Solution of the explosive percolation quest: Scaling functions and critical exponents

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Solution of the explosive percolation quest: Scaling functions and critical exponents

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Percolation refers to the emergence of a giant connected cluster in a disordered system when the number of connections between nodes exceeds a critical value. The percolation phase transitions were believed to be continuous until recently when in a new so-called "explosive percolation" problem for a competition driven process, a discontinuous phase transition was reported. The analysis of evolution equations for this process showed however that this transition is actually continuous though with surprisingly tiny critical exponents. For a wide class of representative models, we develop a strict scaling theory of this exotic transition which provides the full set of scaling functions and critical exponents. This theory indicates the relevant order parameter and susceptibility for the problem, and explains the continuous nature of this transition and its unusual properties.

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The percolation phase transition is one of the central issues for disordered systems [1–4]. Phase transitions in classical percolation problems are very well known to be continuous, that is, the relative size of the percolation cluster S, which is the order parameter for these models, emerges continuously, without a jump at the percolation threshold. As a continuous phase transition, the ordinary percolation transition is characterized by a power-law distribution of cluster sizes at the percolation threshold and a set of standard scaling properties and relations.

This common understanding of percolation was shaken by work [5] that reported a discontinuous percolation phase transition in models whose evolution was driven by local optimization algorithms. Based on a computer experiment for a $512\,,000$ node system [5], it was concluded that the percolation transition for these irreversible processes is discontinuous, and that is why this kind of percolation was termed "explosive percolation". This conclusion was supported by a number of simulations of models of this kind [6–15]. Surprisingly, these and other studies, in addition, reported power-law cluster size distributions at the critical point and scaling features below and above t_c (see Ref. [13–18]), unexpected for discontinuous transitions.

We resolved this contradiction by showing that the explosive percolation transition is actually continuous though with a uniquely small critical exponent β of the percolation cluster size [19]. We obtained this result by analyzing evolution equations for this process in the infinite system size limit. Thanks to the smallness of the exponent β, the continuous transition looks so "sharp" that it is virtually impossible to distinguish it from a discontinuous one in computer experiments even for very large systems [19]. More recently, the fact that this transition is continuous was also supported by mathematicians [23]. Nonetheless, in the physics sense, the quest of the explosive percolation transition actually has not been yet resolved. The main problem is how to explain the nature of this surprising physical phenomenon.

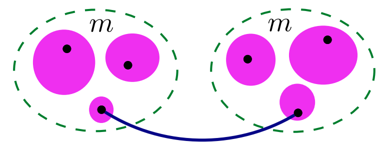

FIG. 1. **Illustration of rules in the model of explosive percolation.** At each step, two sets of m nodes are chosen at random. Within each set, the node in the smallest cluster is selected, and these two nodes are interconnected.

Here, for this explosive percolation transition in a wide set of representative models, we fulfill the following program. We indicate the order parameter and the generalized susceptibility, find the full set of scaling relations and relations between critical exponents, obtain the scaling functions and critical exponents, and get the upper critical dimension (that is, the dimension, above which a mean-field description valid). In short, we develop a scaling theory of this transition.

The main body of this paper is organized as follows. In Sec. I we give the definition of the considered set of models. In Sec. II we derive the evolution equations corresponding to those models. Section III shows the set of scaling relations between critical exponents for this explosive percolation transition. In Sec. IV we indicate the proper order parameter and susceptibility for explosive percolation. Section V shows the set of hyperscaling relations between critical exponents and spatial dimensions. In Sec. VI we outline the derivation of the equations for the scaling functions and describe their solutions, including the precise values of the critical exponents. In Sec. VII we discuss and summarize the results of this paper. In Appendices we give the details of our theory.

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I. THE MODELS

In this work we consider a set of models of evolving networks which generalizes ordinary percolation on classical random graphs. The number N of nodes is fixed. At each time step a new link connecting two nodes is added to the network. The evolution rules define how these nodes must be selected. Initially the network consists of a given set of finite clusters. For example, these may be N unconnected nodes. At each step sample two times (see Fig. 1):
(i) choose $m \geq 1$ nodes uniformly at random and compare the clusters to which these nodes belong; select the node within the smallest of these clusters;
(ii) similarly choose the second sampling of m nodes and, again, as in (i), select the node belonging to the smallest of the m clusters;
(iii) add a link between the two selected nodes thus merging the two smallest clusters.
In particular, if $m = 1$, we arrive at ordinary percolation, in which at each step two randomly selected nodes are interconnected. Importantly, our rules contain the basic element of other explosive percolation models [5, 10, 12, 15, 24] implementing local optimization rules, namely, selection the minimal clusters from a few possibilities. For $m > 1$, this selection is performed more efficiently than in the original explosive percolation model, since, in average, our rules select smaller clusters for merging than the Achlioptas product rule (see Ref. [19]).

In our rules, the selected nodes can belong to the same clusters. This happens frequently when a giant connected component is present in the network. Interestingly, if, in addition to rules (i), (ii), and (iii), we demand that the $2m$ nodes randomly chosen at each step must belong to different clusters (in this case, when samplings (i) and (ii) contain at least two nodes of $2m$ belonging to the same cluster, we reject these samplings and make new ones) then there will be not one but $2m-1$ giant connected components of the same size. We observed this phenomenon in our simulations. Note that multiple giant components were observed in simulations of other explosive percolation models, see for example Refs. [25–27]. In the rest of this paper we only consider the models defined by the rules of steps (i) to (iii), which do not demand that the $2m$ nodes considered at each step belong in different clusters.

II. EVOLUTION EQUATIONS

The evolution processes defined by these models can be treated as consecutive aggregation of clusters. For standard percolation this process can be reversed (i.e., this is actually an equilibrium system), while for $m > 1$ the process is irreversible. In order to describe this specific aggregation process we should find the evolution of the size distribution $P(s)$ for a finite cluster of s nodes to which a randomly chosen node belongs: $P(s) = sn(s)/\langle s\rangle$, where $n(s)$ is the size distribution of clusters (the probability that a uniformly randomly chosen cluster contains s nodes), and $\langle s\rangle$ is the average size for all clusters including the giant connected component. This distribution satisfies the sum rule $\sum_s P(s) = 1 - S$. Here S is the relative size of the percolation cluster. For brevity, we often do not indicate that the distributions are time dependent, where time t is the ratio of the number of links and nodes in the system at a given step. We also introduce the probability $Q(s)$ that if we choose uniformly at random m nodes then the smallest of the clusters to which these nodes belong is of size s. The sum rule here is $\sum_s Q(s) = 1-S^m$. The distribution $Q(s)$ can be easily expressed in terms of $P(s)$. Let us introduce the cumulative distributions $P_{\rm cum}(s) \equiv \sum_{u=s}^{\infty} P(u)$ and $Q_{\rm cum}(s) \equiv \sum_{u=s}^{\infty} Q(u)$, so that $P(s) = P_{\rm cum}(s) - P_{\rm cum}(s+1)$ and $Q(s) = Q_{\rm cum}(s) - Q_{\rm cum}(s+1)$. Then according to probability theory [28], $Q_{\rm cum}(s) + S^m = [P_{\rm cum}(s) + S]^m$, which gives

$$Q(s) = \left[1 - \sum_{u=1}^{s-1} P(u)\right]^m - \left[1 - \sum_{u=1}^{s} P(u)\right]^m = \sum_{k=1}^{m} \binom{m}{k} P(s)^k \left[1 - \sum_{u=1}^{s} P(u)\right]^{m-k}, \qquad (1)$$

that is, $Q(s)$ is determined by $P(s')$ with $s' \leq s$. For the infinite system, the evolution of these coupled distributions are described exactly by the the infinite set of evolution equations:

$$\frac{\partial P(s,t)}{\partial t} = s \sum_{u+v=s} Q(u,t)Q(v,t) - 2sQ(s,t). \qquad (2)$$

We derived these equations in a similar way to ordinary percolation [29, 30]. This is actually a version of Smoluchowski equation [31] for our aggregation process. For ordinary percolation, $Q(s,t) = P(s,t)$, and the problem is explicitly solvable [29, 30]. For $m > 1$, the right-hand side of Eq. (2) is not bilinear, and the explicit solution is not possible. We have solved this system of equations numerically for $s \leq 10^6$ in the case of $m = 2$ [19]. Figure 2a shows the dependence of the relative size of the percolation cluster obtained in this way for $m = 1, 2$, and 3. In the following we present an exact solution of the problem in the critical region around the percolation threshold t_c, where the distributions have scaling form. We will first assume that the transition is continuous, then derive equations for the scaling functions of $Q(s,t)$ and $P(s,t)$. By solving these equations, we will demonstrate that the scaling functions exist and that our assumption is correct and self-consistent.

