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Dynamics of nonlocal discrete G-P equation with defects

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We study the dynamics of dipolar gas in deep lattices described by a nonlocal nonlinear discrete Gross-Pitaevskii equation. The stabilities and the propagation properties of travelling plane waves in the system with defects are discussed in detail. For a clean lattice, both energetic and dynamical stabilities of the travelling plane waves are studied. It shows that the system with attractive local interaction can preserve the stabilities, i.e., the dipoles can stabilize the gas because of repulsive nonlocal dipole-dipole interaction. For a lattice with defects, within a two-mode approximation, the propagation properties of travelling plane waves in the system map on a nonrigid pendulum Hamiltonian with quasimomentum dependent nonlinearity (induced by the nonlocal interactions). Competition between defects, quasimomentum of the gas and nonlocal interaction determines the propagation properties of the travelling plane waves. Critical conditions for crossing from superfluid regime with propagation preserved to normal regime with defects induced damping are obtained analytically and confirmed numerically. Particularly, the critical conditions for supporting the superfluidity strongly depend on the defect type and the quasimomentum of the plane waves. The nonlocal interaction can significantly enhance the superfluidity of the system.

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I. INTRODUCTION

In recent years, the interplay between nonlinearity, discreteness and disorder (i.e., small random impurities or defects) has been the subject of intensive theoretical and experimental investigations[1–11]. The competition between nonlinearity, discreteness and disorder can induce rich phenomena and plays a crucial role in nonlinear discreteness system, such as Anderson localization [12] and disorder induced inhibition of transportation [13–16], etc. Especially, the transportation properties of the disordered nonlinear discrete system have been becoming a challenging issue. A key one is that in such system, the propagation of travelling plane waves experiences a crossover from superfluid regime with propagation preserved to normal regime with disorder induced damping[3], in which nonlinearity plays a crucial role. Because of the controllable of both disorder (can be introduced in the system with a controlled way by using optical means[17], atomic mixtures [18] or inhomogeneous magnetic fields [19]) and nonlinearity (a consequence of interactions between particles can be controlled by the Feshbach technique[20]), ultracold Bosons in deep lattices with defects provides an ideal physical system to study this issue.

At low temperature, Bosons in deep lattices are well described by the nonlinear discrete Gross-Pitaevskii (GP) equation[3, 21], which has played a central role in our understanding of the system. In the discrete GP equation the cubic nonlinearity arising in the case of local interaction is characterized by two-body nonlinear term through a contact interaction that is parameterized by the s-wave scattering length a, whose sign determines the type of interaction, i.e., $a < 0$ indicates the interaction among the particles in the system is attractive, while $a > 0$ indicates the interaction is repulsive. Importantly, systems with dominant attractive local interactions are fundamentally unstable against collapse[22–25]. Up to now, the transportation properties of Bosons in lattices with defects are originally predicted and explored in the context of this local discrete GP equation. The transportation properties of Bosons in disordered nonlocal discrete GP equation are still not clear. Because of the long-range nonlocal character of the dipolar interaction, dipolar condensate trapped in deep optical lattices [26, 27] has opened the door to discuss this issue. Dipolar condensate loaded into the deep lattices can be described by a nonlinear discrete GP equation with nonlocal interaction[28, 29], i.e., a nonlocal nonlinear discrete GP equation. Stable solitons [31–37] and condensate[38–42] should be observable.

In this paper, we investigate the stability and superfluidity of a dipolar condensate in lattices within a nonlocal nonlinear discrete GP equation with and without defects. The stability and the propagation properties of travelling plane waves in the system are discussed in detail. For a clean lattice, both energetic and dynamical stabilities of the travelling plane waves are studied. It is shown that, there is a critical scattering length, a_c, when $a > a_c$, the system is stable. Interestingly, we find that, in a system with nonlocal interaction, a_c is always negative. This is different from the case with only local interaction, where $a_c > 0$. That is, the dipoles can stabilize the condensate because of repulsive nonlocal dipole-dipole interaction. For a lattice with defects, we discuss the superfluidity of the condensate in a deep annular lattice with defects, i.e., the propagation properties of the travelling plane waves in the system with competition between defects and non-

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local interaction. Within a two-mode approximation, the dynamics of the system described by the nonlocal nonlinear discrete GP equation maps on a nonrigid pendulum Hamiltonian. We find there can also exist a critical scattering length a_c that divides the system into two regime: $a > a_c$, a plane wave coherently passes through the defects and the system is in a superfluid state; while $a < a_c$, the system is in a normal regime with defects induced damping. Importantly, a_c and the superfluidity of the system strongly depend on the quasimomentum of the plane waves. Especially, the nonlocal interaction can enhance the superfluidity of the system.

