$$

b. Kink connecting a to b (or $-b$ to $-a$) In this case, $a < \phi < b$ and $b > a$ by convention, hence (9) takes the form

$$\sqrt{2}\lambda x = \int \frac{d\phi}{(\phi^2-a^2)(b^2-\phi^2)}. \tag{14}$$

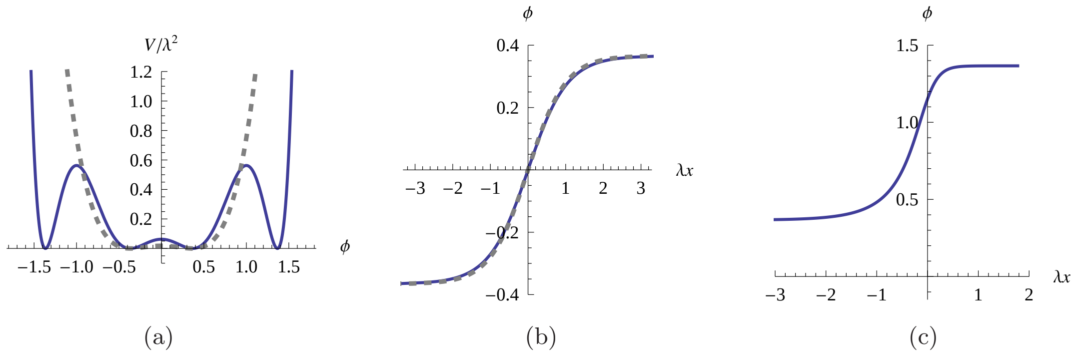


FIG. 2: (Color online.) ϕ^8 field theory at the first-order phase transition, $T = T_c^I$. (a) Example ϕ^8 potential with four degenerate minima (3) and a representative ϕ^4 potential $V(\phi) = \lambda^2(\phi^2 - a^2)^2$ superimposed as the dotted curve. (b) Kink solution (11) connecting $-a$ to $+a$ and the generic ϕ^4 kink $\phi(x) = \frac{1}{2}(\phi_{+\infty} + \phi_{-\infty}) + \frac{1}{2}(\phi_{+\infty} - \phi_{-\infty})\tanh(\lambda x) = a\tanh(\lambda x)$ for $\phi_{\pm\infty} = \pm a$ superimposed as the dotted curve. (c) Kink solution (15) connecting a to b. In all panels, $a = (-1+\sqrt{3})/2$ and $b = (1+\sqrt{3})/2$.

The integral is again evaluated using partial fractions to obtain the implicit solution

$$e^{\mu x} = \left(\frac{\phi - a}{\phi + a}\right)\left(\frac{b+\phi}{b-\phi}\right)^{a/b}, \tag{15}$$

where $\mu = 2\sqrt{2}\lambda a(b^2 - a^2)$ as before. The approach to the asymptotes at $\phi = a, b$ can be shown to be exponential from (15):

$$\phi(x) \simeq \begin{cases} a + 2a\left(\dfrac{b-a}{b+a}\right)^{a/b} e^{\mu x}, & x \to -\infty, \\ b - 2b\left(\dfrac{b-a}{b+a}\right)^{b/a} e^{-\mu x b/a}, & x \to +\infty. \end{cases} \tag{16}$$

Note, however, that the rate at which ϕ asymptotes to a is given by μ, while the rate at which ϕ asymptotes to b is given by $\mu b/a$, hence this kink is asymmetric. The kink's energy is

$$E_k^{(2)} = \frac{2\sqrt{2}}{15}\lambda(b-a)^3(b^2 + 3ab + a^2). \tag{17}$$

Comparing the energies of the two kink solutions [(13) and (17)], we find that $E_k^{(1)} \gtreqless E_k^{(2)}$ if $b/a \lesseqgtr 2/(3-\sqrt{5})$. In particular, for $b/a = 2/(3-\sqrt{5})$, the two kinks have equal energies. It would be of interest to study the interaction between two kinks of the same type as well as two kinks of different types in the case when their energies are equal.

As an illustration, consider the potential (3) with $a^2 + b^2 = 2$ and $a^2b^2 = 1/4$ so that $\alpha_2^c = 1$. This leads to eight possible pairs (a, b) with four of them satisfying $b^2 > a^2$. Without loss of generality, we also take $a > 0$ and $b > 0$, hence $a = (-1+\sqrt{3})/2$ and $b = (1+\sqrt{3})/2$. Figure 2 shows the potential (3) and the two kink solutions (11) and (15). The kink solution from (15) is clearly asymmetric, consistent with the asymptotic behaviors given in (16).

2. $T_c^I < T < T_c^{II}$

For temperatures above the first-order phase transition, the potential (1) can be written as

$$V(\phi) = \lambda^2(\phi^2 - \hat{a}^2)^2[\phi^4 - d\phi^2 + e], \quad d^2 < 4e, \quad \hat{a} < a, \tag{18}$$

and there exists a kink solution connecting the two degenerate minima at $\phi = \pm\hat{a}$, as x goes from $-\infty$ to $+\infty$. As an illustration, consider the potential (7) with $\alpha_2 = 121/128$. In this case, (18) takes the form

$$V(\phi) = \lambda^2[\phi^2 - (1/8)]^2[\phi^4 - (15/4)\phi^2 + (227/64)]. \tag{19}$$

3. $T < T_c^I$

For temperatures below the first-order phase transition, the potential (1) can be written as

$$V(\phi) = \lambda^2(\phi^2 - \hat{b}^2)^2[\phi^4 - d\phi^2 + e], \quad d^2 < 4e, \quad b < \hat{b}, \tag{20}$$

and there exists a kink solution connecting the two degenerate minima at $\phi = \pm\hat{b}$, as x goes from $-\infty$ to $+\infty$. As an illustration, consider the potential with $\alpha_2 = 135/128$. In this case, (20) takes the form

$$V(\phi) = \lambda^2[\phi^2 - (15/8)]^2[\phi^4 - (1/4)\phi^2 + (3/64)]. \tag{21}$$

These kink solutions for $T \gtrless T_c^I$ are illustrated in Fig. 3. Notice that for the case $T < T_c^I$ (dashed kink in right panel of Fig. 3), the kink "feels" the influence of the two local minima at $\phi = \pm\frac{1}{4}\sqrt{3(3-\sqrt{5})} \approx \pm 0.378$, similarly to kinks in certain cases of ϕ^6 field theory [32].

C. Three Degenerate Minima

A ϕ^8 potential with three degenerate minima can have two possible forms. In each case, there exist two kink so-

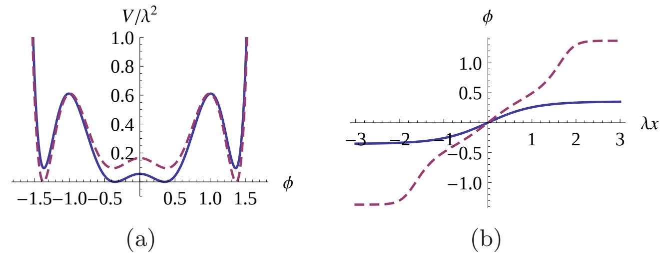


FIG. 3: (Color online.) Away from the first-order phase transition, $T \neq T_c^I$, in ϕ^8 field theory. (a) The potentials (19) (solid, $T > T_c^I$) and (21) (dashed, $T < T_c^I$). (b) The corresponding kinks computed by solving the equation of motion $d\phi/dx = \sqrt{2V(\phi)}$ numerically subject to the symmetry condition $\phi(0) = 0$.

lutions, only one of which is distinct due to the symmetry of the potential.

1. *Case I: $\alpha_2 = 0$*

Let

$$V(\phi) = \lambda^2\phi^4(\phi^2 - a^2)^2, \tag{22}$$

which has degenerate minima at $\phi = 0, \pm a$. In this case, $\alpha_{6,4} > 0$ while $\alpha_{2,0} = 0$. The corresponding kink solution (connecting 0 to a or $-a$ to 0, as x goes from $-\infty$ to $+\infty$), which was also obtained by Lohe [5, Eq. (67)], is given implicitly by

$$\mu x = -\frac{2a}{\phi} + \ln\left(\frac{a+\phi}{a-\phi}\right), \tag{23}$$

where $\mu = 2\sqrt{2}\lambda a^3$. [It should be noted that there is a typographical error in [5, Eq. (67)] that is evident upon comparison with (23).] From (23), the approach to the asymptotes at $\phi = 0, a$ can be shown to be

$$\phi(x) \simeq \begin{cases} -\dfrac{2a}{\mu x}, & x \to -\infty, \\ a - \dfrac{2a}{e^2}e^{-\mu x}, & x \to +\infty. \end{cases} \tag{24}$$

Note that this kink is asymmetric because the asymptotics as $x \to \pm\infty$ differ, specifically the kink decays as $1/x$ as $x \to -\infty$, while it approaches $\phi = a$ exponentially. The corresponding kink energy is

$$E_k = \frac{2\sqrt{2}}{15}\lambda a^5. \tag{25}$$

2. *Case II: $\alpha_2 < 0$*

Let

$$V(\phi) = \lambda^2\phi^2(\phi^2 - a^2)^2(\phi^2 + b^2), \tag{26}$$

which has degenerate minima at $\phi = 0, \pm a$. Note that, in this case,

$$\begin{aligned} \alpha_6 &= b^2 - 2a^2, \\ \alpha_4 &= a^2(a^2 - 2b^2), \\ \alpha_2 &= -a^4b^2, \\ \alpha_0 &= 0. \end{aligned} \tag{27}$$

It can be shown that $\alpha_6 > 0$ as long as $\sqrt{2}a > b > a$, while $\alpha_{4,2} < 0$. The corresponding kink solution (connecting 0 to a or $-a$ to 0 as x goes from $-\infty$ to $+\infty$) is given implicitly by

$$e^{\mu x} = \left(\frac{\sqrt{b^2+\phi^2} - b}{\sqrt{b^2+\phi^2} + b}\right)^{\sqrt{b^2+a^2}/b} \times \left(\frac{\sqrt{b^2+a^2} + \sqrt{b^2+\phi^2}}{\sqrt{b^2+a^2} - \sqrt{b^2+\phi^2}}\right), \tag{28}$$

where $\mu = 2\sqrt{2}\lambda a^2\sqrt{b^2+a^2}$. From (28), the approach to the asymptotes at $\phi = 0, a$ can be shown to be

$$\phi(x) \simeq \begin{cases} 2b\left[1 + \frac{2b}{a^2}(b+\sigma)\right]^{-b/(2\sigma)} e^{\mu x b/(2\sigma)}, & x \to -\infty, \\ a - \frac{2\sigma}{a}\left[1 + \frac{2b}{a^2}(b-\sigma)\right]^{\sigma/b} e^{-\mu x}, & x \to +\infty, \end{cases} \tag{29}$$

where $\sigma = \sqrt{b^2+a^2}$. Note the differing rates $\mu b/(2\sigma)$ and μ at which the asymptotes at $x \to \mp\infty$, respectively, are approached, hence this kink is asymmetric in general. The corresponding kink energy is

$$E_k = \frac{\sqrt{2}}{15}\lambda\left[2(b^2+a^2)^{5/2} - b^3(2b^2+5a^2)\right]. \tag{30}$$

As an illustration, consider $a = 3/4$ and $b = 1$. This kink, as well as the one from the previous subsubsection, are illustrated in Fig. 4. Note that, in both cases, the kinks are asymmetric as shown by the asymptotic expressions given in (24) and (29). For Case I, the mismatch between the ϕ^8 and the ϕ^6 kink is mainly due to the slow algebraic decay (as $x \to -\infty$) of the tail of the kink, see (24).

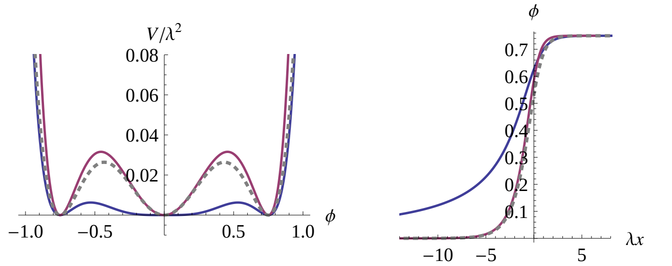
D. Two Degenerate Minima

A ϕ^8 potential with two degenerate minima can have two possible forms. In each case, there exist a kink solution connecting the degenerate minima at $\phi = \pm a$, as x goes from $-\infty$ to $+\infty$.

1. $\alpha_2 = 0$

Let

$$V(\phi) = \lambda^2(\phi^2 - a^2)^4, \tag{31}$$



(a) (b)

FIG. 4: (Color online.) ϕ^8 field theory with three degenerate minima. (a) The potentials (22) (bottom curve, blue online), (26) (top curve, red online), and a representative ϕ^6 potential $V(\phi) = \lambda^2\phi^2(\phi^2 - a^2)^2$ (dotted). (b) The kink solutions (23) (top curve, blue online) and (28) (bottom curve, red online) connecting 0 to a, and the corresponding ϕ^6 kink $\phi(x) = a/\sqrt{1 + e^{-2\sqrt{2}a^2\lambda x}}$ (dotted). In all panels, $a = 3/4$ and $b = 1$.

which has degenerate minima at $\phi = \pm a$. Note that in this case $\alpha_{6,4,2,0} > 0$. The kink solution is given implicitly by

$$\mu x = \frac{2a\phi}{a^2 - \phi^2} + \ln\left(\frac{a+\phi}{a-\phi}\right), \tag{32}$$

where $\mu = 4\sqrt{2}\lambda a^3$. From (32), the approach to the asymptotes at $\phi = \pm a$ can be shown to be algebraic:

$$\phi(x) \simeq \begin{cases} -a - \dfrac{a}{\mu x}, & x \to -\infty, \\ +a - \dfrac{a}{\mu x}, & x \to +\infty, \end{cases} \tag{33}$$

from which it follows that this kink is symmetric. The corresponding kink energy is

$$E_k = \frac{16\sqrt{2}}{15}\lambda a^5. \tag{34}$$

2. $\alpha_2 > 0$

Let

$$V(\phi) = \lambda^2(\phi^2 - a^2)^2(\phi^2 + b^2)^2, \tag{35}$$

which has degenerate minima at $\phi = \pm a$. In this case,

$$\begin{aligned} \alpha_6 &= 2(b^2 - a^2), \\ \alpha_4 &= b^4 - 4a^2b^2 + a^4, \\ \alpha_2 &= 2a^2b^2(b^2 - a^2), \\ \alpha_0 &= a^4b^4. \end{aligned} \tag{36}$$

Clearly, $\alpha_{6,2,0} > 0$ for $b > a$, while $\alpha_4 > 0$ as long as $b\sqrt{2 - \sqrt{3}} > a$.

The kink solution is given implicitly by

$$\mu x = \frac{2a}{b}\tan^{-1}\left(\frac{\phi}{b}\right) + \ln\left(\frac{a+\phi}{a-\phi}\right), \tag{37}$$

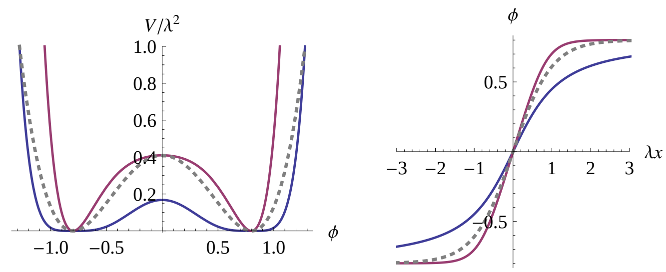
where $\mu = 2\sqrt{2}\lambda a(b^2 + a^2)$. From (37), the approach to the asymptotes at $\phi = \pm a$ can be shown to be exponential:

$$\phi(x) \simeq \begin{cases} -a + 2a\, e^{\mu x + (2a/b)\tan^{-1}(a/b)}, & x \to -\infty, \\ +a - 2a\, e^{-\mu x + (2a/b)\tan^{-1}(a/b)}, & x \to +\infty, \end{cases} \tag{38}$$

from which it follows that this kink is symmetric. The corresponding kink energy is

$$E_k = \frac{4\sqrt{2}}{15}\lambda a^3(a^2 + 5b^2). \tag{39}$$

This kink, as well as the one from the previous subsubsection, are illustrated in Fig. 5.