Equation (2) leads to the following equations for the moments of the distributions and the size of the percolation cluster:

$$\frac{\partial S}{\partial t} = 2S^m \langle s\rangle_Q, \qquad (3)$$

$$\frac{\partial \langle s\rangle_P}{\partial t} = 2\langle s\rangle_Q^2 - 2S^m \langle s^2\rangle_Q, \qquad (4)$$

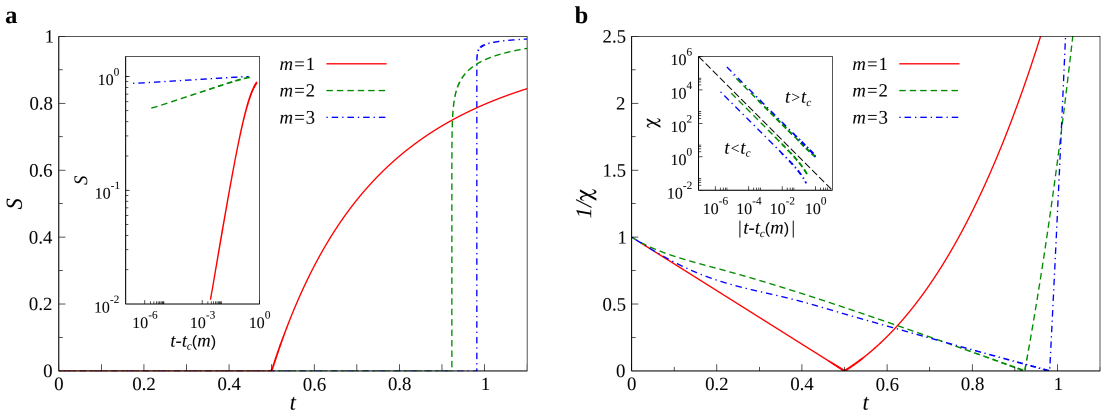

FIG. 2. **Relative size of the percolation cluster S and the inverse susceptibility $1/\chi$ vs. t for $m = 1, 2, 3$.** Here and in Fig. 3 the curves are the result of numerical solution of 10^5 evolution equations. **a**, despite the visually abrupt behavior of $S(t)$ at $m = 2, 3$, the inset suggests a power-law approach to the critical points. The slopes of the three curves in the inset are 1, 0.0555, and 0.0104 for $m = 1, 2$, and 3, respectively. **b**, the main panel and the inset demonstrate the validity of the Curie–Weiss law for the susceptibility which is defined in the text. The black dashed guide line in the inset has slope -1.

where $\langle s^n\rangle_P = \sum_s s^n P(s)$ and $\langle s^n\rangle_Q = \sum_s s^n Q(s)$. We can use these equations to derive relations between critical exponents. Their interpretation is given in Appendix B.

III. BASIC SCALING RELATIONS

In this section we find the basic scaling relations for the explosive percolation models. We assume that in the critical region (both below and above t_c) the distribution function $P(s,t)$ for large s has a scaling form,

$$P(s,t) = s^{1-\tau} f(s\delta^{1/\sigma}), \tag{5}$$

where $\delta = |t - t_c| \ll 1$, τ and σ are critical exponents, and $f(x)$ is a scaling function. Substituting Eq. (5) into the sum rule $\sum_s P(s,t) = 1 - S$ at $t \geq t_c$ and using the equality $\sum_s P(s,t_c) = 1$ at $t = t_c$, we find the size of the giant component,

$$S = \sum_s [P(s,t_c) - P(s,t)] \propto \delta^{\beta}, \tag{6}$$

where the critical exponent β is

$$\beta = (\tau - 2)/\sigma, \tag{7}$$

see Appendix G for detailed derivation and discussion.

The scaling form of the distribution $Q(s,t)$ in the normal phase of the transition is found by substituting Eq. (5) into Eq. (1). Using the fact that

$$Q(s,\delta) \cong m\Big(\int_s^\infty du P(u,\delta)\Big)^{m-1} P(s,\delta), \tag{8}$$

at large s, we obtain

$$Q(s,\delta) = s^{(2m-1)-m\tau} g(s\delta^{1/\sigma}), \tag{9}$$

where $g(x)$ is a scaling function related with $f(x)$.

The critical behavior of the first moments of the distributions, $\langle s\rangle_P = \sum_s sP(s) \sim \delta^{-\gamma_P}$ and $\langle s\rangle_Q = \sum_s sQ(s) \sim \delta^{-\gamma_Q}$, easily follows from Eqs. (5) and (9). We find

$$\gamma_P = (3 - \tau)/\sigma, \tag{10}$$

$$\gamma_Q = (2m + 1 - m\tau)/\sigma. \tag{11}$$

From Eq. (4), we obtain the relation $\gamma_P + 1 = 2\gamma_Q$ which allows us express all the critical exponents in terms of a single unknown exponent, for example, β:

$$\tau = 2 + \frac{\beta}{1 + (2m-1)\beta}, \tag{12}$$

$$1/\sigma = 1 + (2m-1)\beta, \tag{13}$$

$$\gamma_P = 1 + 2(m-1)\beta, \tag{14}$$

$$\gamma_Q = 1 + (m-1)\beta. \tag{15}$$

IV. ORDER PARAMETER AND SUSCEPTIBILITY

For continuous phase transitions the order parameter cannot be chosen in an arbitrary way, by demanding only that it is zero in the normal phase and non-zero in the ordered phase. For these transitions the order parameter must satisfy several strict conditions that are well known in the theory of phase transitions [20]. First, the critical exponent of the order parameter must satisfy basic hyperscaling relations discussed in Sec. V and Appendix D. Second, the order parameter, susceptibility, and pair correlation function are closely related to each other. Indeed, the susceptibility is the derivative of the order parameter with respect to a conjugate field.

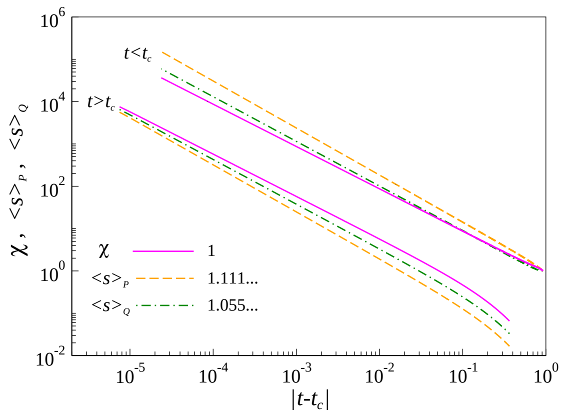

FIG. 3. **Comparison between susceptibility and the first moments of the distributions $P(s)$ and $Q(s)$ vs. deviation from the critical point for $m = 2$.** The numbers show the slopes of the correspondent curves. The susceptibility demonstrates the Curie–Weiss law, while the critical exponents of $\langle s\rangle_P$ and $\langle s\rangle_Q$ agree with relations (14) and (15) when $\beta = 0.055...$.

From this, the relation between susceptibility and pair correlation function follows, which determines a basic relation between the critical exponents of these physical quantities. In the case of ordinary percolation, the relations between the order parameter, susceptibility, and pair correlation function were obtained rigorously by use of the one-state limit of the Potts model [21]. We stress that the order parameter, the susceptibility, and the pair correlation function found here for explosive percolation satisfy all of these basic conditions. In statistical and solid state physics there are many examples (such as spin glasses [22], percolation [1], etc) demonstrating that the search for the order parameter is a nontrivial problem.

For the ordinary percolation phase transition, the relative size of the percolation cluster S is the order parameter, while the average size $\langle s\rangle_P$ of a finite cluster, to which a uniformly randomly chosen node belongs, plays the role of susceptibility. So for ordinary percolation, the exponents β and γ_P are the critical exponents of the order parameter and susceptibility. Let us show that the susceptibility and the order parameter for explosive percolation have a quite different meaning. Here we present heuristic arguments, for a comprehensive consideration see Appendix C.

For percolation problems which we consider, the probability c_2 that a new link interconnects nodes in the same cluster provides both susceptibility χ and the order parameter ϕ of the system: $c_2 = \chi/N + \phi^2$, where the second summand is the probability that both nodes belong to the percolation cluster [1, 2]. To measure the susceptibility in this process experimentally, one should find the fraction of events in which two nodes selected by the specific rule of the model fall into the same finite cluster. The divergence of the susceptibility manifests the critical point of the explosive percolation transition.

For our model of explosive percolation, the probability that two nodes selected by our algorithm belong to the same cluster is

$$c_2 = \frac{1}{N}\sum_s \frac{sQ^2(s)}{P(s)} + S^{2m}. \tag{16}$$

For rigorous derivation of this expression, see Appendix C. The first term on the right-hand side is the probability that both selected nodes belong to the same finite cluster, while the second term is the probability that both selected nodes are in the percolation cluster. To obtain the first term, we divide the probability $Q^2(s)$ that both selected nodes belong to clusters of size s by the number of clusters of s nodes in the system, $n(s)N/\langle s\rangle = P(s)N/s$, and then sum over s. The first term gives the susceptibility for the explosive percolation model (divided by N), the second term gives the square of the order parameter. Consequently the order parameter in these models is S^m and not S as is usually believed. In particular, at $m = 1$, Eq. (16) is reduced to the well-known relation $c_2 = \langle s\rangle_P/N + S^2$ for ordinary percolation. Substituting the scaling forms of the distributions $P(s,t)$ and $Q(s,t)$ near the critical point, Eqs. (5) and (9), respectively, into Eq. (16) immediately gives $\chi \sim \delta^{-\gamma}$, where the critical exponent of susceptibility is $\gamma = 1$. This is the Curie-Weiss law which is valid for cooperative systems above an upper critical dimension, where mean-field theories work. The inset of Fig. 2b confirms this law for $m = 2$, and 3. Notice in Fig. 2b that while for ordinary percolation ($m = 1$), the moduli of the slopes of $1/\chi(t)$ above and below the transition are equal as $t \to t_c$, for higher m they differ drastically from each other. Figure 3 demonstrates the contrast between the critical divergencies of the susceptibility (which diverges according to the Curie-Weiss law) and the first moments $\langle s\rangle_P$ and $\langle s\rangle_Q$ for $m = 2$.

V. HYPERSCALING

Another set of relations between critical exponents contain the dimensionality d of a system, the fractal dimension d_f of clusters at the critical point, the correlation length critical exponent ν, and the Fisher exponent η [32]. These relations are often called hyperscaling relations. In this work we consider infinite-dimensional models, but they can be generalized and formulated for an arbitrary d. For this generalization, one can derive the hyperscaling relations for d below the upper critical dimension d_u in the same way as for ordinary percolation (see Appendix D). The resulting hyperscaling relations at $d < d_u$ are as follows:

$$d_f = 1/(\sigma\nu), \tag{17}$$

$$d_f = d - \beta/\nu, \tag{18}$$

$$d - 2 + \eta = 2\beta^*/\nu, \tag{19}$$

where the order parameter critical exponent β^* equals $m\beta$ for our models, since the order parameter is S^m.

Above the upper critical dimension, one should set in Eqs. (17)–(19) the exponents ν and η to their mean-field theory values, 1/2 and 0, respectively, and d to d_u [32]. The resulting relations together with Eq. (13) lead to the following expressions for the fractal and upper critical dimensions d_f and d_u in terms of the exponent β:

$$d_f = 2[1 + (2m - 1)\beta], \tag{20}$$

$$d_u = 2 + 4m\beta. \tag{21}$$

These relations demonstrate that both d_f and d_u are very close to 2 when $m > 1$. This means that explosive percolation models of this kind defined in two dimensions have critical features very similar to those predicted by the mean-field theory.

VI. SCALING FUNCTIONS AND EXPONENTS

In this section we outline the derivation of the equations for scaling functions and describe their solutions. Above the percolation threshold, Eq. (1) is reduced to $Q(s) \cong mS^{m-1}P(s)$ at large s, which makes the resulting evolution equation for $P(s,t)$ to be similar to that for ordinary percolation. In our work [19] we assumed that the distribution $P(s)$ at the critical point is, asymptotically, a power law, which enabled us to solve Eq. (2) using the initial condition $P(s,t_c) \cong As^{1-\tau}$ (in [19] we show the solution for $m = 2$, for an arbitrary m see Appendix K). This provides the scaling functions $f(x)$ and $g(x)$ on the upper side of the phase transition in terms of the yet unknown critical amplitude A and exponent τ [19]. In the present work we will obtain the distribution at the critical point and verify its power-law form. We will find the critical exponent and amplitude of $P(s,t_c)$, and obtain the scaling functions below the transition. In this way we will completely describe the cluster size distribution in the entire critical region.

Let us approach the critical point from the normal-phase side. To derive equations for scaling functions, we have to remove the non-scaling, low s parts of the distributions from Eqs. (1) and (2) and then substitute their scaling forms of Eqs. (5) and (9). As a result we arrive at a system of nonlinear integro-differential equations of the second order, convenient for analytical and numerical treatment. The details of the derivation and the resulting equations for the scaling functions are presented in Appendix E. In essence, these are nonlinear eigenfunction equations, where eigenfunctions are the scaling functions of our problem and the eigenvalue is one of the critical exponents, e.g., τ. These equations are solved on the one-dimensional interval $0 \leq x < \infty$. At $x = 0$, $f(x)$ and $g(x)$ coincide with the critical amplitudes for the corresponding distributions: $P(s,t_c) \cong f(0)s^{1-\tau}$ and $Q(s,t_c) \cong g(0)s^{(2m-1)-m\tau}$, respectively. The amplitude $g(0)$ can be expressed in terms of $f(0)$ and τ. The critical

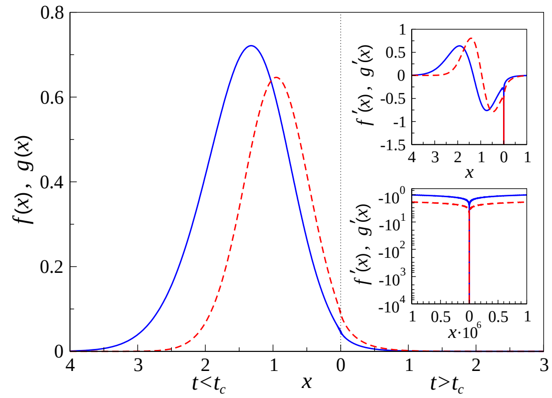

FIG. 4. **Scaling functions $f(x)$ and $g(x)$ for $m = 2$.** The solid lines are for $f(x)$, and the dashed lines are for $g(x)$. The insets showing the respective derivatives highlight the presence of singularities at $x = 0$.

amplitude $f(0)$, as well as the detailed shapes of the scaling functions, is determined by the initial distribution of cluster sizes, $P(s,t = 0)$. In contrast to that, the critical exponents do not depend on initial conditions. So, when searching for the solution of the equation, we can set any convenient value of the critical amplitude $f(0)$. For different values of $f(0)$, the resulting value of the critical exponent τ should be the same, and the scaling functions, while differing from each other, should be qualitatively similar. For a given critical amplitude $f(0)$, the system of first order differential equations for the scaling functions shown in Eq. (E9) can be directly solved numerically. This solution gives the exponent τ together with the scaling functions $f(x)$ and $g(x)$. The unknown critical exponent τ is obtained from the condition that $f(x)$ and $g(x)$ decay to zero as $x \to \infty$, while staying positive (see Appendix I for details of the numerical procedure). These calculations converge rapidly giving the final value of τ and the scaling functions with any desired precision, i.e. exactly in a physics sense.

The resulting scaling functions $f(x)$ and $g(x)$ are shown in Fig. 4 for $m = 2$ (for higher m the scaling functions are qualitatively similar). The plot shows scaling functions in the normal phase, $t < t_c$, and in the phase with the percolation cluster, $t > t_c$. This figure demonstrates a drastic contrast with the scaling function for ordinary percolation above the upper critical dimension, which is symmetric. It is the exponential function $f(x) = e^{-2x}/\sqrt{2\pi}$ both for $t > t_c$ and $t < t_c$. Note that a similar asymmetry is observed at $1 < d < d_u = 6$ in ordinary percolation [33]. The insets in Fig. 4 demonstrate that the scaling functions have singularities at $x = 0$. In Appendix F we show that $f(x) - f(0) \propto g(x) - g(0) \propto x^\sigma$ near $x = 0$ at $m > 1$, where the critical exponent σ is slightly smaller than 1, see Eq. (13). Below t_c for large x we find $f(x) \propto \exp\left(-Cx^{1+\ln m/\ln 2}\right)$ and $g(x) \propto \exp\left(-mCx^{1+\ln m/\ln 2}\right)$, where C is a constant (see Appendix H). Above t_c the scaling functions expo-

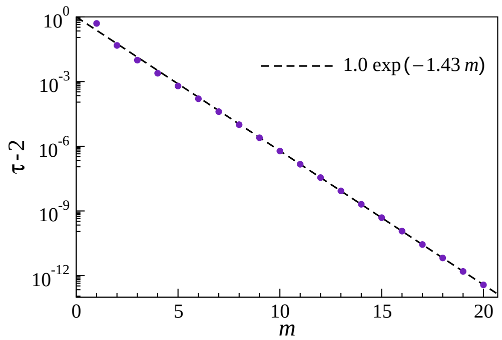

FIG. 5. **Deviation of the critical exponent τ from 2 vs. m.** The dashed line (exponential function) fits well the dots obtained by our numerical procedure. For the corresponding numerical values see Appendix J.

nentially decay to 0.