(a) (b)

FIG. 5: (Color online.) ϕ^8 field theory with two degenerate minima. (a) The potentials (31) (bottom curve, blue online), (35) (top curve, red online) and a representative ϕ^4 potential $V(\phi) = \lambda^2(\phi^2 - a^2)^2$ (dotted). (b) The corresponding kinks (32) (inner curve, blue online) and (37) (outer curve, red online) connecting $-a$ to $+a$, and the corresponding ϕ^4 kink $\phi(x) = a\tanh(\lambda x)$ (dotted). In all panels, $a = 4/5$ and $b = 1$.

E. Phonons

Although we have considered, without loss of generality, only stationary kink solutions, phonon modes superimposed onto the kinks or the equilibrium states (vacua) can be time dependent. Therefore, to study phonons, we must consider the nonlinear Klein–Gordon equation of motion for the field [5, 30]:

$$\Box\phi = -V'(\phi), \tag{40}$$

where $\Box \equiv \partial^2/\partial t^2 - \partial^2/\partial x^2$ is the d'Alembertian operator. This equation can be linearized about any of the equilibrium states ϕ_e discussed above (e.g., $\phi_e = 0$, $\phi_e = \pm a$, etc.) to obtain an equation for the perturbation $\tilde{\phi}$ [30]:

$$\Box\tilde{\phi} = -V''(\phi_e)\tilde{\phi}. \tag{41}$$

Now, seeking harmonic solutions of the form $\tilde{\phi}(x,t) \propto e^{i(qx - \omega_q t)}$, we arrive at the dispersion relation

$$\omega_q^2 - q^2 = V''(\phi_e) \tag{42}$$

for phonon modes. Table I summarizes the possible right-hand sides (RHS) in the dispersion relation (42) for the ϕ^8 field theories with kink solutions studied above. Since, $b > a$ (strictly) by assumption, cases in Table I for which $V''(\phi_e) \neq 0$ represent field theories with only an optical phonon branch, while for cases with $V''(\phi_e) = 0$ there is only an acoustic phonon branch. The latter case indicates the possibility of *nonlinear phonons*. This is a novel feature of higher-than-sixth-order field theories.

TABLE I: Phonon modes of ϕ^8 field theory. DM = degenerate minima. RHS = dispersion relation right-hand side.

potential, V	equilibrium, ϕ_e	RHS, $V''(\phi_e)$
4 DM, Eq. (3)	$\pm a$	$8\lambda^2 a^2(b^2-a^2)^2$
4 DM, Eq. (3)	$\pm b$	$8\lambda^2 b^2(b^2-a^2)^2$
3 DM, Eq. (22)	$\pm a$	$8\lambda^2 a^6$
3 DM, Eq. (22)	0	0
3 DM, Eq. (26)	$\pm a$	$8\lambda^2 a^4(b^2+a^2)$
3 DM, Eq. (26)	0	$2\lambda^2 a^4 b^2$
2 DM, Eq. (31)	$\pm a$	0
2 DM, Eq. (35)	$\pm a$	$8\lambda a^2(b^2+a^2)^2$

III. ϕ^{10} FIELD THEORY

A. The Various Phases

The ϕ^{10} potential (free energy) is given, generically, by

$$V(\phi) = \lambda^2(\phi^{10} - \alpha_8\phi^8 + \alpha_6\phi^6 - \alpha_4\phi^4 + \alpha_2\phi^2 - \alpha_0), \quad (43)$$

where, without loss of generality, we assume the coefficient of ϕ^{10} to be $+1$ in units of λ^2. The coefficients of $\phi^{8,6,4,2}$ are, in general, arbitrary, and there are sixteen different possibilities, depending on whether all four, three, two, one or none of the coefficients are positive. However, if one wants to consider a model describing a succession of two first-order transitions then one must take $\alpha_{8,6,4,2} > 0$ in (43). As before, α_0 in (43) is chosen so that the minimum value of the potential is zero, i.e., $\min_\phi V(\phi) = 0$.

While there are four parameters ($\alpha_{8,6,4,2}$) describing the potential, it can be shown, by scaling arguments, that only three of them are truly independent. It may be noted here that even after taking $\alpha_{8,6,4,2} > 0$ in (43), since there are three free parameters, there is more than one possible "path" to describing successive phase transitions. For example, one possible path is to start from a potential with five degenerate minima at $\phi = 0, \pm a, \pm b$, which is given by

$$V(\phi) = \lambda^2\phi^2\,(\phi^2 - a^2)^2(\phi^2 - b^2)^2. \quad (44)$$

As in the ϕ^8 case, without any loss of generality, we choose $b > a$ throughout this section unless specified otherwise. Now, what happens as α_2 (i.e., coefficient of ϕ^2) is slowly increased or decreased from this critical value (at five degenerate minima)? One finds that when α_2 is increased from this critical value, then $\phi = 0$ is always the absolute minimum while the minima at $\phi = \pm a, \pm b$ are only local minima. On the other hand, if α_2 is decreased from this critical value, then one finds that the potential has absolute minima at $\phi = \pm b$, while the minima at $\phi = \pm a$ and at $\phi = 0$ are now local minima. Thus, even with $\alpha_{8,6,4,2} > 0$ in (43), if one starts with five degenerate minima, then one does not get two first-order transitions in succession.

However, if instead we start with a potential with $\alpha_{8,6,4,2} > 0$ but with four degenerate minima, given by

$$V(\phi) = \lambda^2(\phi^2 + c^2)(\phi^2 - a^2)^2(\phi^2 - b^2)^2, \quad (45)$$

and now we vary α_2, there are indeed two successive first-order transitions. For potentials of this form, (45), there are three parameters, i.e., a, b, c. The four coefficients $\alpha_{8,6,4,2}$ of the potential can be expressed in terms of the parameters a, b, c as

$$\begin{aligned}
\alpha_8 &= 2(b^2 + a^2) - c^2,\\
\alpha_6 &= a^4 + b^4 + 4a^2b^2 - 2c^2(b^2 + a^2),\\
\alpha_4 &= 2a^2b^2(b^2 + a^2) - c^2(a^4 + 4a^2b^2 + b^4),\\
\alpha_2 &= a^4b^4 - 2a^2b^2c^2(b^2 + a^2),\\
\alpha_0 &= -a^4b^4c^2.
\end{aligned} \quad (46)$$

Since the generic ϕ^{10} potential (43) is described by the four coefficients $\alpha_{8,6,4,2}$, there must exist extra constraints on the coefficients $\alpha_{8,6,4,2}$ to ensure a unique mapping from a, b, c to $\alpha_{8,6,4,2}$ [recall the constraints (2) and (5) derived in Section II A]. To this end, let

$$\tilde{a} = b^2 + a^2, \quad \tilde{b}^2 = a^2b^2, \quad \tilde{c} = c^2, \quad (47)$$

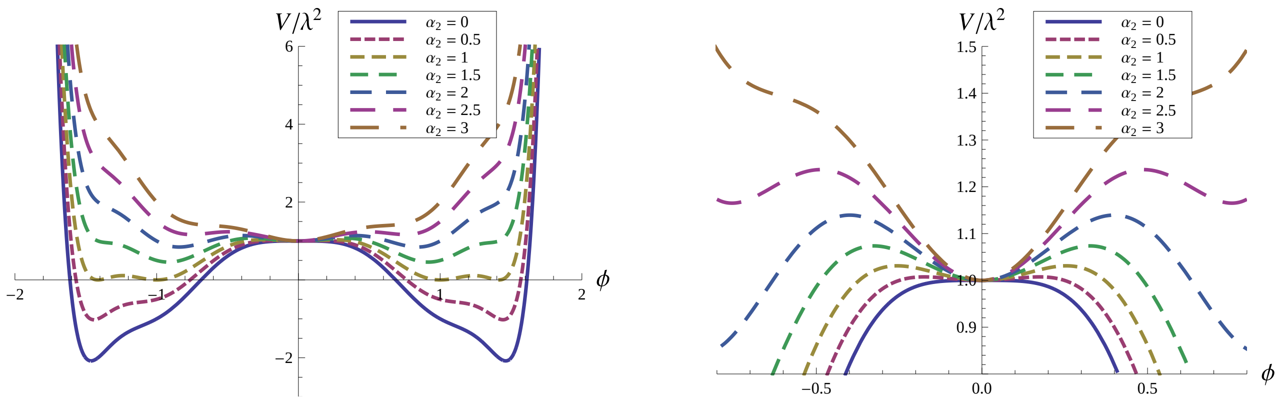
then (46) becomes

$$\begin{aligned}
\alpha_8 &= 2\tilde{a} - \tilde{c},\\
\alpha_6 &= \tilde{a}^2 + 2\tilde{b}^2 - 2\tilde{a}\tilde{c},\\
\alpha_4 &= 2\tilde{a}\tilde{b}^2 - \tilde{c}(\tilde{a}^2 + 2\tilde{b}^2),\\
\alpha_2 &= \tilde{b}^2(\tilde{b}^2 - 2\tilde{a}\tilde{c}),\\
\alpha_0 &= -\tilde{b}^2\tilde{c}.
\end{aligned} \quad (48)$$

By definition, $\tilde{a}^2 > 4\tilde{b}^2$ [i.e., $(a^2+b^2)^2 > 4a^2b^2$ or, equivalently $(b^2-a^2)^2 > 0$], hence

$$\begin{aligned}
4\tilde{b} - \tilde{c} &< \alpha_8 < 2\tilde{a},\\
2(3\tilde{b}^2 - \tilde{a}\tilde{c}) &< \alpha_6 < \tilde{a}^2 + 2\tilde{b}^2 - 4\tilde{b}\tilde{c},\\
4\tilde{b}^3 - \tilde{a}^2\tilde{c} &< \alpha_4 < 2(\tilde{a} - 3\tilde{c})\tilde{b}^2,\\
\tilde{b}^4 - \tilde{c}\tilde{a}^3/2 &< \alpha_2 < (\tilde{a}\tilde{b}^2/4)(\tilde{a} - 8\tilde{c}),
\end{aligned} \quad (49)$$

where $\tilde{b}$ is the positive root of $\tilde{b}^2 = a^2b^2$. This set of inequalities provides the signs of $\alpha_{8,6,4,2}$ in terms of a, b, c.



(a) (b)

FIG. 6: (Color online.) (a) Example potentials of the form (52) for various illustrative values of the coefficient of the quadratic term, α_2, showing the various phases and phase transitions in the ϕ^{10} theory. (b) Zoom-in of (a) near the origin.

Furthermore, we note that $\tilde{b}$ and $\tilde{c}$ can be eliminated between the first four equations in (48) to obtain

$$\begin{aligned} 4\alpha_2 + (5\tilde{a}^2 - \alpha_6 - 2\tilde{a}\alpha_8)(3\tilde{a}^2 + \alpha_6 - 2\tilde{a}\alpha_8) = 0, \\ 5\tilde{a}^3 + \alpha_4 + \tilde{a}\alpha_6 + 2\tilde{a}\alpha_8^2 - (6\tilde{a}^2 + \alpha_6)\alpha_8 = 0. \end{aligned} \tag{50}$$

Then, it is possible to eliminate $\tilde{a}$ between the last two equations to obtain the desired constraint [analogue of (2) for the ϕ^8 field theory with four degenerate minima]:

$$\begin{aligned} &8000\alpha_2^3 + (27\alpha_4^2 + 4\alpha_6^3 - 18\alpha_4\alpha_6\alpha_8 - \alpha_6^2\alpha_8^2 + 4\alpha_4\alpha_8^3) \\ &\times (25\alpha_4^2 - 20\alpha_6^3 - 70\alpha_4\alpha_6\alpha_8 + 37\alpha_6^2\alpha_8^2 + 4\alpha_4\alpha_8^3 - 8\alpha_6\alpha_8^4) \\ &+8\alpha_2\Big[15\alpha_4^2(15\alpha_6+26\alpha_8^2)+2\alpha_4(125\alpha_6^2\alpha_8-262\alpha_6\alpha_8^3+56\alpha_8^5) \\ &\quad + (4\alpha_6 - \alpha_8^2)(35\alpha_6^3 - 66\alpha_6^2\alpha_8^2 + 48\alpha_6\alpha_8^4 - 8\alpha_8^6)\Big] \\ &\quad = 16\alpha_2^2(325\alpha_6^2 + 600\alpha_4\alpha_8 - 440\alpha_6\alpha_8^2 + 88\alpha_8^4). \end{aligned} \tag{51}$$

This constraint ensures that a, b, c can be uniquely mapped to $\alpha_{8,6,4,2}$.

As an illustration, in Fig. 6, we have plotted the potential

$$V(\phi) = \lambda^2[\phi^{10} - 5.75\phi^8 + 11.5\phi^6 - 8.75\phi^4 + \alpha_2\phi^2 + 1], \tag{52}$$

for various values of the parameter α_2, in units of λ^2, to illustrate the structure of the phases. For $\alpha_2 = 1 = \alpha_2^c(II)$ this has four degenerate minima [this is the second first-order transition point, i.e., $T = T_c^I(II)$]. In particular, when $\alpha_2 = 1$, the potential (52) is of the form (45) with $a = 1$, $b = 2$, $c = 1/4$.

If the temperature is increased slightly above $T_c^I(II)$, i.e., α_2 is increased slightly beyond $\alpha_2^c = 1$, then the potential (52) has two absolute minima at $\phi = \pm\hat{a}$ ($\hat{a} < a = 1$), local minima at $\phi = 0, \pm\hat{b}$ ($\hat{b} < b = 2$), and there are four maxima between them. As α_2 is further increased [i.e., T is further increased beyond $T_c^I(II)$], there comes a point [$\alpha_2^c(I) = 2.2$ for the potential (52)], at which the potential has degenerate minima at $\phi = 0$ and at $\phi = \pm a$. Thus, this is the first first-order transition point $T_c^I(I)$. This is because, if the temperature is increased beyond this critical value [i.e., if α_2 is further increased beyond $\alpha_2^c(I)$], then $\phi = 0$ becomes the absolute minimum, while the minima at $\phi = \pm a$ disappear completely.

As far as the two local minima at $\phi = \pm b$ are concerned, they disappear at some point as the temperature is increased beyond $T_c^I(II)$, with the precise value of α_2 depending on the values of the other parameters [in the Fig. 6 they disappear at $\alpha_2 = 4$, much above $T_c^I(I)$].

If, instead, α_2 is decreased from $\alpha_2^c(II) = 1$, [i.e., temperature is lowered below $T_c^I(II)$], then the potential has two absolute minima at $\phi = \pm\hat{b}$ ($\hat{b} > b = 2$), local minima at $\phi = 0, \pm\hat{a}$ ($\hat{a} > a = 1$), and there are four maxima between them. As the temperature is further lowered so that α_2 approaches zero, the local minima at $\phi = \pm a$ disappear. For $\alpha_2 \leq 0$, the potential only has two minima at $\phi = \pm b (= \pm 2)$, a maximum at $\phi = 0$, and this picture persists, no matter how much further the temperature is lowered.

It is insightful to note that the structure near the first first-order transition point $T_c^I(I)$ is similar to that in the ϕ^6 model for a first-order phase transition [26]. Meanwhile, the structure near the second first-order transition point $T_c^I(II)$ is similar to that of the asymmetric double well ϕ^4 model of a first-order phase transition [2].

B. Five Degenerate Minima

Consider the ϕ^{10} potential given in (44). In this case,

$$\begin{aligned}\alpha_8 &= 2(b^2+a^2),\\ \alpha_6 &= a^4+b^4+4a^2b^2,\\ \alpha_4 &= 2a^2b^2(b^2+a^2),\\ \alpha_2 &= a^4b^4,\\ \alpha_0 &= 0.\end{aligned} \tag{53}$$

Clearly, $\alpha_{8,6,4,2}$ are strictly positive. This potential has five degenerate minima at $\phi = 0, \pm a, \pm b$, and, hence, four kink solutions exist, only two of which are distinct due to the symmetry of the potential.