We found τ for m up to 20, see Fig. 5 and Table I, which shows that $\tau - 2$ (as well as β) decreases with m as $\exp(-1.43m)$. All other exponents and upper critical dimensions can be readily found from the relations between critical exponents. In particular, the upper critical dimension decreases rapidly to 2 with increasing m.

VII. DISCUSSION

Our theory reveals that the explosive percolation problem is a direct generalization of ordinary percolation, but this generalization turned out to be principally nontrivial. The complete scaling description which we developed explains the genuine continuous nature of the explosive percolation phase transitions in the investigated class of systems. We completely described all scaling properties of this transition within the framework of the theory of continuous phase transitions. So a complete description of the class of processes considered in this work does not require the introduction of new notions, like "weakly discontinuous transition". Despite of this continuity, we found and highlighted the drastic difference between ordinary and explosive percolation. We found the order parameter and susceptibility of explosive percolation and show that these physical quantities differ strongly from the ones in standard percolation. This explains the principal novelty of critical phenomena associated with this continuous transition and its surprising features including the small values of exponent β. Note that although the actual critical exponent of the order parameter is $m\beta$, Fig. 5 shows that its value is anomalously small for all $m > 1$. Note that this smallness is very unusual but not unprecedented. Small exponents of the order parameter were also observed in other non-equilibrium systems, in specific contact processes [34].

The models which we considered were based on local optimization algorithms, in which each new connection requires a finite amount of information. Our work does not exclude possibility of discontinuity in more sophisticated models, where global optimization is implemented [35–38]. To create new links in these models, one must know their global structure (i.e., all clusters) or globally control them.

In this paper we focused on scaling properties of the transition but not on the critical point value t_c, which is of secondary interest for continuous phase transition and is determined by initial conditions (see Appendix L). In our work [39] we showed how to obtain t_c and $f(0)$ with high precision for any initial conditions, and in Appendix M we show how to get simple estimates.

In summary, by developing the scaling theory for a wide class of models, we explained the continuous nature of the explosive percolation transition and its unusual properties. Our analysis can be extended to other systems and competition driven processes of this kind. We suggest that our work will provide a conceptual and methodological basis for new generalizations of percolation.

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AUTHOR CONTRIBUTIONS

All authors conceived and designed the research, worked out the theory, carried out the numerical calculations, analysed the results, and wrote the manuscript.

ADDITIONAL INFORMATION

The authors declare no competing financial interests. Correspondence and requests for materials should be addressed to S.N.D.

Appendix A: Contents of the Appendices

The following Appendices contain details of the calculations and arguments outlined or mentioned in the main text; a comprehensive and rigorous analysis of the observables for these problems, including the order parameter and susceptibility; discussion of generalizations to other models of this kind; and the table of precise numerical values for the critical exponents τ and β. In the following sections, when it is convenient, we reproduce some of the equations and formulas from the main text.

Appendix B: Equations describing evolution of S, and $\langle s\rangle_P$ and $\langle s\rangle_Q$

The first of the Eqs. (3) and (4) in the main text, which we are reproducing here:

$$\frac{\partial S}{\partial t} = 2S^m \langle s\rangle_Q, \tag{B1}$$

$$\frac{\partial \langle s\rangle_P}{\partial t} = 2\langle s\rangle_Q^2 - 2S^m\langle s^2\rangle_Q, \tag{B2}$$

demonstrates the principal difference of "explosive" percolation from ordinary one. Let us seed a giant component of relative size $h \ll 1$ in the normal phase at some moment $t < t_c$ and consider its evolution. Equation (B1) shows that the growth rate of this component is proportional to h^m, i.e., it is severely suppressed in the entire normal phase if $m > 1$. This suppression results in the delayed transition compared to $m = 1$.

Appendix C: The nature of the order parameter and susceptibility

Here we rigorously introduce the relevant order parameter and susceptibility and generalize them to other explosive percolation models.

In order to define the susceptibility for the explosive percolation problem, we use the relation between the susceptibility and correlation function that follows from the equivalence of the percolation problem to the one-state Potts model. We define the correlation function $C(i,j)$ between vertices i and j as follows. Vertices i and j are correlated if they are connected by at least one path. In this case, $C(i,j) = 1$, otherwise $C(i,j) = 0$. Moreover, by definition, $C(i,i) = 1$. The susceptibility χ equals

$$\chi = \frac{1}{N}\sum_{i,j=1}^{N} C(i,j). \tag{C1}$$

First we find the susceptibility for the ordinary percolation. In a system consisting of finite clusters, each labeled by index α and having size s_α, Eq. (C1) takes a form,

$$\chi = N\sum_{\alpha}\Big(\frac{s_\alpha}{N}\Big)^2. \tag{C2}$$

This equation shows that χ is related to the probability $(s_\alpha/N)^2$ that two randomly chosen vertices belongs to the same cluster α, i.e., they are connected. The probability

$$x_\alpha = s_\alpha/N \tag{C3}$$

that a randomly chosen vertex belongs to cluster α plays the role of an observable. In the phase without a giant component, all clusters are finite. Therefore, at any α, $x_\alpha \to 0$ in the thermodynamic limit $N \to \infty$. In the phase with a giant cluster of relative size S, the corresponding observable $x_{gc} = S$ is nonzero and plays the role of the order parameter.

We can pass in Eq. (C2) from summation over individual clusters to summation over cluster sizes s, which gives for ordinary percolation

$$\chi = N\sum_{\alpha}\Big(\frac{s_\alpha}{N}\Big)^2 = N\sum_{s}\frac{NP(s)}{s}\Big(\frac{s}{N}\Big)^2 = \sum_{s} P(s)s, \tag{C4}$$

where $P(s) = N(s)s/N$ is the probability that a randomly chosen vertex belongs to a cluster of size s, $N(s)$ is the number of clusters of size s. Equation (C4) is actually the standard definition of the susceptibility in ordinary percolation as the average size of the cluster to which a randomly chosen vertex belongs.

Let us now consider the explosive percolation problem. The rule formulated in the main text selects the vertex that belongs to the smallest of the m clusters. The probability that this vertex is in a cluster α equals

$$\begin{aligned} x_\alpha &= m\frac{s_\alpha}{N}\Big[\sum_{s_\beta > s_\alpha}\frac{s_\beta}{N}\Big]^{m-1} \\ &+\frac{m(m-1)}{2!}\frac{s_\alpha}{N}\Big[\frac{N(s_\alpha)s_\alpha}{N}\Big]\Big[\sum_{s_\beta > s_\alpha}\frac{s_\beta}{N}\Big]^{m-2} \\ &+\cdots+\frac{s_\alpha}{N}\Big[\frac{N(s_\alpha)s_\alpha}{N}\Big]^{m-1} \\ &= \frac{1}{N(s_\alpha)}\sum_{k=1}^{m}\binom{m}{k}\Big[\frac{N(s_\alpha)s_\alpha}{N}\Big]^k\Big[\sum_{s_\beta > s_\alpha}\frac{s_\beta}{N}\Big]^{m-k}, \end{aligned} \tag{C5}$$

The first term is the probability that the smallest of m clusters, α, has size s_α while the other $m-1$ clusters are larger. The second term is the probability that apart from cluster α there is one more cluster of the same size while the remaining $m-2$ clusters are larger. The last term is the probability that all m clusters have the same size s_α.

In the normal phase, the observables $x_\alpha \to 0$ in the thermodynamic limit $N \to \infty$. In the phase with a percolation cluster of the relative size S, the observable corresponding to the percolation cluster,

$$x_{\rm pc} = S^m. \tag{C6}$$

is nonzero and plays the role of the order parameter of the explosive percolation transition.

The susceptibility in the explosive percolation problem is a simple generalization of Eq. (C2) in which we replace the probability s_α/N by the probability Eq. (C5),

$$\chi = N\sum_\alpha x_\alpha^2. \tag{C7}$$

This equation relates χ to the probability that two vertices chosen by use of the explosive percolation selection rules belong to the same cluster α, i.e., they are connected. Relation (C7) is valid in the normal phase, i.e., at $t < t_c$. In the phase with a percolation cluster at $t > t_c$, in Eq. (C7) we must subtract the contribution of the giant cluster,

$$\chi = N\Bigl[\sum_\alpha x_\alpha^2 - S^{2m}\Bigr]. \tag{C8}$$

The right-hand sides of Eqs. (C7) and (C8) can be replaced with sums over s accounting only for finite clusters. So both below and above t_c we have

$$\chi = N\sum_s \frac{1}{N(s_\alpha)}\left[\sum_{k=1}^{m}\binom{m}{k}P(s)^k\Bigl[\sum_{u\geq s+1}P(u)\Bigr]^{m-k}\right]^2 = \sum_s \frac{sQ(s)^2}{P(s)}, \tag{C9}$$

which coincides with Eq. (16). Here we used expression (1) for the distribution $Q(s)$. Equations (C6) and (C8) generalize the order parameter and susceptibility to the case of explosive percolation ($m > 1$). At $m = 1$, these equations correspond to the ordinary percolation. At the critical point $t = t_c$, the susceptibility diverges, $\chi \to \infty$, manifesting the explosive percolation transition.

For the original Achlioptas process (product rule) the observable x_α is related to the probability that, a new link connects two vertices already in the same cluster:

$$x_\alpha^2 = 2\Bigl(\frac{s_\alpha}{N}\Bigr)^2\sum_{\beta,\gamma:\, s_\beta\times s_\gamma > s_\alpha^2}\frac{s_\beta}{N}\frac{s_\gamma}{N} + \Bigl(\frac{s_\alpha}{N}\Bigr)^2\sum_{\beta,\gamma:\, s_\beta\times s_\gamma = s_\alpha^2}\frac{s_\beta}{N}\frac{s_\gamma}{N} \tag{C10}$$

The order parameter is S^2 as well as in our model. The susceptibility is given by Eq. (C8):

$$\chi = N\sum_\alpha\Bigl[2\Bigl(\frac{s_\alpha}{N}\Bigr)^2\sum_{\beta,\gamma:\, s_\beta\times s_\gamma > s_\alpha^2}\frac{s_\beta}{N}\frac{s_\gamma}{N} + \Bigl(\frac{s_\alpha}{N}\Bigr)^2\sum_{\beta,\gamma:\, s_\beta\times s_\gamma = s_\alpha^2}\frac{s_\beta}{N}\frac{s_\gamma}{N}\Bigr]. \tag{C11}$$

The first sum is over all clusters α, excluding the giant component. For a general selection rule minimizing $f(s,s')$, the summation over $\beta,\gamma\colon s_\beta{\times}s_\gamma{>}s_\alpha^2$ and over $\beta,\gamma\colon s_\beta{\times}s_\gamma{=}s_\alpha^2$ is replaced with summation over $\beta,\gamma\colon f(s_\beta,s_\gamma){>}f(s_\alpha,s_\alpha)$ and over $\beta,\gamma\colon f(s_\beta,s_\gamma){=}f(s_\alpha,s_\alpha)$, respectively.

The square of the order parameter is the probability that a new link is inside of the percolation cluster. For this, all four randomly chosen nodes must be in the percolation cluster, which gives S^4. That is, the order parameter for the Achlioptas process is S^2 both for the product and sum rules (as well as for any other rule involving four nodes).

For the rule in which two optimal clusters from three are interlinked, we have the order parameter $S^{3/2}$. If this rule imposes selection of the pair with the smallest $f(s_\alpha, s_\beta)$, we have for the susceptibility:

$$\chi = N\sum_\alpha\Bigl[3\Bigl(\frac{s_\alpha}{N}\Bigr)^2\sum_{\beta:\, f(s_\alpha,s_\beta)>f(s_\alpha,s_\alpha)}\frac{s_\beta}{N} + \Bigl(\frac{s_\alpha}{N}\Bigr)^3\Bigr]. \tag{C12}$$

Appendix D: Hyperscaling relations

Let us present hyperscaling relations for ordinary percolation (relations between scaling exponents including spatial dimensions) below the upper critical dimension d_u:

$$1/d_f = \sigma\nu, \tag{D1}$$

$$d_f = d - \beta/\nu, \tag{D2}$$

$$d - 2 + \eta = 2\beta/\nu, \tag{D3}$$

where d is the number of spatial dimensions and d_f is the fractal dimension. The corresponding relations above d_u are obtained by substituting d_u for d, 1/2 for ν, and 0 for η. Here 1/2 and 0 are the mean-field theory values of the critical exponents ν and η, respectively.

Let us recall how these relations were derived [1].

(i) Relation (D1).

According to the scaling form of the distribution $P(s,t)$, see Eq. (5), the critical features are determined by cluster sizes $s \sim \delta^{-1/\sigma}$. In the critical region, the clusters are fractals,

$$\delta^{-1/\sigma} \sim s \sim \xi^{d_f} \sim \delta^{-d_f\nu}, \tag{D4}$$

where ξ is the correlation length, $\xi \sim \delta^{-\nu}$, so we have relation (D1). One can see that this derivation is actually relevant for our explosive problem.

(ii) Relation (D2).

Consider a hyper-cube of L^d nodes and estimate the number of nodes $M(L)$ of the percolation cluster falling inside this hyper-cube above t_c. It is easy to see that for $L \ll \xi$, this number is $M \sim L^{d_f}$, while for $L \gg \xi$ it is $M \sim SL^d$. So for $L \sim \xi$,

$$\xi^{d_f} \sim \delta^{\beta}\xi^{d}, \tag{D5}$$

which gives relation (D2). One can see that this derivation is also relevant for our explosive problem.

(iii) Relation (D3).

In general, the spin–spin correlation function near a continuous phase transition decays as $r^{-(d-2-\eta)}$ until the spin separation r approaches the correlation radius ξ. So we can estimate

$$\xi^{-(d-2-\eta)} \sim \phi^2 \sim \delta^{2\beta}, \tag{D6}$$

which gives relation (D3), if the order parameter $\phi = S \sim \delta^{\beta}$. For our model of explosive percolation, the order parameter $\phi = S^m$, i.e., $\phi \sim \delta^{\beta^*} \sim \delta^{m\beta}$, so for explosive percolation it should be

$$d - 2 + \eta = 2\beta^*/\nu. \tag{D7}$$

After substitution of the mean-field theory values 1/2 for ν, 0 for η, and d_u for d we arrive at

$$d_u - 2 = 4\beta^* = 4m\beta. \tag{D8}$$

In addition we have

$$1/d_f = \sigma/2 = \frac{1}{2[1+(2m-1)\beta]}, \tag{D9}$$

$$d_f = d_u - 2\beta. \tag{D10}$$

We emphasize that only two of the last three relations are independent. If, for example, we express d_f and d_u in terms of β by using Eqs. (D8) and (D9) and then substitute the result into Eq. (D10), we will arrive at the identity.

The upper critical dimension d_u also describes the finite size effect for a continuous phase transition in systems above d_u, namely,

$$t_c(\infty) - t_c(N) \propto N^{-2/d_u}. \tag{D11}$$

Here $t_c(\infty)$ is the critical point value in the infinite system (in which the transition is well defined) and $t_c(N)$ is, in particular, the position of the maximum of the susceptibility for the system of N nodes.

Appendix E: Derivation of equations for scaling functions

In this section we show in detail how to derive equations for scaling functions from the evolution equations. We suggest that the ideas implemented in this derivation will be useful for numerous generalizations of percolation.

In our work of Ref. [19], we have shown that if it is known that the distribution $P(s)$ at the critical point is, asymptotically, power-law with some given critical exponent and amplitude, $P(s,t_c) \cong As^{-\tau+1}$, as it should be for a continuous phase transition, then from Eq. (2), immediately follows the power law $S \cong B\delta^{\beta}$, where $\beta = (\tau-2)/[1-(2m-1)(\tau-2)]$ as in Eq. (12) and the coefficient B is expressed in terms of A and τ. (Here the critical amplitude A is determined by the initial form of the distribution $P(s,t=0)$.) Furthermore, this assumption allows us to find the scaling functions $f(x)$ and $g(x)$ on the upper side of the phase transition, i.e. at $t > t_c$. The form of these functions turns out to be close to exponential, similarly to ordinary percolation above an upper critical dimension. The derivation detailed in Appendix K exploits the convenient simplification of the equations above the critical point, where S differs from zero. In this region, at large s, Eq. (1) is reduced asymptotically to $Q(s) \cong mS^{m-1}P(s)$, which makes the resulting evolution equation for $P(s,t)$ to be similar to that for ordinary percolation and so easily solvable with the initial condition $P(s,t_c) \cong As^{-\tau+1}$. Therefore our present more difficult task is to find the distribution at the critical point, which we just used in that derivation, its critical exponent (if this distribution will appear to be power-law), and the scaling functions on the normal phase side of the phase transition, i.e. at $t < t_c$. So, simultaneously we verify that the transition is continuous.