1. *Kink connecting* 0 *to* a *(or* $-a$ *to* 0*)*

This kink solutions is given implicitly by

$$e^{\mu x} = \frac{\phi^{2(\gamma-1)}(b^2-\phi^2)}{(a^2-\phi^2)^\gamma}, \tag{54}$$

where $\mu = 2\sqrt{2}\lambda b^2(b^2-a^2)$ and $\gamma = b^2/a^2$ (>1 by assumption). From (54), the approach to the asymptotes at $\phi = 0, a$ can be shown to be exponential:

$$\phi(x) \simeq \begin{cases} \dfrac{a^{\gamma/(\gamma-1)}}{b^{1/(\gamma-1)}}e^{\mu x/[2(\gamma-1)]}, & x\to-\infty,\\[2ex] a - \dfrac{a(b^2-a^2)^{1/\gamma}}{2a^{2/\gamma}}e^{-\mu x/\gamma}, & x\to+\infty.\end{cases} \tag{55}$$

Note, however, that the rate at which ϕ asymptotes to 0 is given by $\mu/[2(\gamma-1)]$, while the rate at which ϕ asymptotes to a is given by μ/γ, hence this kink is asymmetric. The corresponding kink energy is

$$E_k^{(1)} = \frac{\sqrt{2}}{12}\lambda a^4(3b^2-a^2). \tag{56}$$

2. *Kink connecting* a *to* b *(or* $-b$ *to* $-a$*)*

In this case, the kink solution is given implicitly by

$$e^{\mu x} = \frac{(\phi^2-a^2)^\gamma}{\phi^{2(\gamma-1)}(b^2-\phi^2)}, \tag{57}$$

where $\mu = 2\sqrt{2}\lambda b^2(b^2-a^2)$ and $\gamma = b^2/a^2$. From (57), the approach to the asymptotes at $\phi = a, b$ can be shown to be exponential:

$$\phi(x) \simeq \begin{cases} a + \dfrac{(b^2-a^2)^{1/\gamma}}{2a^{(2-\gamma)/\gamma}}e^{\mu x/\gamma}, & x\to-\infty,\\[2ex] b - \dfrac{(b^2-a^2)^\gamma}{2b^{2\gamma-1}}e^{-\mu x}, & x\to+\infty.\end{cases} \tag{58}$$

Note, however, that the rate at which ϕ asymptotes to a is given by μ/γ, while the rate at which ϕ asymptotes to b is given by μ, hence this kink is asymmetric. The corresponding kink energy is

$$E_k^{(2)} = \frac{\sqrt{2}}{12}\lambda(b^2-a^2)^3. \tag{59}$$

Note that $E_k^{(1)} \gtreqqless E_k^{(2)}$ for $b/a \lesseqqgtr \sqrt{3}$. As for the similar ϕ^8 case (Section II B 1 b), it would be of interest to study the interaction energy between two kinks of the same type as well as two kinks of different types but with equal energies.

As an illustration, consider $a = 1/2$ and $b = 1$. This kink, as well as the one from the previous subsubsection, are illustrated in Fig. 7. Since the potential (44) has five degenerate minima, it is possible to fit a ϕ^8 potential with four degenerate minima (at $\phi = \pm a$ and $\phi = \pm b$) to it, and also a ϕ^6 potential with three degenerate minima (at $\phi = 0$ and $\phi = \pm a$). As can be seen in Fig. 7, for the parameters chosen, the shapes of the corresponding kink solutions from the lower-order field theories closely match those of the ϕ^{10} theory.

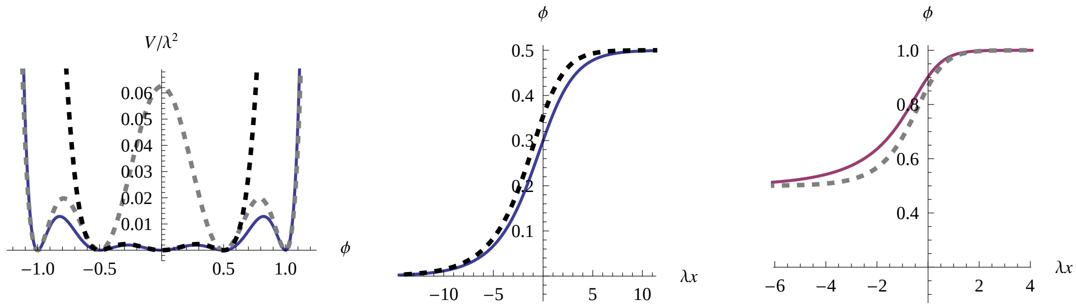
C. Four Degenerate Minima

1. $T = T_c^I(II)$

Consider the ϕ^{10} potential given in (45). This potential has four degenerate minima at $\phi = \pm a, \pm b$, and, hence, three kink solutions exist, only two of which are distinct due to the symmetry of the potential.

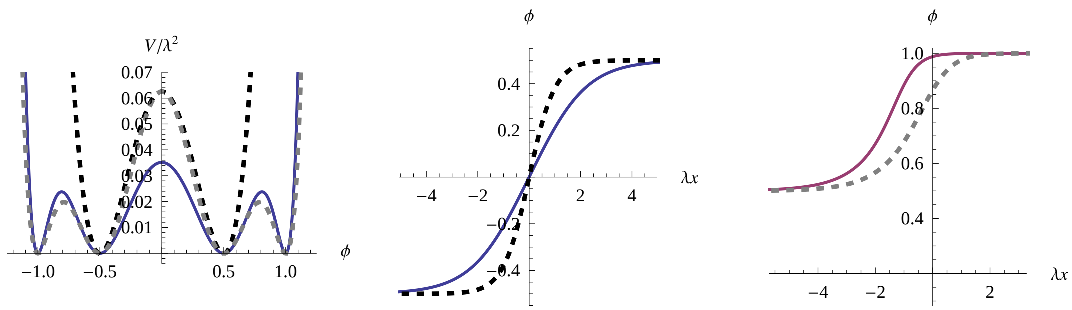
a. Kink connecting $-a$ *to* $+a$ This kink solution is given implicitly by

$$\mu x = \left\{\sinh^{-1}\left[\frac{c+\alpha\phi}{\alpha(a-\phi)}\right] - \sinh^{-1}\left[\frac{c-\alpha\phi}{\alpha(a+\phi)}\right]\right\} + \frac{\alpha\sqrt{1+\alpha^2}}{\beta\sqrt{1+\beta^2}}\left\{\sinh^{-1}\left[\frac{c-\beta\phi}{\beta(b+\phi)}\right] - \sinh^{-1}\left[\frac{c+\beta\phi}{\beta(b-\phi)}\right]\right\}, \tag{60}$$



(a) (b) (c)

FIG. 7: (Color online.) ϕ^{10} field theory with five degenerate minima. (a) The potential (44) (solid), a representative ϕ^8 potential with four degenerate minima (3) (gray, dotted) and a representative ϕ^6 potential with three degenerate minima $V(\phi) = \lambda^2\phi^2(\phi^2 - a^2)^2$ (black, dotted). (b) The kink solution (54) (solid) connecting 0 to a and the corresponding ϕ^6 kink $\phi(x) = a/\sqrt{1 + e^{-2\sqrt{2}a^2\lambda x}}$ (black, dotted). (c) The kink solution (57) (solid) connecting a to b and the corresponding ϕ^8 kink (15) (gray, dotted). In all panels, $a = 1/2$ and $b = 1$.



(a) (b) (c)

FIG. 8: (Color online.) ϕ^{10} field theory at the second first-order phase transition, $T = T_c^I(II)$. (a) The potential (45) (solid), a representative ϕ^8 potential with four degenerate minima (3) (gray, dotted) and a representative ϕ^4 potential $V(\phi) = \lambda^2(\phi^2 - a^2)^2$ (black, dotted). (b) The kink solution (60) (solid) connecting $-a$ to $+a$ and the corresponding ϕ^4 kink $\phi(x) = a\tanh(\lambda x)$ (black, dotted). (c) The kink solution (63) (solid) connecting a to b and the corresponding ϕ^8 kink (15) (gray, dotted). In all panels, $a = 1/2$, $b = 1$ and $c = 3/4$.

where $\mu = 2\sqrt{2}\lambda\alpha\sqrt{1+\alpha^2}(\beta^2 - \alpha^2)c^4$, $\beta = b/c$ and $\alpha = a/c$ with $\beta > \alpha$ by assumption. From (60), the approach to the asymptotes at $\phi = \pm a$ can be shown to be exponential:

$$\phi(x) \simeq \begin{cases} -a + \frac{2(c+\alpha a)}{\alpha}\exp\left(\sinh^{-1}\left[\frac{1}{2}(1-\alpha^{-2})\right] + \frac{\alpha\sqrt{1+\alpha^2}}{\beta\sqrt{1+\beta^2}}\left\{\sinh^{-1}\left[\frac{c-\beta a}{\beta(b+a)}\right] - \sinh^{-1}\left[\frac{c+\beta a}{\beta(b-a)}\right]\right\}\right)e^{\mu x}, & x \to -\infty, \\ +a - \frac{2(c+\alpha a)}{\alpha}\exp\left(\sinh^{-1}\left[\frac{1}{2}(1-\alpha^{-2})\right] + \frac{\alpha\sqrt{1+\alpha^2}}{\beta\sqrt{1+\beta^2}}\left\{\sinh^{-1}\left[\frac{c-\beta a}{\beta(b+a)}\right] - \sinh^{-1}\left[\frac{c+\beta a}{\beta(b-a)}\right]\right\}\right)e^{-\mu x}, & x \to +\infty. \end{cases} \tag{61}$$

Clearly, this kink is symmetric. The kink's energy is

$$E_k^{(1)} = \frac{\sqrt{2}}{24}\lambda\left\{\alpha\sqrt{1+\alpha^2}(12a^2b^2 - 4a^2c^2 - 6b^2c^2 - 4a^4 - 3c^4) + 3c^2[8a^2b^2 + 2(b^2+a^2)c^2 + c^4]\sinh^{-1}\alpha\right\}. \tag{62}$$

b. Kink connecting a to b (or $-b$ to $-a$) In this case, the kink solution is given implicitly by

$$\mu x = -\left\{\sinh^{-1}\left[\frac{c+\alpha\phi}{\alpha(\phi-a)}\right] - \sinh^{-1}\left[\frac{c-\alpha\phi}{\alpha(a+\phi)}\right]\right\} - \frac{\alpha\sqrt{1+\alpha^2}}{\beta\sqrt{1+\beta^2}}\left\{\sinh^{-1}\left[\frac{c-\beta\phi}{\beta(b+\phi)}\right] - \sinh^{-1}\left[\frac{c+\beta\phi}{\beta(b-\phi)}\right]\right\}, \tag{63}$$

where μ, β and α are defined below (60). From (63), the approach to the asymptotes at $\phi = a, b$ can be shown to be exponential:

$$\phi(x) \simeq \begin{cases} a + \frac{2(c+\alpha a)}{\alpha} \exp\left(\sinh^{-1}\left[\frac{\alpha^2-1}{2\alpha^2}\right] + \frac{\alpha\sqrt{1+\alpha^2}}{\beta\sqrt{1+\beta^2}}\left\{\sinh^{-1}\left[\frac{c-\beta a}{\beta(b+a)}\right] - \sinh^{-1}\left[\frac{c+\beta a}{\beta(b-a)}\right]\right\}\right) e^{\mu x}, & x \to -\infty, \\ b - \frac{2(c+\beta b)}{\beta} \exp\left(\sinh^{-1}\left[\frac{\beta^2-1}{2\beta^2}\right] + \frac{\beta\sqrt{1+\beta^2}}{\alpha\sqrt{1+\alpha^2}}\left\{\sinh^{-1}\left[\frac{c-\alpha b}{\alpha(b+a)}\right] + \sinh^{-1}\left[\frac{c+\alpha b}{\alpha(b-a)}\right]\right\}\right) e^{-\mu\frac{\beta\sqrt{1+\beta^2}}{\alpha\sqrt{1+\alpha^2}}x}, & x \to +\infty. \end{cases} \tag{64}$$

Note, however, that the rate at which ϕ asymptotes to a is given by μ, while the rate at which ϕ asymptotes to b is given by $\mu\beta\sqrt{1+\beta^2}/(\alpha\sqrt{1+\alpha^2})$, hence this kink is asymmetric. The kink's energy is

$$E_k^{(2)} = \frac{\sqrt{2}}{48}\lambda\Big\{\alpha\sqrt{1+\alpha^2}(12a^2b^2 - 4a^2c^2 - 6b^2c^2 - 4a^4 - 3c^4) - \beta\sqrt{1+\beta^2}(12a^2b^2 - 4b^2c^2 - 6a^2c^2 - 4b^4 - 3c^4) + 3c^2[8a^2b^2 + 2(b^2+a^2)c^2 + c^4]\left(\sinh^{-1}\alpha - \sinh^{-1}\beta\right)\Big\}. \tag{65}$$

Figure 8 shows the kink solutions from the previous subsubsection. Note that, unlike the ϕ^8 case in Fig. 2, the match between the ϕ^{10} and ϕ^4 theories for the symmetric kink connecting $-a$ to $+a$ is not very good for the chosen parameters. The agreement between the two is determined by the curvature of the potential near $\phi = 0$, which is controlled by c; for other values of c, these can be made more similar. Specifically, as $c \to 1$ the ϕ^{10} and ϕ^4 kinks match well [Fig. 8(b)], while as $c \to 0$, the ϕ^{10} and ϕ^8 kinks match better [Fig. 8(c].

2. $T_c^I(II) < T < T_c^I(I)$

For temperatures between the two first-order phase transitions [i.e., $1 < \alpha_2 < 2.2$ for the example potential (52)], the potential can be rewritten as

$$V(\phi) = \lambda^2(\phi^2 - \hat{a}^2)^2(\phi^2 + \beta)[\phi^4 - \gamma\phi^2 + \delta], \tag{66}$$

with $\beta, \gamma > 0$, $\hat{a}^2 < a^2$ and $\delta > 4\gamma$ so that the minimum of the potential is indeed at 0. We expect a kink solution exists connecting the two degenerate minima $\phi = \pm\hat{a}$, as x goes from $-\infty$ to $+\infty$.

As an illustration, consider the factorized potential

$$V(\phi) = \lambda^2(\phi^2 - 0.9)^2(\phi^2 + 0.2)[\phi^4 - 4.15\phi^2 + 4.45]. \tag{67}$$

This potential has absolute minima at $\phi = \pm\hat{a} = \pm\sqrt{0.9}$ and local minima at $\phi = 0$ and at $\phi = \pm\hat{b}$ ($\hat{b}^2 < b^2 = 2$).

3. $T < T_c^I(II)$

Below the second first-order phase transition [i.e., $\alpha_2 < 1$ for the example potential (52)], the potential can be rewritten as

$$V(\phi) = \lambda^2(\phi^2 - \hat{b}^2)^2(\phi^2 + \beta)[\phi^4 - \gamma\phi^2 + \delta], \tag{68}$$

with $\beta, \gamma > 0$, $\hat{b}^2 > 2$ and $\delta > 4\gamma$ so that the minimum of the potential is indeed at 0. We expect a kink solution exists connecting the degenerate minima $\phi = \pm\hat{b}$, as x goes from $-\infty$ to $+\infty$.

As an illustration, consider the factorized potential

$$V(\phi) = \lambda^2(\phi^2 - 2.05)^2(\phi^2 + 0.3)[\phi^4 - 1.97\phi^2 + 1.15]. \tag{69}$$

This potential has absolute minima at $\phi = \pm\hat{b} = \pm\sqrt{2.05}$ and local minima at $\phi = 0$ and at $\phi = \pm\hat{a}$ ($\hat{a}^2 > a^2 = 1$).

This kink solution for $T < T_c^I(II)$ and the previous one for $T_c^I(II) < T < T_c^I(I)$ are illustrated in Fig. 9. Notice that for the case $T < T_c^I(II)$ [dashed curve in Fig. 9(b)], the kink "feels" the influence of the two local minima at $\phi \approx \pm 1.17101$, similarly to kinks in certain cases of ϕ^6 field theory [32], and the kink near the first-order phase transition in ϕ^8 field theory (recall Section II B 3). However, for these choices of γ and δ, neither set of kinks in Fig. 9 appears to "feel" the influence of the local minimum at $\phi = 0$.

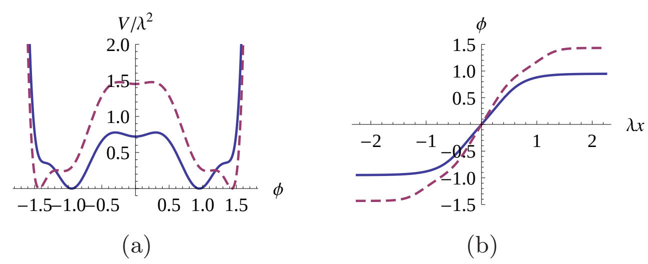


FIG. 9: (Color online.) Kink solutions between the first and second first-order phase transition [$T_c^I(II) < T < T_c^I(I)$] and below the second first-order phase transition [$T < T_c^I(II)$] in ϕ^{10} field theory. (a) The potentials (67) (solid) and (69) (dashed). (b) The corresponding kinks computed by solving the equation of motion $d\phi/dx = \sqrt{2V(\phi)}$ numerically subject to the symmetry condition $\phi(0) = 0$.

D. Three Degenerate Minima

There are four possible forms of the ϕ^{10} potential with three degenerate minima for which kink solutions can be constructed. These potentials have three degenerate minima, and, hence, two kink solutions exist, only one of which is distinct due to the symmetry of the potential.

1. Case I

First, consider the potential

$$V(\phi) = \lambda^2\phi^2(\phi^2-a^2)^2[\phi^4 - b\phi^2 + c], \quad b^2 < 4c, \tag{70}$$

with $b > 0$, so that the potential has degenerate minima at $\phi = 0, \pm a$. In this case,

$$\begin{aligned}
\alpha_8 &= 2a^2 + b,\\
\alpha_6 &= a^2(a^2+2b) + c,\\
\alpha_4 &= a^2(a^2 b + 2c),\\
\alpha_2 &= a^4 c,\\
\alpha_0 &= 0.
\end{aligned} \tag{71}$$

Clearly, $\alpha_{8,6,4,2}$ are strictly positive.

The corresponding kink solution connecting 0 to $+a$ (or $-a$ to 0) is given implicitly by

$$\mu x = \frac{\sqrt{c}}{\sqrt{c+a^4}} \sinh^{-1}\left[\frac{2c + ba^2 + (2a^2-b)\phi^2}{(a^2-\phi^2)\sqrt{4a^2b - b^2 + 4c}}\right] - \sinh^{-1}\left(\frac{2c - b\phi^2}{\phi^2\sqrt{4c-b^2}}\right), \tag{72}$$

where $\mu = 2\sqrt{2}\lambda a^2\sqrt{c}$. From (72), the approach to the asymptotes at $\phi = 0, a$ can be shown to be exponential:

$$\phi(x) \simeq \begin{cases} \frac{2\sqrt{c}}{(4c-b^2)^{1/4}} \exp\left(-\frac{\sqrt{c}}{2\sqrt{c+a^4}}\sinh^{-1}\left[\frac{2c+ba^2}{a^2\sqrt{4a^2b-b^2+4c}}\right]\right) e^{\mu x/2}, & x \to -\infty,\\ a - \frac{2(c+a^4)}{a\sqrt{4a^2b-b^2+4c}} \exp\left(-\frac{\sqrt{c+a^4}}{\sqrt{c}}\sinh^{-1}\left[\frac{2c-ba^2}{a^2\sqrt{4c-b^2}}\right]\right) e^{-\mu x\sqrt{c+a^4}/\sqrt{c}}, & x \to +\infty. \end{cases} \tag{73}$$

Note, however, that the rate at which ϕ asymptotes to a is given by $\mu/2$, while the rate at which ϕ asymptotes to b is given by $\mu\sqrt{c+a^4}/\sqrt{c}$, hence this kink is asymmetric. The kink's energy is

$$E_k = \frac{\sqrt{2}}{96}\lambda\Big\{2(3b^2 + 4a^4 - 4a^2b - 8c)\sqrt{a^4 - ba^2 + c} + 16c^{3/2} + 6b(2a^2-b)\sqrt{c} + 3(b^2-4c)(2a^2-b) \times \ln\left[\frac{-b+2\sqrt{c}}{2a^2 - b + 2\sqrt{a^4-ba^2+c}}\right]\Big\}. \tag{74}$$

2. Case II

Now, let

$$V(\phi) = \lambda^2\phi^2(\phi^2-a^2)^2(\phi^2+b^2)^2. \tag{75}$$

This potential has three degenerate minima at $\phi = 0, \pm a$. In this case,

$$\begin{aligned}
\alpha_8 &= 2(b^2-a^2),\\
\alpha_6 &= b^4 + a^4 - 4a^2b^2,\\
\alpha_4 &= 2a^2b^2(b^2-a^2),\\
\alpha_2 &= a^4b^4,\\
\alpha_0 &= 0.
\end{aligned} \tag{76}$$

Clearly, $\alpha_{8,4,2} > 0$ for $b > a$, while $\alpha_6 > 0$ as long as $b\sqrt{2-\sqrt{3}} > a$.

The corresponding kink solution connecting 0 to $+a$ (or $-a$ to 0) is given implicitly by

$$e^{\mu x} = \frac{\phi^2}{(a^2-\phi^2)^{b^2/(b^2+a^2)}(b^2+\phi^2)^{a^2/(b^2+a^2)}}, \tag{77}$$

where $\mu = 2\sqrt{2}a^2b^2\lambda$. From (77), it can be shown that the approach to the asymptotes at $\phi = 0, a$ is exponential:

$$\phi(x) \simeq \begin{cases} a^{b^2/(b^2+a^2)} b^{a^2/(b^2+a^2)} e^{\mu x/2}, & x \to -\infty,\\ a - \frac{a^{1+2a^2/b^2}}{2(a^2+b^2)^{a^2/b^2}} e^{-\mu(b^2+a^2)x/b^2}, & x \to +\infty. \end{cases} \tag{78}$$

Consequently, this kink is asymmetric due to its different growth rates as $x \to \pm\infty$. The kink's energy is

$$E_k = \frac{\sqrt{2}}{12}\lambda a^4 (a^2 + 3b^2). \tag{79}$$

3. Case III

Next, consider

$$V(\phi) = \lambda^2 \phi^6 (\phi^2 - a^2)^2, \tag{80}$$

In this case $\alpha_{8,6} > 0$, while $\alpha_{4,2,0} = 0$.

The corresponding kink solution connecting 0 to $+a$ (or $-a$ to 0) is given implicitly by

$$\mu x = -\frac{a^2}{\phi^2} + \ln\left(\frac{\phi^2}{a^2 - \phi^2}\right), \tag{81}$$

where $\mu = 2\sqrt{2}\lambda a^4$. From (81), it can be shown that the approach to the asymptotes at $\phi = 0, a$ is of mixed type:

$$\phi(x) \simeq \begin{cases} \dfrac{a}{\sqrt{-\mu x}}, & x \to -\infty, \\ a - \frac{1}{2} a e^{-\mu x - 1}, & x \to +\infty. \end{cases} \tag{82}$$

Consequently, this kink is asymmetric due to the algebraic versus exponential approach as $x \to \pm\infty$, respectively. The kink's energy is

$$E_k = \frac{\sqrt{2}}{12}\lambda a^6. \tag{83}$$

4. Case IV

Finally, consider the potential

$$V(\phi) = \lambda^2 \phi^4 (\phi^2 - a^2)^2 (\phi^2 + b^2), \tag{84}$$

for which

$$\begin{aligned} \alpha_8 &= 2a^2 - b^2, \\ \alpha_6 &= a^2(a^2 - 2b^2), \\ \alpha_4 &= -a^4 b^2, \\ \alpha_2 &= \alpha_0 = 0. \end{aligned} \tag{85}$$

In this case, $\alpha_8, \alpha_6 > 0$ as long as $a > \sqrt{2}b$, and $\alpha_4 < 0$.

The corresponding kink solution connecting 0 to $+a$ (or $-a$ to 0) is given implicitly by

$$\begin{aligned} \mu x = &-\frac{2a\sqrt{b^2 + a^2}\sqrt{\phi^2 + b^2}}{b^2 \phi} \\ &+ \sinh^{-1}\left[\frac{b^2 + a\phi}{b(a - \phi)}\right] - \sinh^{-1}\left[\frac{b^2 - a\phi}{b(a + \phi)}\right], \end{aligned} \tag{86}$$

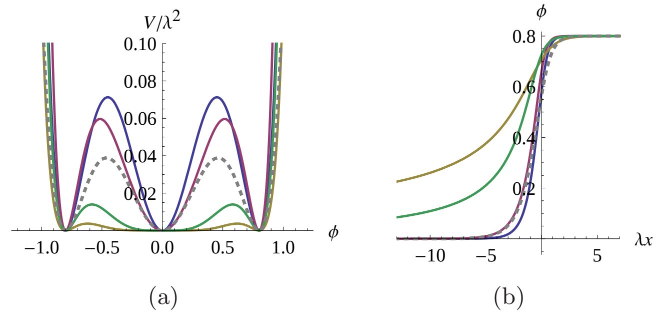


FIG. 10: (Color online.) ϕ^{10} field theory with three degenerate minima. (a) The potentials: (70) (first curve from top to bottom, blue online), (75) (second curve from top to bottom, red online), (80) (fourth curve from top to bottom, yellow online), (84) (third curve from top to bottom, green online), and a representative ϕ^6 potential $V(\phi) = \lambda^2\phi^2(\phi^2 - a^2)^2$ (dotted). (b) The kink solutions connecting 0 to a: (72) (fourth curve from top to bottom, blue online), (77) (third curve from top to bottom, red online), (81) (first curve from top to bottom, yellow online), (86) (second curve from top to bottom, green online), and the corresponding ϕ^6 kink $\phi(x) = a/\sqrt{1 + e^{-2\sqrt{2}a^2\lambda x}}$ (dotted). In all panels, $a = 4/5$, $b = 1$ and $c = 1$.

where $\mu = 2\sqrt{2}\lambda a^3\sqrt{a^2 + b^2}$. From (86), it can be shown that the approach to the asymptotes at $\phi = 0, a$ is of mixed type:

$$\phi(x) \simeq \begin{cases} -\dfrac{\sqrt{2}}{2a^2 b x}, & x \to -\infty, \\ a - \dfrac{2a}{b^2}(b^2 + a^2) e^{-\mu x - 2 - 2a^2/b^2}, & x \to +\infty. \end{cases} \tag{87}$$

Consequently, this kink is asymmetric due to the algebraic versus exponential approach as $x \to \pm\infty$, respectively. The kink's energy is

$$\begin{aligned} E_k = \frac{\sqrt{2}}{48}\lambda\Big[&a\sqrt{b^2 + a^2}(4a^4 + 4a^2b^2 + 3b^4) \\ &- 3b^4(2a^2 + b^2)\sinh^{-1}(a/b)\Big]. \end{aligned} \tag{88}$$

All four kinks from this subsection are illustrated in Fig. 10. Note that the plots for Cases III and IV are distinct from those for Cases I and II in part due to the algebraic decay of the corresponding kink solutions as $\phi \to 0$ [recall (82) and (87)].

E. Two Degenerate Minima

There are three possible forms of the ϕ^{10} potential with two degenerate minima at $\phi = \pm a$ for which kink solutions can be constructed.

1. Case I

Let

$$V(\phi) = \lambda^2(\phi^2 - a^2)^2(\phi^2 + b^2)^3. \tag{89}$$

In this case,

$$\begin{aligned}
\alpha_8 &= a^2 - 3b^2, \\
\alpha_6 &= 3b^4 - 6a^2b^2 + a^4, \\
\alpha_4 &= b^2(6a^2b^2 - 3a^4 - b^4), \\
\alpha_2 &= a^2b^4(3a^2 - 2b^2), \\
\alpha_0 &= -a^4b^6,
\end{aligned} \tag{90}$$

It can be shown that, $\alpha_8 < 0$ and $\alpha_0 < 0$, while $\alpha_{6,4} > 0$ and $\alpha_2 < 0$ as long as $a < b\sqrt{3-\sqrt{6}} < \sqrt{3}a$.

The kink solution is given implicitly by

$$\begin{aligned}
\mu x = &\frac{2a\phi\sqrt{b^2+a^2}}{b^2\sqrt{b^2+\phi^2}} \\
&+ \sinh^{-1}\left[\frac{b^2+a\phi}{b(a-\phi)}\right] - \sinh^{-1}\left[\frac{b^2-a\phi}{b(a+\phi)}\right],
\end{aligned} \tag{91}$$

where $\mu = 2\sqrt{2}\lambda a(b^2+a^2)^{3/2}$. From (91), the approach to the asymptotes at $\phi = \pm a$ can be shown to be exponential:

$$\phi(x) \simeq \begin{cases} -a + \dfrac{2a}{b^2}(b^2+a^2)e^{\mu x + 2a^2/b^2}, & x \to -\infty, \\ +a - \dfrac{2a}{b^2}(b^2+a^2)e^{-\mu x - 2a^2/b^2}, & x \to +\infty. \end{cases} \tag{92}$$

Clearly, this kink is symmetric. The kink's energy is

$$\begin{aligned}
E_k = \frac{\sqrt{2}}{24}\lambda\Big[&a\sqrt{b^2+a^2}(4a^4 + 16a^2b^2 - 3b^4) \\
&+ 3b^4(6a^2+b^2)\sinh^{-1}(a/b)\Big].
\end{aligned} \tag{93}$$

2. Case II

Let

$$V(\phi) = \lambda^2(\phi^2 - a^2)^4(\phi^2 + b^2). \tag{94}$$

In this case,

$$\begin{aligned}
\alpha_8 &= 4a^2 - b^2, \\
\alpha_6 &= a^2(6a^2 - 4b^2), \\
\alpha_4 &= 2a^4(2a^2 - 3b^2), \\
\alpha_2 &= a^6(a^2 - 4b^2), \\
\alpha_0 &= -a^8b^2,
\end{aligned} \tag{95}$$

It can be shown that, $\alpha_{4,2,0} < 0$, while $\alpha_{8,6} > 0$ as long as $a\sqrt{6}/2 > b > a$.

The kink solution is given implicitly by

$$\begin{aligned}
\mu x = &\frac{2\phi a\sqrt{b^2+\phi^2}}{(a^2-\phi^2)\sqrt{b^2+a^2}} + \left(\frac{2a^2+b^2}{b^2+a^2}\right) \\
&\times \left\{\sinh^{-1}\left[\frac{b^2+a\phi}{b(a-\phi)}\right] - \sinh^{-1}\left[\frac{b^2-a\phi}{b(a+\phi)}\right]\right\},
\end{aligned} \tag{96}$$

where $\mu = 4\sqrt{2}\lambda a^3\sqrt{b^2+a^2}$. From (96), the approach to the asymptotes at $\phi = \pm a$ can be shown to be as $1/x$:

$$\phi(x) \simeq \begin{cases} -a - \dfrac{a}{x}, & x \to -\infty, \\ +a - \dfrac{a}{x}, & x \to +\infty. \end{cases} \tag{97}$$

Clearly, this kink is symmetric. The kink's energy is

$$\begin{aligned}
E_k = \frac{\sqrt{2}}{24}\lambda\Big[&a\sqrt{b^2+a^2}(8a^4 - 10a^2b^2 - 3b^4) \\
&+ 3b^2(8a^4 + 4a^2b^2 + b^4)\sinh^{-1}(a/b)\Big].
\end{aligned} \tag{98}$$

3. Case III

Let

$$V(\phi) = \lambda^2|\phi^2 - a^2|^5, \tag{99}$$

In this case, $\alpha_{8,6,4,2,0} > 0$. The kink solution is given implicitly by

$$\mu x = \frac{\phi(3a^2 - 2\phi^2)}{(a^2-\phi^2)^{3/2}}, \tag{100}$$

where $\mu = 3\sqrt{2}\lambda a^4$. From (100), the approach to the asymptotes at $\phi = \pm a$ can be shown to be algebraic:

$$\phi(x) \simeq \begin{cases} -a + \dfrac{a}{2(-\mu x)^{2/3}}, & x \to -\infty, \\ +a - \dfrac{a}{2(\mu x)^{2/3}}, & x \to +\infty. \end{cases} \tag{101}$$

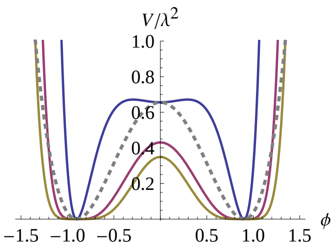
Clearly, this kink is symmetric. The kink's energy is

$$E_k = \frac{5\sqrt{2}\pi}{16}\lambda a^6. \tag{102}$$

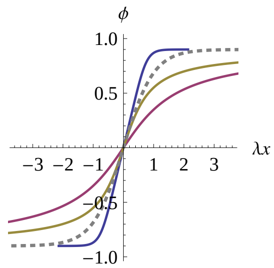
All three kinks from this subsection are illustrated in Fig. 11.

F. Phonons

The discussion from Section II E applies here as well. Table II summarizes the properties of the phonon dispersion relation (42) for the ϕ^{10} field theories with kink solutions studied above. As was the case for the ϕ^8 field theories considered above, there are once again potentials for which the RHS of the dispersion relation vanishes; but, it cannot vanish in the other cases due to our assumption $b > a$.



(a)



(b)

FIG. 11: (Color online.) ϕ^{10} field theory with two degenerate minima. (a) The potentials: (89) (top curve, blue online), (94) (middle curve, red online), (99) (bottom curve, yellow online), and a representative ϕ^4 potential $V(\phi) = \lambda^2(\phi^2 - a^2)^2$ (dotted). (b) The kink solutions connecting $-a$ to a: (91) (outer curve, blue online), (96) (inner curve, red online), (100) (middle curve, yellow online), and the corresponding ϕ^4 kink $\phi(x) = a\tanh(\lambda x)$ (dotted). In all panels, $a = 9/10$ and $b = 1$.