First we derive equation for scaling functions approaching the critical point from the normal-phase side by using Eqs. (2) and (8). The direct substitution of the scaling forms of the distributions $P(s,\delta)$ and $Q(s,\delta)$ into the evolution Eq. (2) is impossible, since these forms are valid at large s, while the contribution from the region of small s to the sum in Eq. (2) is nonzero. Let us rewrite Eq. (2) to eliminate this contribution from the sum and so to remove the non-scaling, low s parts of the distribution from consideration. We substitute $Q(u) = Q(s)+[Q(u)-Q(s)]$ into the evolution equation, which leads to the following equation:

$$\frac{\partial P(s)}{\partial t} = -s(s-1)Q^2(s) + 2sQ(s)[1 - \sum_{u=s}^{\infty} Q(u)]$$
$$+s\sum_{u=1}^{s-1}[Q(u)-Q(s)][Q(s-u)-Q(s)] - 2sQ(s), \tag{E1}$$

in which we can safely substitute integrals for the sums. The resulting equation is

$$\frac{\partial P(s)}{\partial t} \cong -s^2Q^2(s) - 2sQ(s)\int_s^{\infty} du\, Q(u)$$
$$+s\int_0^s du\,[Q(u)-Q(s)][Q(s-u)-Q(s)]. \tag{E2}$$

The scaling form of the distribution $P(s,t)$ for large s in the critical region is

$$P(s,t) = s^{1-\tau} f(s\delta^{1/\sigma}) = \delta^{(\tau-1)/\sigma} \tilde{f}(s\delta^{1/\sigma}), \qquad \text{(E3)}$$

where $\delta = |t - t_c| \ll 1$, and $f(x)$ and $\tilde{f}(x)$ are scaling functions, $f(x) = x^{\tau-1}\tilde{f}(x)$, τ and σ are critical exponents. These two functions, $f(x)$ and $\tilde{f}(x)$, provide two equivalent representations of scaling. In the following, $\tilde{f}(x)$ turned out to be more convenient for us. On the other hand, the scaling form of the $Q(s,t)$ distribution is

$$Q(s,t) = s^{(2m-1)-m\tau} g(s\delta^{1/\sigma}) = \delta^{[m\tau-(2m-1)]/\sigma} \tilde{g}(s\delta^{1/\sigma}), \qquad \text{(E4)}$$

where $g(x) = x^{m\tau-(2m-1)}\tilde{g}(x)$. Substituting these scaling forms of the distributions into Eqs. (E2) and (8), and equating the powers of δ in all the terms, we arrive at the following equation for the scaling functions:

$$-\frac{\tau-1}{\sigma}\tilde{f}(x) - \frac{1}{\sigma}x\tilde{f}'(x)$$
$$= -x^2\tilde{g}^2(x) - 2x\tilde{g}(x)\int_x^\infty dy\,\tilde{g}(y)$$
$$+x\int_0^x dy\,[\tilde{g}(y) - \tilde{g}(x)][\tilde{g}(x-y) - \tilde{g}(x)] \qquad \text{(E5)}$$

$$\tilde{g}(x) = m\left[\int_x^\infty dy\,\tilde{f}(y)\right]^{m-1}\tilde{f}(x), \qquad \text{(E6)}$$

where

$$\sigma = 1 - (2m-1)(\tau-2). \qquad \text{(E7)}$$

The last relation for the critical exponents follows from the condition that all factors containing powers of δ must cancel each other. Equation (E5), with substituted $\tilde{g}(x)$ from Eq. (E6) can be treated as a nonlinear integral differential eigenfunction equation for the scaling function $\tilde{f}(x)$, in which a critical exponent, say τ, plays the role of the eigenvalue. Note that Eqs. (E5) and (E6) inconveniently contain integrals with integration over different intervals, (x,∞) and $(0,x)$. To avoid this inconvenience, we must exclude the integrals over the interval (x,∞). For that, in both Eqs. (E5) and (E6) we move the integrals over (x,∞) to the left-hand sides of the equations and move everything else to the right-hand sides, and then take the derivatives of the both sides. The derivation removes the integrals $\int_x^\infty$, but, unfortunately, produces new divergencies within the remaining integrals $\int_0^x$. To avoid these divergencies, it is sufficient first to pass from the integral over the interval $(0,x)$ to integration over $(0,x/2)$ in Eq. (E5), namely

$$\int_0^x dy\,[\tilde{g}(y) - \tilde{g}(x)][\tilde{g}(x-y) - \tilde{g}(x)]$$
$$= 2\int_0^{x/2} dy\,[\tilde{g}(y) - \tilde{g}(x)][\tilde{g}(x-y) - \tilde{g}(x)]. \qquad \text{(E8)}$$

The resulting system of two equations contains $\tilde{f}''(x)$, $\tilde{f}'(x)$, $\tilde{f}(x)$, $\tilde{g}'(x)$, and $\tilde{g}(x)$. Introducing $\tilde{u}(x) = \tilde{f}'(x)$, we obtain the system of three first order equations for $\tilde{f}(x)$, $\tilde{g}(x)$, and $\tilde{u}(x)$:

$$\tilde{f}''(x) = \tilde{u}'(x) = \frac{\tau-1}{x}\Big[\frac{\tilde{f}(x)}{x} - \tilde{f}'(x)\Big]$$
$$+\frac{\tilde{g}'(x)}{\tilde{g}(x)}\Big[\frac{(\tau-1)\tilde{f}(x)}{x} + \tilde{f}'(x)\Big] - \sigma\tilde{g}^2(x/2)$$
$$+\frac{2\sigma}{\tilde{g}(x)}\int_0^{x/2} dy\,\tilde{g}(y)[\tilde{g}'(x)\tilde{g}(x-y) - \tilde{g}(x)\tilde{g}'(x-y)]$$
$$\tilde{g}'(x) = \frac{\tilde{f}'(x)\tilde{g}(x)}{\tilde{f}(x)} - m(m-1)\tilde{f}^2(x)\Big[\frac{\tilde{g}(x)}{m\tilde{f}(x)}\Big]^{(m-2)/(m-1)}$$
$$\tilde{f}'(x) = \tilde{u}(x), \qquad \text{(E9)}$$

where the exponent σ is related with τ according to (E7).

Appendix F: Singularities of scaling functions at zero

One can verify that at small x, the solution of this system has the following expansion:

$$f(x) = x^{\tau-1}\tilde{f}(x) = f(0) + a_1x^\sigma + a_2x^{2\sigma} + ...,$$
$$g(x) = x^{m\tau-(2m-1)}\tilde{g}(x) = g(0) + b_1x^\sigma + b_2x^{2\sigma} + ..., \qquad \text{(F1)}$$

where $f(0)$ and $g(0)$ are the critical amplitudes of the distributions, $P(s,t_c) \cong f(0)s^{1-\tau}$ and $Q(s,t_c) \cong g(0)s^{(2m-1)-m\tau}$, respectively. One can easily find that $g(0)$ and all other coefficients in these series are expressed in terms of only $f(0)$ and τ. For example, from the relation

$$\int_x^\infty dy\,y^{(2m-1)-m\tau}g(y) = \left[\int_x^\infty dy\,y^{1-\tau}f(y)\right]^m, \qquad \text{(F2)}$$

we immediately obtain

$$g(0) = \frac{m}{(\tau-2)^{m-1}}f^m(0). \qquad \text{(F3)}$$

Similarly, we obtain the next coefficients using relations (E5) and (F2),

$$a_1 = -\frac{g(0)^2\Gamma[-m(\tau-2)]^2}{\Gamma[-2m(\tau-2)]},$$

$$b_1 = \frac{a_1 g(0)[1-(3m-1)(\tau-2)]}{f(0)[1-2m(\tau-2)]},$$

$$a_2 = \frac{a_1 b_1}{2g(0)} + g(0)b_1\Gamma[-m(\tau-2)] \times\left(\frac{4^{m(\tau-2)}\sqrt{\pi}}{\Gamma[1/2-m(\tau-2)]} - \frac{\Gamma[1-(3m-1)(\tau-2)]}{\Gamma[1-(4m-1)(\tau-2)}\right),$$

$$b_2 = \frac{g(0)[(5m-2)(\tau-2)-2]}{f(0)} \times\left(\frac{a_1^2(m-1)(\tau-2)}{2f(0)[1-2m(\tau-2)]^2} - \frac{a_2}{\tau-4m(\tau-2)}\right), \qquad \text{(F4)}$$

and so on. We do not show here the next two pairs of coefficients a_k and b_k which we have also obtained using Mathematica since they are too cumbersome. If we know $f(0)$ and τ, these Taylor series provide the solutions $f(x)$ and $g(x)$ only at sufficiently small x (below the maxima of the scaling functions).