TABLE II: Phonon modes of ϕ^{10} field theory. DM = degenerate minima. RHS = dispersion relation right-hand side.

potential, V	equilibrium, ϕ_e	RHS, $V''(\phi_e)$
5 DM, Eq. (44)	0	$2\lambda^2 a^4 b^4$
5 DM, Eq. (44)	$\pm a$	$8\lambda^2 a^4(b^2-a^2)^2$
5 DM, Eq. (44)	$\pm b$	$8\lambda^2 b^4(b^2-a^2)^2$
4 DM, Eq. (45)	$\pm a$	$8\lambda^2 a^2(b^2-a^2)^2(c^2+a^2)$
4 DM, Eq. (45)	$\pm b$	$8\lambda^2 b^2(b^2-a^2)^2(c^2+b^2)$
3 DM, Eq. (70)	0	$2\lambda^2 c a^4$
3 DM, Eq. (70)	$\pm a$	$8\lambda^2 a^4(a^4-ba^2+c)$
3 DM, Eq. (75)	0	$2\lambda^2 a^4 b^4$
3 DM, Eq. (75)	$\pm a$	$8\lambda^2 a^4(b^2+a^2)^2$
3 DM, Eq. (80)	0	$2\lambda^2 a^4 b^4$
3 DM, Eq. (80)	$\pm a$	$8\lambda^2 a^8$
3 DM, Eq. (84)	0	$2\lambda^2 a^4 b^4$
3 DM, Eq. (84)	$\pm a$	$8\lambda^2 a^6(b^2+a^2)$
2 DM, Eq. (89)	$\pm a$	$8\lambda^2 a^2(b^2+a^2)^3$
2 DM, Eq. (94)	$\pm a$	0
2 DM, Eq. (99)	$\pm a$	0

G. Classical Free Energy Using the Transfer Matrix Technique

Using the transfer matrix technique, it was shown by Scalapino et al. [33, 34] that, in the thermodynamic limit, the classical free energy of a given field theory is essentially given by the ground state energy of the Schrödinger-like equation whose potential is given by the field theory's potential $V(\phi)$. Now, it is well known that while the ground state energy cannot be obtained analytically if the leading term of the potential is of the form ϕ^{4n} with $n = 1, 2, \ldots$. On the other hand, if the leading term in the potential is instead of the form ϕ^{4n+2}, then it leads to a quasi-exactly solvable (QES) problem for which the eigenstates of the first few levels can be obtained analytically. For example, this has been demonstrated for the ϕ^6 field theory in [26]. We would now like to show that there is a specific set of coefficients of our ϕ^{10} potential of form (43) that lead to the classical free energy and probability distribution function (PDF) being obtainable analytically at a given temperature.

In particular, the Schrödinger-like eigenvalue problem takes the form ($m = c = \hbar = 1$):

$$-\frac{d^2\psi}{d\phi^2} + 2V(\phi)\psi = 2E\psi, \tag{103}$$

with potential V given by (43). Then, it is easily shown that the exact ground state energy eigenvalue and eigenfunction can be obtained exactly for some special cases of the coefficients α_i of the potential V.

First,

$$\begin{aligned} E_0 &= \frac{C}{2}, \\ \psi_0(\phi) &= \exp\left[-\frac{\lambda\phi^6}{3\sqrt{2}} + \frac{B\phi^4}{4} - \frac{C\phi^2}{2}\right] \end{aligned} \tag{104}$$

satisfy (103) provided B and C are related to λ, $\alpha_{8,6,4,2}$ via

$$\begin{aligned} \alpha_8 &= \frac{\sqrt{2}B}{\lambda}, \\ \alpha_6 &= \frac{B^2 + 2\sqrt{2}C\lambda}{2\lambda^2}, \\ \alpha_4 &= \frac{2BC + 5\sqrt{2}\lambda}{2\lambda^2}, \\ \alpha_2 &= \frac{C^2 + 3B}{2\lambda^2}, \\ \alpha_0 &= 0. \end{aligned} \tag{105}$$

This solution corresponds to a ground state.

Second,

$$\begin{aligned} E_1 &= \frac{C - F}{2}, \\ \psi_1(\phi) &= (\phi^2 + D)\exp\left[-\frac{\lambda\phi^6}{3\sqrt{2}} + \frac{B\phi^4}{4} - \frac{C\phi^2}{2}\right] \end{aligned} \tag{106}$$

satisfy (103) provided B, C and D are related to λ, $\alpha_{8,6,4,2}$ via

$$\begin{aligned} \alpha_8 &= \frac{\sqrt{2}B}{\lambda}, \\ \alpha_6 &= \frac{B^2 + 2\sqrt{2}C\lambda}{2\lambda^2}, \\ \alpha_4 &= \frac{2BC + 9\sqrt{2}\lambda}{2\lambda^2}, \\ \alpha_2 &= \frac{C^2 + 3B + G}{2\lambda^2}, \\ \alpha_0 &= 0, \end{aligned} \tag{107}$$

where we have set $F = 2/D$ and $G = -2(1+2CD)/D^2$ for convenience. In addition, D must satisfy the cubic equation

$$2\sqrt{2}\lambda D^3 + 2BD^2 + 2CD + 1 = 0. \tag{108}$$

It is clear that as long as $B, C > 0$ (so that $\alpha_{8,6} > 0$), then $D < 0$, and the solution (106) is for the second excited state (having two roots at $\phi = \pm\sqrt{D}$). On the other hand, if we allow $B < 0$, then $D > 0$ is possible for certain values of C, and the solution corresponds to another ground state. Note, however, that in that case $\alpha_8 < 0$.

Third,

$$E_2 = \frac{C-G}{2},$$
$$\psi_2(\phi) = (\phi^4 + D\phi^2 + J)\exp\left[-\frac{\lambda\phi^6}{3\sqrt{2}} + \frac{B\phi^4}{4} - \frac{C\phi^2}{2}\right] \tag{109}$$

satisfy (103) provided B, C, D and J are related to λ, $\alpha_{8,6,4,2}$ via

$$\begin{aligned}
\alpha_8 &= \frac{\sqrt{2}B}{\lambda},\\
\alpha_6 &= \frac{B^2 + 2\sqrt{2}C\lambda}{2\lambda^2},\\
\alpha_4 &= \frac{2BC + 13\sqrt{2}\lambda}{2\lambda^2},\\
\alpha_2 &= \frac{C^2 + 3B + H}{2\lambda^2},\\
\alpha_0 &= 0,
\end{aligned} \tag{110}$$

where we have set $G = 2D/J$ and $H = (6/D - 2D - G/2)G$ for convenience. In addition, D and J must satisfy

$$D^2 + 2(JC + \sqrt{2}\lambda J^2)D + 2J(2BJ - 3) = 0, \tag{111}$$

$$D^3 + 2CJD^2 + (2BJ - 7)DJ + 4J^2(\sqrt{2}\lambda J - C) = 0. \tag{112}$$

The PDF for the classical field ϕ is just the square of the (normalized) ground state eigenfunctions given in (104), (106) or (109).

IV. ϕ^{12} FIELD THEORY

Finally, there are systems in which phase transitions are only captured by going to the ϕ^{12} field theory (e.g., highly piezoelectric perovskite materials [11, 12]). Depending on the form of the potential, it can have six, five, four, three or two degenerate minima, hence five, four, three, two or one kink solution(s) exist, respectively. In this section, we discuss these cases separately. However, we do not provide a discussion of the various phases of the ϕ^{12} theory because its complexity necessarily makes such a discussion quite lengthy.

A. Six Degenerate Minima

Consider the potential

$$V(\phi) = \lambda^2(\phi^2 - a^2)^2(\phi^2 - b^2)^2(\phi^2 - c^2)^2, \tag{113}$$

where $c > b > a$ without loss of generality. This potential has six degenerate minima at $\phi = \pm a, \pm b, \pm c$ and, hence, five kink solutions exist. Out of these five, only three are distinct due to the symmetry of the potential.

1. Kink connecting $-a$ to $+a$

In this case, the kink solution is given impicitly by

$$e^{\mu x} = \left(\frac{a+\phi}{a-\phi}\right)^{(c^2-b^2)/a}\left(\frac{b-\phi}{b+\phi}\right)^{(c^2-a^2)/b}\left(\frac{c+\phi}{c-\phi}\right)^{(b^2-a^2)/c}, \tag{114}$$

where $\mu = 2\sqrt{2}\lambda(b^2-a^2)(c^2-b^2)(c^2-a^2)$. From (114), the approach to the asymptotes at $\phi = \pm a$ can be shown to be exponential:

$$\phi(x) \simeq \begin{cases} -a + 2a\left[\left(\frac{b+a}{b-a}\right)^{(c^2-a^2)/b}\left(\frac{c-a}{c+a}\right)^{(b^2-a^2)/c}\right]^{-a/(c^2-b^2)} e^{\mu a x/(c^2-b^2)}, & x \to -\infty,\\ +a - 2a\left[\left(\frac{b-a}{b+a}\right)^{(c^2-a^2)/b}\left(\frac{c+a}{c-a}\right)^{(b^2-a^2)/c}\right]^{a/(c^2-b^2)} e^{-\mu a x/(c^2-b^2)}, & x \to +\infty. \end{cases} \tag{115}$$

Clearly, this kink is symmetric. The corresponding kink energy is

$$E_k^{(1)} = \frac{4\sqrt{2}}{105}\lambda a^3 \left[3a^4 - 7(b^2+c^2)a^2 + 35b^2c^2\right]. \tag{116}$$

2. Kink connecting a to b (or $-b$ to $-a$)

In this case, the kink solution is given implicitly by

$$e^{\mu x} = \left(\frac{\phi - a}{\phi + a}\right)^{(c^2-b^2)/a} \left(\frac{b+\phi}{b-\phi}\right)^{(c^2-a^2)/b} \left(\frac{c+\phi}{c-\phi}\right)^{(b^2-a^2)/c}, \tag{117}$$

where μ is given below (114). From (117), the approach to the asymptotes at $\phi = a, b$ can be shown to be exponential:

$$\phi(x) \simeq \begin{cases} a + 2a\left[\left(\dfrac{b+a}{b-a}\right)^{(c^2-a^2)/b}\left(\dfrac{c+a}{c-a}\right)^{(b^2-a^2)/c}\right]^{-a/(c^2-b^2)} e^{\mu a x/(c^2-b^2)}, & x \to -\infty, \\ b - 2b\left[\left(\dfrac{b-a}{b+a}\right)^{(c^2-b^2)/a}\left(\dfrac{c+b}{c-b}\right)^{(b^2-a^2)/c}\right]^{b/(c^2-a^2)} e^{-\mu b x/(c^2-a^2)}, & x \to +\infty. \end{cases} \tag{118}$$

Due to the different growth rates, $\mu b/(c^2-a^2)$ versus $\mu a/(c^2-b^2)$ as $x \to \pm\infty$, respectively, this kink is asymmetric. The corresponding kink energy is

$$E_k^{(2)} = \frac{2\sqrt{2}}{105}\lambda(b-a)^3\left[7c^2(b^2+3ab+a^2) - (3b^4+9b^3a+11b^2a^2+9ba^3+3a^4)\right]. \tag{119}$$

3. Kink connecting b to c (or $-c$ to $-b$)

In this case, the kink solution is given implicitly by

$$e^{\mu x} = \left(\frac{\phi + a}{\phi - a}\right)^{(c^2-b^2)/a} \left(\frac{\phi - b}{\phi + b}\right)^{(c^2-a^2)/b} \left(\frac{c+\phi}{c-\phi}\right)^{(b^2-a^2)/c}, \tag{120}$$

where μ is given below (114). From (120), the approach to the asymptotes at $\phi = b, c$ can be shown to be exponential:

$$\phi(x) \simeq \begin{cases} b + 2b\left[\left(\dfrac{b+a}{b-a}\right)^{(c^2-b^2)/a}\left(\dfrac{c+b}{c-b}\right)^{(b^2-a^2)/c}\right]^{-b/(c^2-a^2)} e^{\mu b x/(c^2-a^2)}, & x \to -\infty, \\ c - 2c\left[\left(\dfrac{c+a}{c-a}\right)^{(c^2-b^2)/a}\left(\dfrac{c-b}{c+b}\right)^{(c^2-a^2)/b}\right]^{c/(b^2-a^2)} e^{-\mu c x/(b^2-a^2)}, & x \to +\infty. \end{cases} \tag{121}$$

Due to the different growth rates, $\mu c/(b^2-a^2)$ versus $\mu b/(c^2-a^2)$ as $x \to \pm\infty$, respectively, this kink is asymmetric.

The corresponding kink energy is

$$E_k^{(3)} = \frac{2\sqrt{2}}{105}\lambda(c-b)^3\big[3c^4 + 9c^3b + 11c^2b^2 + 9cb^3 + 3b^4 - 7a^2(c^2+3bc+b^2)\big]. \tag{122}$$

To the best of our knowledge, this is the first instance in which three kink solutions exist for the same values of the potential's parameters. It would be of interest to determine values of the parameters a, b and c for which $E_k^{(1)} = E_k^{(2)} = E_k^{(3)}$.

All three kinks from this subsection are illustrated in Fig. 12 and compared to the kinks from the ϕ^8 and ϕ^4 field theories. Note that, while the agreement between the ϕ^8 kink connecting $-a$ to $+a$ and the corresponding ϕ^4 one was quite good in Fig. 2, the agreement between the ϕ^{12} kink connecting $-a$ to $+a$ and the corresponding ϕ^4 is not. As in the previous examples, this is mainly due to the curvatures of the potentials near $\phi = 0$ being quite different, hence the kinks having different widths. The

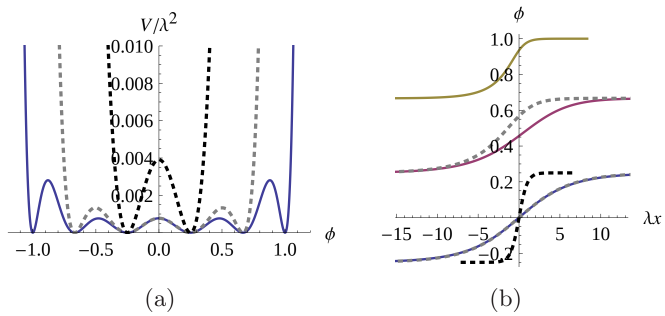


FIG. 12: (Color online.) ϕ^{12} field theory with six degenerate minima. (a) The potential (113) (solid), a representative ϕ^8 potential (3) (gray, dotted), and a representative ϕ^4 potential $V(\phi) = \lambda^2(\phi^2 - a^2)^2$ (black, dotted). (b) Kink solutions connecting $-a$ to $+a$ (114) (bottom curve, blue online), a to b (117) (middle curve, red online), b to c (120) (top curve, yellow online), the corresponding ϕ^8 kinks connecting $-a$ to $+a$ (11) and a to b (15) (both gray, dotted), and the corresponding ϕ^4 kink $\phi(x) = a\tanh(\lambda x)$ (black, dotted). In all panels, $a = 1/4$, $b = 2/3$ and $c = 1$.

agreement between the ϕ^8 and ϕ^{12} kinks connecting $-a$ to $+a$, however, is so good that they are nearly indistinguishable for the chosen parameters. On the other hand, the ϕ^8 and ϕ^{12} kinks connecting a to b do not match as well.

B. Five Degenerate Minima

There are two possible forms of the ϕ^{12} potential with five degenerate minima for which we are able to obtain the kink solutions. In this subsection, we discuss these separately.