Note that the case of ordinary percolation, i.e. $m=1$, is special for series (F1) in the following sense. It turns out that for $m=1$, the odd coefficients in these series are zero. For example, one can easily check in Eq. (F4) that in this case, $a_1 = b_1 = 0$. This feature also follows from the form of the scaling function for ordinary percolation, $f(x) = e^{-2x}/\sqrt{2\pi}$.

Appendix G: Relation between the critical exponents β, τ, and σ

Let us analyze the critical singularity of the giant cluster size S. Let deviations from the critical point be small, $\delta = t - t_c \ll t_c$. Substituting the scaling form of the distribution $P(s,t)$, Eq. (5), into Eq. (6) and replacing summation with integration, we find

$$S \approx \int_1^\infty s^{1-\tau}[f(0) - f(s\delta^{1/\sigma})]ds$$
$$= \delta^{(\tau-2)/\sigma}\int_{\delta^{1/\sigma}}^\infty x^{1-\tau}[f(0)-f(x)]dx. \qquad \text{(G1)}$$

Integration by parts leads to

$$S \approx \frac{\delta^{(\tau-2)/\sigma}}{\tau-2}\Big\{-x^{2-\tau}[f(0)-f(x)]\Big|_0^\infty - \int_0^\infty x^{2-\tau}\frac{f(x)}{dx}dx\Big\}. \qquad \text{(G2)}$$

This equation shows that $S \sim \delta^\beta$ with the critical exponent

$$\beta = (\tau-2)/\sigma$$

if the scaling function $f(x)$ satisfies the following conditions. First, $x^{2-\tau}[f(0)-f(x)] \to 0$ at $x \to 0$. Second, the integral in Eq. (G2) is finite. These assumptions impose conditions on the value of τ and the behavior of $f(x)$ at $x \ll 1$ and $x \gg 1$ [2]. In particular, using the lowest term of the series (F1), we find that the equality $2 < \tau < 2+\sigma$ must be satisfied at $m \geq 2$ in contrast to $2 < \tau < 3$ for ordinary percolation ($m=1$) [2]. The inequality $\tau < 2+\sigma$ substituted into relation (E7) for the critical exponents σ and τ results in the condition $2 < \tau < 2 + 1/(2m)$ for $m \geq 2$. (As we mentioned above, the case of $m=1$ is special here, since in this case, the odd coefficients in the series (F1) are zero.) Our solution presented in Appendix F and in Table I satisfies these conditions. This evidences the self-consistency of the solution and the assumptions used when obtaining the scaling relations.

Appendix H: Asymptotics of scaling functions

Let us find the asymptotic behavior of the scaling functions explicitly. Tending $x \to \infty$, and taking the leading terms on each side of Eqs. (E5) and (E6), one can easily check that they have the following rapidly decaying asymptotics:

$$\tilde{f}(x) \cong Ax^\lambda \exp\left[-Cx^{1+\ln m/\ln 2}\right],$$

$$\tilde{g}(x) \cong \frac{mA^m x^{m\lambda-(m-1)\ln m/\ln 2}}{[C(1+\ln m/\ln 2)]^{m-1}} \exp\left[-mCx^{1+\ln m/\ln 2}\right], \qquad \text{(H1)}$$

where

$$\lambda = \left(1 + \frac{\ln m}{\ln 2}\right)\left(1 + \frac{1}{4m-2}\right) - \frac{2m}{2m-1}.$$

This procedure also gives a relation between constants A and C,

$$A^{2m-1} = \left(\frac{\ln m}{\pi \ln 2}\right)^{1/2} \times\frac{(1+\ln m/\ln 2)^{2m-1/2}2^{\lambda+1+\ln m/\ln 2}}{\sigma m^{3/2}}C^{2m-1/2}, \qquad \text{(H2)}$$

however, does not fix them. A and C are determined by $f(0)$, which is in its turn determined by the initial distribution $P(s, t=0)$.

Appendix I: Solving equations for scaling functions numerically

The Taylor series (F1) provide the solutions $f(x)$ and $g(x)$ at small x in terms of yet unknown $f(0)$ and τ. The critical amplitude $f(0)$, as well as the detailed shapes of the scaling functions, are determined by the initial distribution of cluster sizes, $P(s, t=0)$. In contrast to that, the critical exponent values do not depend on initial conditions, if $P(s, t = 0)$ decays sufficiently rapidly (see below). So, when searching for the solution of the system of Eqs. (E9), we can set any convenient value of the critical amplitude $f(0)$. For different values of $f(0)$, the resulting value of the critical exponent τ should be the same, and the scaling functions, while differing from each other, should be qualitatively similar. For a given critical amplitude $f(0)$, the system of first order differential Eqs. (E9) can be directly solved numerically. This solution should give the exponent τ together with the scaling functions $f(x)$ and $g(x)$. The unknown critical exponent τ is obtained from the condition that $f(x)$ and $g(x)$ decay rapidly to zero as x approaches infinity.

We use the following procedure. For the sake of convenience, set the value of the critical amplitude $f(0)$ such that the maxima $f(x)$ and $g(x)$ are of the order of 1 (with this choice, the numerical solution takes minimum time). First try some reasonable value of τ. Insert this pair, $f(0)$ and τ into truncated series (F1) and use them at some small x_0 as initial conditions for the first order Eqs. (E9). With these initial conditions, find the numerical solution of the system (E9) up to sufficiently large x at which the asymptotics of the solutions are already visible. Since the value of τ, which we used in this first attempt, surely deviates from the correct one, the obtained solutions will not show a proper decay to zero. Instead, they may decay more slowly than exponentially or even become negative, oscillate, and so on. Then solve equation numerically with a different value of τ, and repeat this procedure again and again, adjusting progressively the value of τ in such a way that the solutions $f(x)$ and $g(x)$ decay to zero more and more rapidly, staying positive. These calculations converge rapidly giving the final value of τ with any desired precision and the scaling functions $f(x)$ and $g(x)$, see Fig. 4.

Appendix J: Critical exponents for m from 1 to 20

The list of values of the exponent τ for m from 1 to 20 plotted in Fig. 5 is presented in Table I. These values were obtained in the way described in Appendix I. Table I also contains the values of exponent β, obtained from τ using the following relation:

$$\beta = \frac{\tau - 2}{1 - (2m-1)(\tau - 2)}.$$

In our work [39] we found the values of t_c, $f(0)$, and $P(1, t_c)$ for $m = 2$, 3, and 4 in the case of $P(1, t{=}0) = 1$, that is, the initial configuration consisting of isolated nodes.

TABLE I. Critical exponents τ and β for m from 1 to 20.

m	τ	β
1	2.5	1
2	2.04763044(2)	$5.557106(2) \times 10^{-2}$
3	2.00991188(1)	$1.042872(1) \times 10^{-2}$
4	2.002438330(5)	$2.480671(5) \times 10^{-3}$
5	2.000625199(1)	$6.28737(1) \times 10^{-4}$
6	2.0001601191(4)	$1.604016(4) \times 10^{-4}$
7	2.0000404460(1)	$4.04673(1) \times 10^{-5}$
8	2.00001006831(5)	$1.006983(5) \times 10^{-5}$
9	2.00000247685(5)	$2.47695(5) \times 10^{-6}$
10	2.00000060412(2)	$6.0412(2) \times 10^{-7}$
11	2.00000014639(1)	$1.4639(1) \times 10^{-7}$
12	2.000000035313(5)	$3.5313(5) \times 10^{-8}$
13	2.000000008489(2)	$8.489(2) \times 10^{-9}$
14	2.0000000020355(2)	$2.0355(2) \times 10^{-9}$
15	2.0000000004870(1)	$4.870(1) \times 10^{-10}$
16	2.00000000011634(4)	$1.1634(4) \times 10^{-10}$
17	2.00000000002776(2)	$2.776(2) \times 10^{-11}$
18	2.000000000006617(5)	$6.617(5) \times 10^{-12}$
19	2.000000000001575(2)	$1.575(2) \times 10^{-12}$
20	2.0000000000003746(8)	$3.746(8) \times 10^{-13}$

Appendix K: Above the transition

In this section we show that for $t > t_c$, where the percolation cluster is present, the evolution equations become similar to those for ordinary percolation. In the critical region near t_c, this enables us to perform a complete analysis of the problem using the known critical distribution as an initial condition.