1. *Case I*

Consider the potential

$$V(\phi) = \lambda^2\phi^2(\phi^2 - a^2)^2(\phi^2 - b^2)^2(\phi^2 + c^2). \tag{123}$$

This potential has five degenerate minima at $\phi = 0, \pm a, \pm b$ and hence four kink solutions, two of which are distinct due to the symmetry of the potential. As before, we take $b > a$ without any loss of generality.

a. Kink connecting 0 *to* a *(or* $-a$ *to* 0*)* This kink solution is given implicitly by

$$e^{\mu x} = \left(\frac{\sqrt{c^2+\phi^2}-c}{\sqrt{c^2+\phi^2}+c}\right)\left(\frac{\sqrt{c^2+b^2}-\sqrt{c^2+\phi^2}}{\sqrt{c^2+b^2}+\sqrt{c^2+\phi^2}}\right)^{a^2c/(b^2-a^2)\sqrt{c^2+b^2}}\left(\frac{\sqrt{c^2+a^2}+\sqrt{c^2+\phi^2}}{\sqrt{c^2+a^2}-\sqrt{c^2+\phi^2}}\right)^{b^2c/(b^2-a^2)\sqrt{c^2+a^2}}, \tag{124}$$

where $\mu = 2\sqrt{2}\lambda a^2 b^2 c$. From (124), it can be shown that the approach to the asymptotes at $\phi = 0, a$ is

$$\phi(x) \simeq \begin{cases} 2c\left[\frac{a^2+2c(c+\sqrt{c^2+a^2})}{a^2}\right]^{-b^2c/2(b^2-a^2)\sqrt{c^2+a^2}}\left[\frac{b^2+2c(c-\sqrt{c^2+b^2})}{b^2}\right]^{-a^2c/2(b^2-a^2)\sqrt{c^2+b^2}} e^{\mu x/2}, & x \to -\infty, \\ a - \frac{2(c^2+a^2)}{a}\left(\frac{\sqrt{c^2+b^2}-\sqrt{c^2+a^2}}{\sqrt{c^2+b^2}+\sqrt{c^2+a^2}}\right)^{\frac{a^2\sqrt{c^2+a^2}}{b^2\sqrt{c^2+b^2}}}\left[\frac{a^2+2c(c-\sqrt{c^2+a^2})}{a^2}\right]^{(b^2-a^2)\sqrt{c^2+a^2}/b^2c} e^{-\mu x(b^2-a^2)\sqrt{c^2+a^2}/b^2c}, & x \to +\infty. \end{cases} \tag{125}$$

Consequently, this kink is asymmetric due to the different growth rates as $\phi \to 0, a$. The kink's energy is

$$E_k^{(1)} = \frac{\sqrt{2}}{105}\lambda\left[2(c^2+a^2)^{3/2}(4c^4+7b^2-3a^2) - c^3(35a^2b^2+14a^2c^2+14b^2c^2+8c^4)\right]. \tag{126}$$

b. Kink connecting a *to* b *(or* $-b$ *to* $-a$*)* This kink solution is given implicitly by

$$e^{\mu x} = \left(\frac{\sqrt{c^2+\phi^2}-c}{\sqrt{c^2+\phi^2}+c}\right)\left(\frac{\sqrt{c^2+\phi^2}-\sqrt{c^2+a^2}}{\sqrt{c^2+\phi^2}+\sqrt{c^2+a^2}}\right)^{b^2c/(b^2-a^2)\sqrt{c^2+a^2}}\left(\frac{\sqrt{c^2+\phi^2}+\sqrt{c^2+b^2}}{\sqrt{c^2+b^2}-\sqrt{c^2+\phi^2}}\right)^{a^2c/(b^2-a^2)\sqrt{c^2+b^2}}, \tag{127}$$

where $\mu = 2\sqrt{2}\lambda a^2 b^2 c$. From (127), it can be shown that the approach to the asymptotes at $\phi = a, b$ is

$$\phi(x) \simeq \begin{cases} a + \frac{2(c^2+a^2)}{a^2}\left(\frac{\sqrt{c^2+b^2}-\sqrt{c^2+a^2}}{\sqrt{c^2+b^2}+\sqrt{c^2+a^2}}\right)^{\frac{a^2\sqrt{c^2+a^2}}{b^2\sqrt{c^2+b^2}}}\left[\frac{a^2+2c(c-\sqrt{c^2+a^2})}{a^2}\right]^{-(b^2-a^2)\sqrt{c^2+a^2}/b^2c} e^{\mu x(b^2-a^2)\sqrt{c^2+a^2}/b^2c}, & x \to -\infty, \\ b - \frac{2(c^2+b^2)}{b^2}\left(\frac{\sqrt{c^2+b^2}-\sqrt{c^2+a^2}}{\sqrt{c^2+b^2}+\sqrt{c^2+a^2}}\right)^{\frac{b^2\sqrt{c^2+b^2}}{a^2\sqrt{c^2+a^2}}}\left[\frac{b^2+2c(c-\sqrt{c^2+b^2})}{b^2}\right]^{(b^2-a^2)\sqrt{c^2+b^2}/a^2c} e^{-\mu x(b^2-a^2)\sqrt{c^2+b^2}/a^2c}, & x \to +\infty. \end{cases} \tag{128}$$

Consequently, this kink is asymmetric due to the different growth rates as $\phi \to a, b$. The kink's energy is

$$E_k^{(2)} = \frac{2\sqrt{2}}{105}\lambda\Big[(a^2+c^2)^{3/2}(4c^2+7b^2-3a^2) - (b^2+c^2)^{3/2}(4c^2-3b^2+7a^2)\Big]. \quad (129)$$

2. Case II

Now, consider the potential

$$V(\phi) = \lambda^2\phi^4(\phi^2-a^2)^2(\phi^2-b^2)^2. \quad (130)$$

In this case,

$$\begin{aligned} \alpha_{10} &= 2(b^2+a^2), \\ \alpha_8 &= b^4+2b^2a^2+a^4, \\ \alpha_6 &= 2a^2b^2(b^2+a^2), \\ \alpha_4 &= b^4a^4, \\ \alpha_2 &= \alpha_0 = 0. \end{aligned} \quad (131)$$

Clearly, $\alpha_{10,8,6,4} > 0$.

a. Kink connecting 0 *to* a *(or* $-a$ *to* 0*)* This kink solution is given implicitly by

$$\mu x = -\frac{2a(b^2-a^2)}{b^2\phi} + \ln\left[\left(\frac{a+\phi}{a-\phi}\right)\left(\frac{b-\phi}{b+\phi}\right)^{a^3/b^3}\right], \quad (132)$$

where $\mu = 2\sqrt{2}\lambda a^3(b^2-a^2)$. From (132), it can be shown that the approach to the asymptotes at $\phi = 0, a$ is

$$\phi(x) \simeq \begin{cases} \dfrac{2a(b^2-a^2)}{b^2(-\mu x)}, & x \to -\infty, \\ a - 2a\left(\frac{b-a}{b+a}\right)^{a^3/b^3} e^{2a^2/b^2-\mu x-2}, & x \to +\infty. \end{cases} \quad (133)$$

Consequently, this kink is asymmetric due to the different growth types (algebraic versus exponential) as $\phi \to 0, a$, respectively. The kink's energy is

$$E_k^{(1)} = \frac{2\sqrt{2}}{105}\lambda a^5(7b^2-3a^2). \quad (134)$$

b. Kink connecting a *to* b *(or* $-b$ *to* $-a$*)* This kink is given implicitly by

$$\mu x = \frac{2a(b^2-a^2)}{b^2\phi} + \ln\left[\left(\frac{\phi-a}{\phi+a}\right)\left(\frac{b+\phi}{b-\phi}\right)^{a^3/b^3}\right], \quad (135)$$

where $\mu = 2\sqrt{2}\lambda a^3(b^2-a^2)$. From (135), it can be shown that the approach to the asymptotes at $\phi = a, b$ is

$$\phi(x) \simeq \begin{cases} a + 2a\left(\frac{b-a}{b+a}\right)^{a^3/b^3} e^{\mu x - 2a(b^2-a^2)/b^a}, & x \to -\infty, \\ b - 2b\left(\frac{b-a}{b+a}\right)^{b^3/a^3} e^{b^{3-b}(2b^2/a^2-\mu x b^b/a^3-2)}, & x \to +\infty. \end{cases} \quad (136)$$

Consequently, this kink is asymmetric due to the different growth rates $\mu b^3/a^3$ versus μ as $x \to \pm\infty$, repsectively. The kink's energy is

$$E_k^{(2)} = \frac{2\sqrt{2}}{105}\lambda(b-a)^3\left[3b^4+9b^3a+11b^2a^2+9ba^3+3a^4\right]. \quad (137)$$

Comparing the energies of the two kink solutions [(134) and (137)], we find that $E_k^{(1)} \gtreqqless E_k^{(2)}$ if $b/a \lesseqqgtr \sqrt{7/3}$.

All four kinks from this subsection are illustrated in Fig. 13.

C. Four Degenerate Minima

There are three possible forms of the ϕ^{12} potential with four degenerate minima for which we are able to obtain the kink solutions. In this subsection, we discuss these separately.

1. Case I

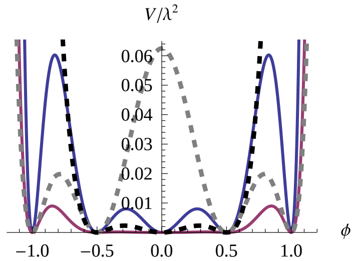
Consider the potential

$$V(\phi) = \lambda^2(\phi^2-a^2)^2(\phi^2-b^2)^2(\phi^2+c^2)^2, \quad (138)$$

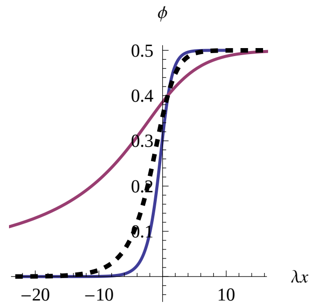
which has four degenerate minima at $\phi = \pm a, \pm b$ ($b > a$ as before) and, hence, three kink solutions, only two of which are distinct due to the symmetry of the potential.

a. Kink connecting $-a$ *to* $+a$ This kink solution is given implicitly by

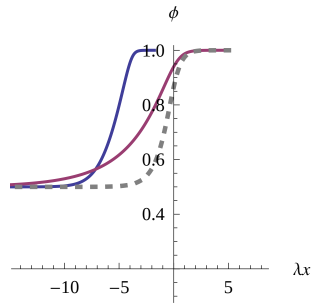
$$\mu x = \frac{2a(b^2-a^2)}{c(c^2+b^2)}\tan^{-1}\left(\frac{\phi}{c}\right) + \frac{a(a^2+c^2)}{b(c^2+b^2)}\ln\left[\left(\frac{a+\phi}{a-\phi}\right)\left(\frac{b-\phi}{b+\phi}\right)\right], \quad (139)$$



(a)



(b)



(c)

FIG. 13: (Color online.) ϕ^{12} field theory with five degenerate minima. (a) The potentials: (123) (top curve, blue online), (130) (bottom curve, red online), a representative ϕ^8 potential with four degenerate minima (3) (gray, dotted), and a representative ϕ^6 potential with three degenerate minima $V(\phi) = \lambda^2\phi^2(\phi^2 - a^2)^2$ (black, dotted). (b) The kink solutions connecting 0 to a: (124) (outer curve, blue online), (132) (inner curve, red online), and the corresponding ϕ^6 kink $\phi(x) = a/\sqrt{1+e^{-2\sqrt{2}a^2\lambda x}}$ (black, dotted). (c) The kink solutions connecting a to b: (127) (outer curve, blue online), (135) (inner curve, red online), and the corresponding ϕ^8 kink (15) (gray, dotted). In all panels, $a = 1/2$, $b = 1$ and $c = 2$.

where $\mu = 2\sqrt{2}\lambda a(c^2+a^2)(b^2-a^2)$. From (139), it can be shown that the approach to the asymptotes at $\phi = \pm a$ is exponential:

$$\phi(x) \simeq \begin{cases} -a + 2a\left(\dfrac{b-a}{b+a}\right)\exp\left[\dfrac{(b^2+c^2)\mu x b/a + 2(b^2-a^2)\tan^{-1}(a/c)b/c}{c^2+a^2}\right], & x \to -\infty, \\ +a - 2a\left(\dfrac{b-a}{b+a}\right)\exp\left[\dfrac{-(b^2+c^2)\mu x b/a + 2(b^2-a^2)\tan^{-1}(a/c)b/c}{c^2+a^2}\right], & x \to +\infty. \end{cases} \tag{140}$$

Clearly, this kink is symmetric. The kink's energy is

$$E_k^{(1)} = \frac{4\sqrt{2}}{105}\lambda a^3[35b^2c^2 - 7a^2c^2 + 7a^2b^2 - 3a^4]. \tag{141}$$

b. Kink connecting a to b (or $-b$ to $-a$) This kink solution is given implicitly by

$$\mu x = -\frac{2a(b^2-a^2)}{c(c^2+b^2)}\tan^{-1}\left(\frac{\phi}{c}\right) + \frac{a(a^2+c^2)}{b(c^2+b^2)}\ln\left[\left(\frac{\phi-a}{\phi+a}\right)\left(\frac{b+\phi}{b-\phi}\right)\right], \tag{142}$$

where $\mu = 2\sqrt{2}\lambda a(c^2+a^2)(b^2-a^2)$. From (142), it can be shown that the approach to the asymptotes at $\phi = a, b$ is exponential:

$$\phi(x) \simeq \begin{cases} a + 2a\left(\dfrac{b-a}{b+a}\right)\exp\left[\dfrac{(b^2+c^2)\mu x b/a + 2(b^2-a^2)\tan^{-1}(a/c)b/c}{c^2+a^2}\right], & x \to -\infty, \\ b - 2b\left(\dfrac{b-a}{b+a}\right)\exp\left[\dfrac{-(b^2+c^2)\mu x b/a - 2(b^2-a^2)\tan^{-1}(b/c)b/c}{c^2+a^2}\right], & x \to +\infty. \end{cases} \tag{143}$$

Clearly, this kink is symmetric. The kink's energy is

$$E_k^{(2)} = \frac{2\sqrt{2}}{105}\lambda(b-a)^3\big[3b^4 + 9b^3a + 11b^2a^2 + 9ba^3 + 3a^4 + 7c^2(b^2+3ab+a^2)\big]. \tag{144}$$

2. Case II

Consider the potential

$$V(\phi) = \lambda^2(\phi^2-a^2)^4(\phi^2-b^2)^2. \tag{145}$$

In this case,

$$\begin{aligned}\alpha_{10} &= 2(b^2+2a^2),\\ \alpha_8 &= b^4+8b^2a^2+6a^4,\\ \alpha_6 &= 4a^2(b^4+3a^2b^2+a^4),\\ \alpha_4 &= a^4(6b^4+8a^2b^2+a^4),\\ \alpha_2 &= 2a^6b^2(a^2+2b^2),\\ \alpha_0 &= a^8b^4.\end{aligned} \tag{146}$$

Clearly, $\alpha_{10,8,6,4,2,0} > 0$.

a. Kink connecting $-a$ and $+a$ In this case, the kink solution is given implicitly by

$$\mu x = \frac{b\phi(b^2-a^2)}{a^2(a^2-\phi^2)} + \ln\left(\frac{b+\phi}{b-\phi}\right) - \frac{b(3a^2-b^2)}{2a^3}\ln\left(\frac{a+\phi}{a-\phi}\right), \tag{147}$$

where $\mu = 2\sqrt{2}\lambda b(b^2-a^2)^2$. From (147), it can be shown that the approach to the asymptotes at $\phi = \pm a$ is algebraic:

$$\phi(x) \simeq \begin{cases} -a - \dfrac{b(b^2-a^2)}{2a^2\mu x}, & x\to-\infty,\\ +a - \dfrac{b(b^2-a^2)}{2a^2\mu x}, & x\to+\infty.\end{cases} \tag{148}$$

Clearly, this kink is symmetric. The kink's energy is

$$E_k^{(1)} = \frac{16\sqrt{2}}{105}\lambda a^5(7b^2-a^2). \tag{149}$$

b. Kink connecting a to b (or $-b$ to $-a$) In this case, the kink solution is given implicitly by

$$\mu x = -\frac{b\phi(b^2-a^2)}{a^2(\phi^2-a^2)} + \ln\left(\frac{b+\phi}{b-\phi}\right) + \frac{b(3a^2-b^2)}{2a^3}\ln\left(\frac{\phi-a}{\phi+a}\right), \tag{150}$$

where $\mu = 2\sqrt{2}\lambda b(b^2-a^2)^2$. From (150), it can be shown that the approach to the asymptotes at $\phi = a, b$ is of mixed type:

$$\phi(x) \simeq \begin{cases} a - \dfrac{b(b^2-a^2)}{2a^2\mu x}, & x\to-\infty,\\ b - 2b\left(\frac{b+a}{b-a}\right)^{(\kappa^{-2}-3)/(2\kappa)} e^{-\mu x-\kappa^{-2}}, & x\to+\infty,\end{cases} \tag{151}$$

where $\kappa = a/b$. Consequently, the kink is asymmetric due to the different growth types as $x\to\pm\infty$. The kink's energy is

$$E_k^{(2)} = \frac{2\sqrt{2}}{105}\lambda(b-a)^4(3b^3+12b^2a+16ba^2+4a^3). \tag{152}$$

3. Case III

Consider the potential

$$V(\phi) = \lambda^2(\phi^2-a^2)^2(\phi^2-b^2)^4. \tag{153}$$

In this case,

$$\begin{aligned}\alpha_{10} &= 2(b^2+2a^2),\\ \alpha_8 &= b^4+8a^2b^2+6a^4,\\ \alpha_6 &= 4a^2b^2(b^2+3a^2)+4a^6,\\ \alpha_4 &= a^4(6b^4+8a^2b^2+a^4),\\ \alpha_2 &= a^6b^2(4b^2+2a^2),\\ \alpha_0 &= a^8b^4.\end{aligned} \tag{154}$$

Clearly, $\alpha_{10,8,6,4,2,0} > 0$.

a. Kink connecting $-a$ and $+a$ In this case, the kink solution is given implicitly by

$$\mu x = -\frac{a\phi(b^2-a^2)}{b^2(b^2-\phi^2)} + \ln\left(\frac{a+\phi}{a-\phi}\right) - \frac{a(3b^2-a^2)}{2b^3}\ln\left(\frac{b+\phi}{b-\phi}\right), \tag{155}$$

where $\mu = 2\sqrt{2}\lambda a(b^2-a^2)^2$. From (155), it can be shown that the approach to the asymptotes at $\phi = \pm a$ is exponential:

$$\phi(x) \simeq \begin{cases} -a + 2a\left(\frac{b+a}{b-a}\right)^{\kappa(\kappa^2-3)/2} e^{\mu x-\kappa^2}, & x\to-\infty,\\ +a - 2a\left(\frac{b+a}{b-a}\right)^{\kappa(\kappa^2-3)/2} e^{-\mu x-\kappa^2}, & x\to+\infty,\end{cases} \tag{156}$$

where $\kappa = a/b$. Clearly, this kink is symmetric. The corresponding kink energy is

$$E_k^{(1)} = \frac{4\sqrt{2}}{105}\lambda a^3(35b^4-14a^2b^2+3a^4). \tag{157}$$

b. Kink connecting a to b (or $-b$ to $-a$) In this case, the kink solution is given implicitly by

$$\mu x = \frac{a\phi(b^2-a^2)}{b^2(b^2-\phi^2)} + \ln\left(\frac{\phi-a}{\phi+a}\right) + \frac{a(3b^2-a^2)}{2b^3}\ln\left(\frac{b+\phi}{b-\phi}\right), \tag{158}$$

where $\mu = 2\sqrt{2}\lambda a(b^2-a^2)^2$. From (158), it can be shown that the approach to the asymptotes at $\phi = a, b$ is of mixed type:

$$\phi(x) \simeq \begin{cases} a - 2a\left(\frac{b-a}{b+a}\right)^{\kappa(\kappa^2-3)/2} e^{\mu x-\kappa^2}, & x\to-\infty,\\ b - \dfrac{a(b^2-a^2)}{2b^2\mu x}, & x\to+\infty.\end{cases} \tag{159}$$

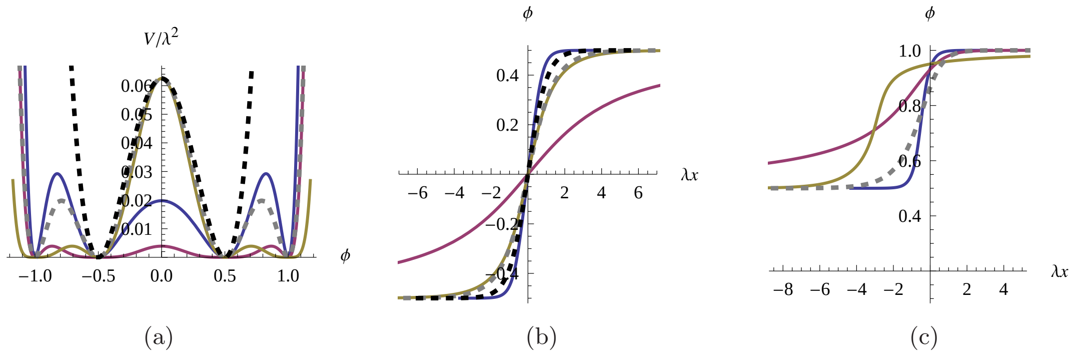


FIG. 14: (Color online.) ϕ^{12} field theory with four degenerate minima. (a) The potentials: (138) (second curve from top to bottom, blue online), (145) (third curve from top to bottom, red online), (153) (first curve from top to bottom, yellow online), a representative ϕ^8 potential with four degenerate minima (3) (gray, dotted), and a representative ϕ^4 potential $V(\phi) = \lambda^2(\phi^2 - a^2)^2$ (black, dotted). (b) The kink solutions connecting $-a$ to $+a$: (139) (outer curve, blue online), (147) (inner curve red, online), (155) (middle curve, yellow online), the corresponding ϕ^8 kink (11) (gray, dotted), and the corresponding ϕ^4 kink $\phi(x) = a\tanh(\lambda x)$ (black, dotted). (c) The kink solutions connecting a to b: (142) (outer curve, blue online), (150) (inner curve, red online), (158) (middle curve, yellow online), and the corresponding ϕ^8 kink (15) (gray, dotted). In all panels, $a = 1/2$, $b = 1$ and $c = 3/4$.

where $\kappa = a/b$. Consequently, this kink is asymmetric due to the different growth types as $x \to \pm\infty$. The kink's energy is

$$E_k^{(2)} = \frac{2\sqrt{2}}{105}\lambda(b-a)^4(4b^3 + 16b^2 a + 12ba^2 + 3a^3). \quad (160)$$

All six kinks from this subsection are illustrated in Fig. 14. Note that the kink connecting a to b in Case I is symmetric [see (143)], unlike the corresponding ϕ^8 kink (15) [see also (16)].

D. Three Degenerate Minima

There are five possible forms of the ϕ^{12} potential with three degenerate minima for which kink solutions can be obtained analytically. In this subsection, we discuss these cases separately.

1. Case I

Consider the potential

$$V(\phi) = \lambda^2\phi^8(\phi^2 - a^2)^2, \quad (161)$$

which has three degenerate minima at $\phi = 0, \pm a$. In this case, $\alpha_{10,8} > 0$, while $\alpha_{6,4,2,0} = 0$. The kink solution, which connects 0 to a (or $-a$ to 0), as x goes from $-\infty$ to $+\infty$, is given implicitly by

$$\mu x = -\frac{2a}{\phi} - \frac{2a^3}{3\phi^3} + \ln\left(\frac{a+\phi}{a-\phi}\right), \quad (162)$$

where $\mu = 2\sqrt{2}\lambda a^5$. From (162), it can be shown that the approach to the asymptotes at $\phi = 0, a$ is of mixed type:

$$\phi(x) \simeq \begin{cases} \dfrac{2^{1/3}a}{(-3\mu x)^{1/3}}, & x \to -\infty, \\ a - 2ae^{-\mu x - 8/3}, & x \to +\infty. \end{cases} \quad (163)$$

Consequently, this kink is asymmetric due to the different growth types as $x \to \pm\infty$. The kink's energy is

$$E_k = \frac{2\sqrt{2}}{35}\lambda a^7. \quad (164)$$

2. Case II

Consider the potential

$$V(\phi) = \lambda^2\phi^4(\phi^2 - a^2)^4. \quad (165)$$

In this case, $\alpha_{10,8,6,4} > 0$, while $\alpha_{2,0} = 0$. The kink solution, which connects 0 to a (or $-a$ to 0), as x goes from $-\infty$ to $+\infty$, is given implicitly by

$$\mu x = \frac{2a(3\phi^2 - 2a^2)}{3\phi(a^2 - \phi^2)} + \ln\left(\frac{a+\phi}{a-\phi}\right), \quad (166)$$

where $\mu = (4/3)\sqrt{2}\lambda a^5$. From (166), it can be shown that the approach to the asymptotes at $\phi = 0, a$ is algebraic:

$$\phi(x) \simeq \begin{cases} -\dfrac{a}{\sqrt{2}a^5\lambda x}, & x \to -\infty, \\ a - \dfrac{a}{4\sqrt{2}a^5\lambda x}, & x \to +\infty. \end{cases} \quad (167)$$

Consequently, this kink is asymmetric due to the different growth rates as $x \to \pm\infty$. The kink's energy is

$$E_k = \frac{8\sqrt{2}}{105}\lambda a^7. \quad (168)$$

3. Case III

Consider the potential

$$V(\phi) = \lambda^2\phi^4(\phi^2 - a^2)^2(\phi^2 + b^2)^2. \tag{169}$$

In this case,

$$\begin{aligned}
\alpha_{10} &= 2(b^2 - a^2),\\
\alpha_8 &= b^4 - 4a^2b^2 + a^4,\\
\alpha_6 &= 2a^2b^2(b^2 - a^2),\\
\alpha_4 &= a^4b^4,\\
\alpha_2 &= \alpha_0 = 0.
\end{aligned} \tag{170}$$

It can be shown that $\alpha_{10,8,6,4} > 0$ as long as $b\sqrt{2-\sqrt{3}} > a$.

The kink solution, which connects 0 to a (or $-a$ to 0), as x goes from $-\infty$ to $+\infty$, is given implicitly by

$$\mu x = -\frac{2a(b^2+a^2)}{b^2\phi} - \frac{2a^3}{b^3}\tan^{-1}\left(\frac{\phi}{b}\right) + \ln\left(\frac{a+\phi}{a-\phi}\right), \tag{171}$$

where $\mu = 2\sqrt{2}\lambda a^3(b^2+a^2)$. From (171), it can be shown that the approach to the asymptotes at $\phi = 0, a$ is of mixed type:

$$\phi(x) \simeq \begin{cases} -\dfrac{1}{\sqrt{2}b^2a^2\lambda x}, & x \to -\infty,\\ a - 2ae^{-\mu x - 2(1+\kappa^2+\kappa^3\tan^{-1}\kappa)}, & x \to +\infty, \end{cases} \tag{172}$$

where $\kappa = a/b$. Consequently, this kink is asymmetric due to the different growth types as $x \to \pm\infty$. The kink's energy is

$$E_k = \frac{2\sqrt{2}}{105}\lambda a^5(7b^2 + 3a^2). \tag{173}$$

4. Case IV

Consider the potential

$$V(\phi) = \lambda^2\phi^2(\phi^2 - a^2)^2(\phi^2 + b^2)^3. \tag{174}$$

In this case,

$$\begin{aligned}
\alpha_{10} &= 2a^2 - 3b^2,\\
\alpha_8 &= 3b^4 - 6a^2b^2 + a^4,\\
\alpha_6 &= b^2(6a^2b^2 - 3a^4 - b^4),\\
\alpha_4 &= a^2b^4(3a^2 - 2b^2),\\
\alpha_2 &= -a^4b^6,\\
\alpha_0 &= 0.
\end{aligned} \tag{175}$$

It can be shown that $\alpha_{10,2} < 0$, while $\alpha_{8,6} > 0$ and $\alpha_4 < 0$ as long as $a < b\sqrt{3-\sqrt{6}} < \sqrt{3}a$.

The kink solution, which connects 0 to a (or $-a$ to 0) as x goes from $-\infty$ to $+\infty$, is given implicitly by

$$\mu x = \frac{2a^2\sqrt{b^2+a^2}}{b^2\sqrt{b^2+\phi^2}} + \frac{(b^2+a^2)^{3/2}}{b^3}\ln\left[\left(\frac{\sqrt{b^2+a^2}+\sqrt{b^2+\phi^2}}{\sqrt{b^2+a^2}-\sqrt{b^2+\phi^2}}\right)\left(\frac{\sqrt{b^2+\phi^2}-b}{\sqrt{b^2+\phi^2}+b}\right)\right], \tag{176}$$

where $\mu = \sqrt{2}\lambda(b^2+a^2)^{3/2}$. From (176), it can be shown that the approach to the asymptotes at $\phi = 0, a$ is exponential:

$$\phi(x) \simeq \begin{cases} \dfrac{2ab}{\sqrt{a^2+2b\left(b+\sqrt{b^2+a^2}\right)}}\exp\left[-\dfrac{a^2}{b^2+a^2} + \dfrac{b^3\mu x}{2\left(b^2+a^2\right)^{3/2}}\right], & x \to -\infty,\\ \dfrac{a}{\left(b+\sqrt{b^2+a^2}\right)^2}\left\{a^2 + 2b\left(b+\sqrt{b^2+a^2}\right) - 2\left(b^2+a^2\right)\exp\left[\dfrac{2a^2b - b^3\mu x}{\left(b^2+a^2\right)^{3/2}}\right]\right\}, & x \to +\infty. \end{cases} \tag{177}$$

Consequently, this kink is asymmetric due to the different growth rates as $x \to \pm\infty$. The kink's energy is

$$E_k = \frac{\sqrt{2}}{35}\lambda\left[2(b^2+a^2)^{7/2} - b^5(7a^2+b^2)\right]. \tag{178}$$

5. Case V

Consider the potential

$$V(\phi) = \lambda^2\phi^6(\phi^2 - a^2)^2(\phi^2 + b^2), \tag{179}$$

which has three degenerate minima at $\phi = 0, \pm a$. In this case,

$$\begin{aligned}\alpha_{10} &= b^2 - 2a^2,\\ \alpha_8 &= a^4 - 2a^2 b^2,\\ \alpha_6 &= -a^4 b^2,\\ \alpha_4 &= \alpha_2 = \alpha_0 = 0.\end{aligned} \tag{180}$$

It can be shown that $\alpha_{10} > 0$ as long as $\sqrt{2}a > b > a$, while $\alpha_{8,6} < 0$.

The kink solution, which connects 0 to a (or $-a$ to 0), as x goes from $-\infty$ to $+\infty$, is given implicitly by

$$\mu x = -\frac{a^2\sqrt{b^2+a^2}\sqrt{b^2+\phi^2}}{b^2\phi^2} + \frac{(2b^2-a^2)(b^2+a^2)^{1/2}}{b^3}\ln\left[\left(\frac{\sqrt{b^2+\phi^2}+\sqrt{b^2+a^2}}{\sqrt{b^2+a^2}-\sqrt{b^2+\phi^2}}\right)\left(\frac{\sqrt{b^2+\phi^2}-b}{\sqrt{b^2+\phi^2}+b}\right)\right], \tag{181}$$

where $\mu = 2\sqrt{2}\lambda a^4(b^2+a^2)^{1/2}$. From (181), it can be shown that the approach to the asymptotes at $\phi = 0, a$ is of mixed type:

$$\phi(x) \simeq \begin{cases} \dfrac{a(b^2+a^2)^{1/4}}{\sqrt{-b\mu x}}, & x \to -\infty,\\ a - a^{-3}\left[2a^4 + 6a^2b^2 + 4b^4 - 4b(b^2+a^2)^{3/2}\right] e^{-\mu x - 1 - a^2/b^2}, & x \to +\infty. \end{cases} \tag{182}$$

Consequently, this kink is asymmetric due to the different growth types as $x \to \pm\infty$. The kink's energy is

$$E_k = \frac{2\sqrt{2}}{105}\lambda\left[(4b^2+7a^2)b^5 - (4b^2-3a^2)(b^2+a^2)^{5/2}\right]. \tag{183}$$

All five kinks from this subsection are illustrated in Fig. 15. Note that the plots for Cases II and IV are distinct from those for Cases I, III and V in part due to pure algebraic and pure exponential versus mixed type, respectively, decay of the corresponding kinks' tails as $\phi \to 0, a$ [recall (167) and (177)].

E. Two Degenerate Minima

There are three possible forms of the ϕ^{12} potential with two degenerate minima for which we can obtain a kink solution that connects $\phi = -a$ to $\phi = +a$, as x goes from $-\infty$ to $+\infty$. We discuss these separately.