Let us recall the expression of the distribution $Q(s)$ in terms of $P(s)$:

$$Q(s) = P(s) \sum_{k=0}^{m-1} \binom{m}{k+1} P(s)^k \left[1 - \sum_{u \le s} P(s)\right]^{m-1-k}. \tag{K1}$$

Above the percolation threshold t_c, where a giant component is present, the large s asymptotic behavior of expression (K1) is determined by the first term of the sum on the right-hand side (the term $k = 0$) in which the factor $\left(1 - \sum_{u \le s} P(u)\right)^{m-1}$ can be substituted by S^{m-1}. The relation between asymptotic distributions, above t_c, becomes

$$Q(s) \cong m S^{m-1} P(s).$$

Let us introduce the generating functions of the distributions:

$$\rho(z) \equiv \sum_{s=1}^{\infty} P(s) z^s \tag{K2}$$

and

$$\sigma(z) \equiv \sum_{s=1}^{\infty} Q(s) z^s. \tag{K3}$$

Then for z close to 1, taking into account the normalization condition $1 - \sum_s Q(s) = S^m$, we can write the relation between generation functions (K2) and (K3) as

$$1 - S^m - \sigma(z) = \sum_s Q(s)[1 - z^s]$$

$$\cong \sum_s m S^{m-1} P(s)[1 - z^s] = m S^{m-1}[1 - S - \rho(z)],$$

so

$$1 - \sigma(z) = m S^{m-1}\left[1 - \rho(z) - \frac{m-1}{m} S\right]. \tag{K4}$$

Substituting the last relation into the evolution equation

$$\frac{\partial P(s,t)}{\partial t} = s \sum_{u+v=s} Q(u,t) Q(v,t) - 2 s Q(s,t) \tag{K5}$$

we obtain the partial differential equation for any m:

$$\frac{\partial \rho(z,t)}{\partial t} = 2m^2 [S(t)]^{2(m-1)} \times \left[\rho(z,t) - 1 + \frac{m-1}{m} S(t)\right] \frac{\partial \rho(z,t)}{\partial \ln z}. \tag{K6}$$

We use the power-law asymptotics of the distribution $P(s,t_c) \cong f(0) s^{1-\tau}$ as the initial condition for Eq. (K6). This corresponds to the following singularity of the generating function at $z = 1$:

$$1 - \rho(z,t_c) = \text{analytic terms} - f(0)\Gamma(2-\tau)(1-z)^{\tau-2}. \tag{K7}$$

We substitute $S(t) = B(t - t_c)^\beta$ into Eq. (K6), and rewrite it in terms of the transformed variables $\epsilon \equiv (t - t_c)^{(m-1)2\beta+1}$ and $x \equiv \ln z$:

$$\frac{\partial \rho}{\partial \epsilon} = \frac{2m^2 B^{2(m-1)}}{1 + (m-1)2\beta}\left(\rho - 1 + \frac{m-1}{m} B \epsilon^{\beta/[1+(m-1)2\beta]}\right)\frac{\partial \rho}{\partial x}. \tag{K8}$$

To solve this equation, we use the hodograph transformation approach. We pass from $\rho = \rho(x,\epsilon)$ to $x = x(\rho,\epsilon)$, which leads to a simple linear partial differential equation for $x(\rho,\epsilon)$ and enables us to find the general solution

$$\ln z = \frac{2m^2 B^{2(m-1)}}{1 + (m-1)2\beta} \times \left[1 - \rho - \frac{m-1}{m} B \frac{(t-t_c)^\beta}{1+\beta/[1+(m-1)2\beta]}\right](t-t_c)^{1+(m-1)2\beta} + F(\rho), \tag{K9}$$

where the function $F(\rho)$ is obtained from the initial condition (K7), which gives the solution:

$$\ln z = \frac{2m^2 B^{2(m-1)}}{1 + (m-1)2\beta} \times \left[1 - \rho - \frac{m-1}{m} B \frac{(t-t_c)^\beta}{1+\beta/[1+(m-1)2\beta]}\right](t-t_c)^{1+(m-1)2\beta} - [f(0)]^{-1/(\tau-2)} |\Gamma(2-\tau)|^{-1/(\tau-2)} [1-\rho]^{1/(\tau-2)}. \tag{K10}$$

Setting $z = 1$ and taking into account the relation $1 - \rho(t,1) = S(t) = B(t - t_c)^\beta$ and comparing resulting powers and coefficients in Eq. (K10), we obtain relations between critical exponents

$$\tau = 2 + \frac{\beta}{1 + (2m-1)\beta}, \tag{K11}$$

and between critical amplitudes B and $f(0)$:

$$B = [f(0)|\Gamma(2-\tau)|]^{1/[1-(2m-1)(\tau-2)]} \times \left[2m \frac{[1-(2m-1)(\tau-2)][1+(m-1)(\tau-2)]}{3-\tau}\right]^{\frac{\tau-2}{1-(2m-1)(\tau-2)}}, \tag{K12}$$

for an arbitrary m.

One can easily show that Eqs. (E5) and (E6) for the scaling functions $\tilde{f}(x)$ and $\tilde{g}(x)$ derived for the normal phase are also valid for the percolation phase ($t > t_c$) after the following modification: one must invert signs of each term of Eq. (E5) which contains $\tilde{f}(x)$ or its derivatives. The same applies to the equation for $\tilde{f}''(x)$ in system (E9). Then, similarly to the normal phase, we find that the derivatives of $f(x)$ and $g(x)$ diverge approaching $x = 0$ also from above. This singular behavior is described by series (F1), that were written for the disordered phase, but, in fact, hold on both sides of the transition. It is clear that $f(0)$ and $g(0)$ should be equal on both phases. Moreover, the series coefficients of $f(x)$ and $g(x)$ above t_c, can be found similarly to $t < t_c$. In the percolation phase, the coefficients a_k and b_k are given by Eq. (F4), which was derived for the normal phase, after the following transformation $a_k \to (-1)^k a_k$ and $b_k \to (-1)^k b_k$.

Appendix L: Power-law initial conditions

Here we show that if the initial size distribution of clusters decays sufficiently slowly, the transition takes place at the initial moment.

Let us assume that the initial distribution is power-law, $P(s, t=0) \sim s^{1-\tau_0}$, where the exponent τ_0 defines the initial condition. This distribution results in divergent susceptibility if, according to Eq. (16) and Eq. (C9),

$$\int_{\text{const}}^{\infty} ds\, [s^{(2m-1)-m\tau_0}]^2 / s^{-\tau_0} = \infty, \tag{L1}$$

that is if

$$\tau_0 \leq 2 + 1/(2m-1). \tag{L2}$$

The divergent susceptibility indicates the presence of the continuous transition exactly at the point of divergence. So, if this condition is satisfied, then the transition occurs at the initial instant, i.e. $t_c = 0$. We will describe this case in detail elsewhere. On the other hand, if $\tau_0 > 2 + 1/(2m-1)$, then we arrive at the situation described in the previous sections, namely, $t_c > 0$ (t_c depends on τ_0), and the critical exponent values (independent of τ_0) presented in Table I.

Appendix M: Simple estimates for the percolation threshold

Let us estimate t_c in the case of $m = 2$ assuming that the process starts from isolated nodes, i.e., $P(1, t = 0) = 1$. The numerical solution of evolution equations for $P(s, t)$ showed that for sufficiently small m, including $m = 2$, the asymptotic power-law at the critical point, $P(s, t_c) \cong f(0)s^{1-\tau}$, is still approximately valid even at small s, and, moreover, $f(0)$ deviates from $P(s = 1, t_c)$ only by a small number of the order of $\tau - 2$ if all nodes initially were isolated. In this special case, we can approximate $P(s, t_c)$ in the sum rule $\sum_{s=1}^{\infty} P(s, t_c) = 1$ by $P(s = 1, t_c)s^{1-\tau}$ at any $s \geq 1$, which gives

$$P(1, t_c)\zeta(\tau - 1) \approx 1, \tag{M1}$$

where $\zeta(x) \equiv \sum_{s=1}^{\infty} s^{-x}$ is the Riemann zeta function. We find $P(1, t)$ explicitly in the full range of t by solving the master Eq. (2) with the initial condition $P(1, 0) = 1$. Let, e.g., $m = 2$. Then the result is

$$P(1, t) = \frac{2}{1 + e^{4t}}, \tag{M2}$$

so we have

$$\frac{2}{1 + e^{4t_c}}\zeta(\tau - 1) \approx 1, \tag{M3}$$

and finally

$$t_c \approx \frac{1}{4}\ln[2\zeta(\tau - 1) - 1]. \tag{M4}$$

Substituting $\tau = 2.04763044$, which we obtained above for $m = 2$ into this formula, we finally find an estimate for t_c, namely $t_c \approx 0.935$. This estimate is close to a precise value $t_c = 0.92320750930(2)$ obtained in our work [39].

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