1. *Case I*

Consider the potential

$$V(\phi) = \lambda^2(\phi^2 - a^2)^2(\phi^2 + b^2)^4. \tag{184}$$

In this case,

$$\begin{aligned}\alpha_{10} &= 2(2b^2 - a^2),\\ \alpha_8 &= 6b^4 - 8a^2b^2 + a^4,\\ \alpha_6 &= 4b^2(b^2+ab-a^2)(a^2+ab-b^2),\\ \alpha_4 &= b^4(b^4 - 8a^2b^2 + 6a^4),\\ \alpha_2 &= 2a^2b^6(b^2-2a^2),\\ \alpha_0 &= a^4b^8.\end{aligned} \tag{185}$$

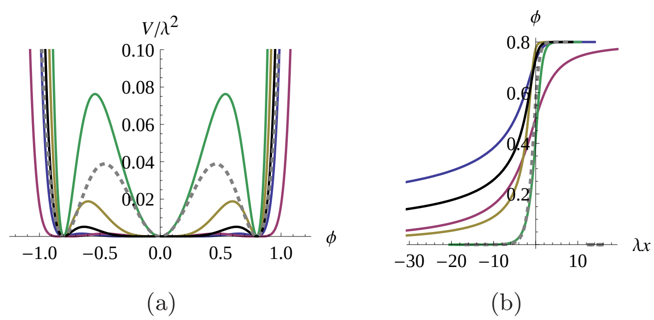


FIG. 15: (Color online.) ϕ^{12} field theory with three degenerate minima. (a) The potentials: (161) (fourth curve from top to bottom, blue online), (165) (fifth curve from top to bottom, red online), (169) (second curve from top to bottom, yellow online), (174) (first curve from top to bottom, green online), (179) (third curve from top to bottom, black online), and a representative ϕ^6 potential $V(\phi) = \lambda^2\phi^2(\phi^2 - a^2)^2$ (gray, dotted). (b) The kink solutions connecting 0 to a: (162) (first curve from top to bottom, blue online), (166) (third curve from top to bottom, red online), (171) (fourth curve from top to bottom, yellow online), (176) (fifth curve from top to bottom, green online), (181) (second curve from top to bottom, black online), and the corresponding ϕ^6 kink $\phi(x) = a/\sqrt{1 + e^{-2\sqrt{2}a^2\lambda x}}$ (gray, dotted). In all panels, $a = 4/5$ and $b = 1$.

It can be shown that $\alpha_{10,0} > 0$, while $\alpha_{8,6,2} > 0$ and $\alpha_4 < 0$ as long as $2a/(\sqrt{5}-1) > b > \sqrt{2}a$.

The kink solution is given implicitly by

$$\mu x = \frac{a(b^2+a^2)\phi}{b^2(b^2+\phi^2)} + \frac{a(a^2+3b^2)}{b^3}\tan^{-1}\left(\frac{\phi}{b}\right) + \ln\left(\frac{a+\phi}{a-\phi}\right), \quad (186)$$

where $\mu = 2\sqrt{2}\lambda a(b^2+a^2)^2$. From (186), the approach to the asymptotes at $\phi = \pm a$ can be shown to be exponential:

$$\phi(x) \simeq \begin{cases} -a + 2ae^{\mu x + \kappa^2 + (\kappa^2+3)\kappa \tan^{-1}\kappa}, & x \to -\infty, \\ +a - 2ae^{-\mu x + \kappa^2 + (\kappa^2+3)\kappa \tan^{-1}\kappa}, & x \to +\infty, \end{cases} \quad (187)$$

where $\kappa = a/b$. Clearly, this kink is symmetric. The kink's energy is

$$E_k = \frac{4\sqrt{2}}{105}\lambda a^3(35b^4 + 14a^2b^2 + 3a^4). \quad (188)$$

2. Case II

Consider the potential

$$V(\phi) = \lambda^2(\phi^2-a^2)^4(\phi^2+b^2)^2. \quad (189)$$

In this case,

$$\begin{aligned} \alpha_{10} &= 2(2a^2-b^2), \\ \alpha_8 &= b^4 - 8a^2b^2 + 6a^4, \\ \alpha_6 &= 4a^2(a^2-ab-b^2)(a^2+ab-b^2), \\ \alpha_4 &= a^4(6b^4 - 8a^2b^2 + a^4), \\ \alpha_2 &= 2a^6b^2(2b^2-a^2), \\ \alpha_0 &= a^8b^4. \end{aligned} \quad (190)$$

It can be shown that $\alpha_{10} < 0$ and $\alpha_{8,6,4,2} > 0$ as long as $b\sqrt{4-\sqrt{10}} > \sqrt{6}a$, while $\alpha_0 > 0$.

The kink solution is given implicitly by

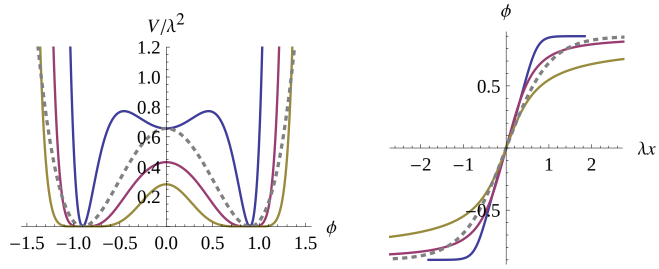
$$\mu x = \frac{2a(b^2+a^2)\phi}{(3a^2+b^2)(a^2-\phi^2)} + \frac{4a^3}{b(3a^2+b^2)}\tan^{-1}\left(\frac{\phi}{b}\right) + \ln\left(\frac{a+\phi}{a-\phi}\right), \quad (191)$$

where $\mu = 4\sqrt{2}\lambda(b^2+a^2)^2/(3a^2+b^2)$. From (191), the approach to the asymptotes at $\phi = \pm a$ can be shown to be algebraic:

$$\phi(x) \simeq \begin{cases} -a - \dfrac{a}{4\sqrt{2}(b^2+a^2)\lambda x}, & x \to -\infty, \\ +a - \dfrac{a}{4\sqrt{2}(b^2+a^2)\lambda x}, & x \to +\infty. \end{cases} \quad (192)$$

Clearly, this kink is symmetric. The kink's energy is

$$E_k = \frac{16\sqrt{2}}{105}\lambda a^5(7b^2+a^2). \quad (193)$$



(a) (b)

FIG. 16: (Color online.) ϕ^{12} field theory with two degenerate minima. (a) The potentials: (184) (top curve, blue online), (189) (middle curve, red online), (194) (bottom curve, yellow online), and a representative ϕ^4 potential $V(\phi) = \lambda^2(\phi^2-a^2)^2$ (dotted). (b) The kink solutions connecting $-a$ to $+a$: (186) (outer curve, blue online), (191) (middle curve, red online), (195) (inner curve, yellow online), and the corresponding ϕ^4 kink $\phi(x) = a\tanh(\lambda x)$ (dotted). In all panels, $a = 9/10$ and $b = 1$.

3. Case III

Consider the potential

$$V(\phi) = \lambda^2(\phi^2-a^2)^6. \quad (194)$$

In this case, $\alpha_{10,8,6,4,2,0} > 0$. The kink solution is given implicitly by

$$\mu x = \frac{a(7a^2-3\phi^2)\phi}{3(a^2-\phi^2)^2} + \ln\left(\frac{a+\phi}{a-\phi}\right), \quad (195)$$

where $\mu = (16/3)\sqrt{2}a^5\lambda$. From (195), the approach to the asymptotes at $\phi = \pm a$ can be shown to be algebraic:

$$\phi(x) \simeq \begin{cases} -a + \dfrac{a}{\sqrt{-3\mu x}}, & x \to -\infty, \\ +a - \dfrac{a}{\sqrt{3\mu x}}, & x \to +\infty. \end{cases} \quad (196)$$

Clearly, this kink is symmetric. The kink's energy is

$$E_k = \frac{32\sqrt{2}}{35}\lambda a^7. \quad (197)$$

All three kinks from this subsection are illustrated in Fig. 16.

F. Phonons

The discussion from Section II E applies here as well. Table III summarizes the properties of the phonon dispersion relation (42) for the ϕ^{12} field theories with kink solutions studied above. As was the case for the ϕ^8 and ϕ^{10} field theories considered above, there are once again potentials for which the RHS of the dispersion relation vanishes; but, it cannot vanish in the other cases due to our assumption $c > b > a$.

TABLE III: Phonon modes of ϕ^{12} field theory. DM = degenerate minima. RHS = dispersion relation right-hand side.

potential, V	equilibrium, ϕ_e	RHS, $V''(\phi_e)$
6 DM, Eq. (113)	$\pm a$	$8\lambda^2 a^2(b^2-a^2)^2(c^2-a^2)^2$
6 DM, Eq. (113)	$\pm b$	$8\lambda^2 b^2(b^2-a^2)^2(c^2-b^2)^2$
6 DM, Eq. (113)	$\pm c$	$8\lambda^2 c^2(c^2-a^2)^2(c^2-b^2)^2$
5 DM, Eq. (123)	0	$2\lambda^2 a^4 b^4 c^2$
5 DM, Eq. (123)	$\pm a$	$8\lambda^2 a^4(b^2-a^2)^2(c^2+a^2)$
5 DM, Eq. (123)	$\pm b$	$8\lambda^2 b^4(b^2-a^2)^2(c^2+b^2)$
5 DM, Eq. (130)	0	0
5 DM, Eq. (130)	$\pm a$	$8\lambda^2 a^6(b^2-a^2)^2$
5 DM, Eq. (130)	$\pm b$	$8\lambda^2 b^6(b^2-a^2)^4$
4 DM, Eq. (138)	$\pm a$	$8\lambda^2 a^2(b^2-a^2)^2(c^2+a^2)^2$
4 DM, Eq. (138)	$\pm b$	$8\lambda^2 b^2(b^2-a^2)^2(c^2+b^2)^2$
4 DM, Eq. (145)	$\pm a$	0
4 DM, Eq. (145)	$\pm b$	$8\lambda^2 b^2(b^2-a^2)^4$
4 DM, Eq. (153)	$\pm a$	$8\lambda^2 a^2(b^2-a^2)^4$
4 DM, Eq. (153)	$\pm b$	0
3 DM, Eq. (161)	0	0
3 DM, Eq. (161)	$\pm a$	$8\lambda^2 a^{10}$
3 DM, Eq. (165)	0	0
3 DM, Eq. (165)	$\pm a$	0
3 DM, Eq. (169)	0	0
3 DM, Eq. (169)	$\pm a$	$8\lambda^2 a^6(b^2+a^2)^2$
3 DM, Eq. (174)	0	$2\lambda^2 a^4 b^6$
3 DM, Eq. (174)	$\pm a$	$8\lambda^2 a^4(b^2+a^2)^3$
3 DM, Eq. (179)	0	0
3 DM, Eq. (179)	$\pm a$	$8\lambda^2 a^8(b^2+a^2)$
2 DM, Eq. (184)	$\pm a$	$8\lambda^2 a^2(b^2+a^2)^4$
2 DM, Eq. (189)	$\pm a$	0
2 DM, Eq. (194)	$\pm a$	0

V. LIMITING BEHAVIORS AS $n \to \infty$

As the degree of the even polynomial field theories considered herein becomes large, there are two limiting cases to be considered. The potentials have the general form

$$V_{2m}(\phi) = \lambda^2 \sum_{i=0}^{m} (-1)^{m-i}\alpha_{2i}\phi^{2i}, \tag{198}$$

where for $m = 2n$ (even) we obtain the ϕ^4, ϕ^8, ϕ^{12}, etc. field theories, whereas for $m = 2n+1$ (odd) we obtain the ϕ^6, ϕ^{10}, etc. field theories.

Now, there are two paths to obtaining the limiting field theory as $m \to \infty$. First, for $m = 2n$ (even), we choose $\alpha_0 = 2$ and $\alpha_{2i} = 1/(2i)!$, then

$$\lim_{n\to\infty} V_{4n}(\phi) = \lambda^2(1+\cos\phi), \tag{199}$$

which satisfies both $\min_\phi V(\phi) = 0$, $\alpha_{2i} > 0$ for all i, and the coefficient of ϕ^{4n} is $\alpha_{4n} > 0$ as needed to ensure $V_{4n}(\phi) \to +\infty$ as $|\phi| \to \infty$. For these theories, the maximum number of degenerate minima is even and, hence, there is no degenerate minimum at $\phi = 0$, *unlike* the sine-Gordon theory.

Second, for $m = 2n+1$ (odd), we choose $\alpha_0 = 0$ and $\alpha_{2i} = 1/(2i)!$, then

$$\lim_{n\to\infty} V_{4n+2}(\phi) = \lambda^2(1-\cos\phi) \equiv V_{\text{sine-Gordon}}(\phi), \tag{200}$$

which satisfies both $\min_\phi V(\phi) = 0$, $\alpha_{2i} > 0$ for all i, and the coefficient of ϕ^{4n+2} is $\alpha_{4n+2} > 0$ as needed to ensure $V_{4n+2}(\phi) \to +\infty$ as $|\phi| \to \infty$. For these theories, the maximum number of degenerate minima is odd and, hence, there is a degenerate minimum at $\phi = 0$, as in the sine-Gordon theory.

Lohe [5] argued that both the ϕ^{4n} and ϕ^{4n+2} field theories limit onto the sine-Gordon theory with potential $\lambda^2(1-\cos\phi)$, while we showed that they limit onto field theories with potentials $\lambda^2(1\pm\cos\phi)$, respectively. This is because Lohe [5] only considered ϕ^{4n} field theories with a degenerate minimum at $\phi = 0$, i.e., $V_{4n}(\phi) = \phi^2 V_{4n-2}(\phi)$ (see [5, Eq. (10)]), while this does not have to be the case in general [recall, e.g., the ϕ^{12} potential in (138)]. Nevertheless, the the two limiting theories are indeed equivalent, as shown below.

The limiting kink structures are easily found to be

$$\tan(\phi/4) = \begin{cases} \tanh(\lambda x/2), & V(\phi) = \lambda^2(1+\cos\phi), \\ e^{\lambda x}, & V(\phi) = \lambda^2(1-\cos\phi), \end{cases} \tag{201}$$

and both field theories are fully-integrable. These two kinks are illustrated in Fig. 17. The two limiting theories are equivalent through the transformation $\phi \mapsto \phi - \pi$, since $\cos(\phi-\pi) = -\cos\phi$ and $\tan(\phi/4 - \pi/4) = [\tan(\phi/4)-1]/[1+\tan(\phi/4)]$, which upon equating to $\tanh(\lambda x/2)$ and solving gives $\tan(\phi/4) = e^{\lambda x}$. Similarly, kink lattice solutions can also be obtained.

We expect the corresponding statistical mechanics, correlation functions and PDFs of the ϕ^{4n+2} theories to approach, asymptotically as $n \to \infty$, those of the sine-Gordon theory derived in [29].

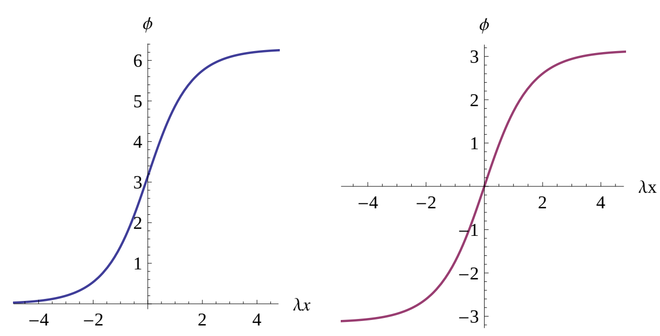


FIG. 17: (Color online.) Kink solutions (201) of the $n \to \infty$ limiting field theories.

VI. CONCLUSION

We have systematically studied high-order polynomial field theories (specifically, ϕ^8, ϕ^{10} and ϕ^{12}) describing successions of phase transitions, and we obtained exact analytical (albeit implicit) kink solutions in different special cases of the theories with degenerate minima. In view of [31], steadily-translating kink solutions (with velocity v and initial location x_0) can be obtained from the static kinks found herein through the Lorentz boost

$$\{x,t\} \mapsto \left\{ \frac{x - x_0 - vt}{\sqrt{1-v^2}}, t \right\}. \tag{202}$$

Similarly anti-kink solutions can be obtained through the transformation

$$\{x,\phi\} \mapsto -\{x,\phi\}. \tag{203}$$

Some novel features of the kink solutions found herein include asymmetry, power-law decay of their tails, possibility of different kink types to have equal energy, and nonlinear phonons. The tail asymptotics that we derived for the kinks above could be used, in conjunction with Manton's approach [35], to compute (asymptotically, for large separations) kink–kink and kink–anti-kink interaction energies [36]. It would also be of interest to determine whether the implicit kink solutions can be used to study interactions via the collective-coordinate variational approximation techniques previously applied to the ϕ^4 [37, 38], ϕ^6 [39] and sine-Gordon [40, 41] field theories.

The field theories considered above also possess pulse solutions confined to individual minima of the relevant potentials, however, such pulse solutions are beyond the scope of this work.

Beyond meson physics [5, 14], the kink solutions obtained here correspond to domain walls in different ferroic materials such as ferroelectric and ferroelastic ones [3, 7–9, 11, 12]. It would be instructive to explore how asymmetric domain walls and nonlinear phonons affect the thermodynamic and physical properties of these materials.

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