The paper is organized as follows. In Sec. II, we present the physical model for the dipolar condensate in a deep one-dimensional lattice. In Sec. III, by using the perturbative approximation, we analyze the stabilities of the clean system. In Sec. IV, within a two-mode approximation, the dipolar condensate is mapped onto a nonrigid pendulum Hamiltonian. We study the dynamical properties of the system with a single defects and Gaussian defects. Finally, Sec. V is our discussion and conclusion.

II. MODEL

We consider a dipolar condensate trapped in deep 1D lattice, with magnetic dipolar moment $\overrightarrow{\mu}$ oriented particular to the lattice by an external magnetic $\overrightarrow{B}$. By using the tight-binding approximation, the system can be described by the dimensionless nonlocal discrete nonlinear G-P equation [28–30]:

$$i\frac{\partial\psi_n}{\partial\tau} = -\frac{1}{2}(\psi_{n-1}+\psi_{n+1}) + \epsilon_n\psi_n + [(a\chi + C_{DD0})|\psi_n|^2 + C_{DD1}(|\psi_{n+1}|^2+|\psi_{n-1}|^2) + C_{DD2}(|\psi_{n+2}|^2+|\psi_{n-2}|^2)]\psi_n, \quad (1)$$

where ψ_n is the wave function of condensate in the nth site of the array, $n = 1, ..., \mathcal{N}$ ($\mathcal{N}$ the number of sites). The first term in the RHS of Eq. (1) is the tunneling term, it denotes the tunneling between the adjacent sites. ϵ_n, proportional to any external field superimposing on the lattice (i.e., $\epsilon_n \propto \int d\overrightarrow{r}[(\hbar^2/2mJ)|\nabla\phi_n|^2 + V_{ext}|\phi_n|^2]$, where ϕ_n are wave functions localized in each site of the periodic potential), is the on-site energy. For a clean lattice, ϵ_n is a constant; for a defected lattice, ϵ_n in each lattice is different and expresses the defect distribution. The defects ϵ_n can be created by the additional lasers and/or magnetic fields. In the physical systems we have discussed, the defects ϵ_n can be spatially localized or extended. In Eq. (1), the nonlinearity is induced by the atomic contact interaction a, the on-site dipolar interaction C_{DD0}, the nearest-neighbor dipolar interaction C_{DD1} and the next-nearest-neighbor interaction C_{DD2}. a is the s-wave scattering length in the units of the Bohr radius a_0. The local on-site dipolar interaction C_{DD0} and the nonlocal inter-site dipolar interaction C_{DDj} ($j = 1, 2$) are given in Ref. [28], i.e. $C_{DD0} = \frac{\mu_0\mu^2}{4\pi J}\frac{1}{l_\perp^3 c^3}\sqrt{\frac{2}{\pi}}[\frac{c(3-c^2)}{3\sqrt{1-c^2}} - \arcsin(c)]$, $C_{DDj} = \frac{\mu_0\mu^2}{4\pi J}\frac{1}{3l_\perp^3}\sqrt{\frac{2}{\pi}}F(c, \frac{jb}{l_\perp})$ ($j = 1, 2$), where $\chi = \frac{4\pi\hbar^2}{mJ}\frac{a_0}{(2\pi)^{3/2}l_\perp^2 l}$, $l_\perp = \sqrt{\hbar/mw_\perp}$ and $l = bs^{-1/4}/\pi$, $b = \pi/k_L$ being the lattice step, $w_\perp = 290$ Hz is the vertical trapping frequency and $k_L = 2\pi/\lambda$ is the laser wave vector ($\lambda = 1064\mathtt{nm}$). $J = \frac{4}{\sqrt{\pi}}s^{3/4}e^{-2\sqrt{s}}Er$, $Er = \hbar^2\pi^2/2Md^2$ is the recoil energy of the optical lattices, d is the lattice period ($d = \lambda/2$), and s is the strength of the optical lattice, $c = \sqrt{1 - l^2/l_\perp^2}$. μ_0 is the vacuum permeability and μ is the magnetic dipole moment ($\mu = 6\mu_B$ for $^{52}C_r$ with μ_B the Bohr magneton). Here $F(u, \nu) = \int_0^1 ds\frac{3s^2-1}{(1-u^2s^2)^{3/2}}(1 - \frac{\nu^2 s^2}{1-u^2s^2})e^{-[\nu^2s^2/2(1-u^2s^2)]}$. In this article, we study the propagation of a plan wave $\psi_n(\tau = 0) = e^{ikn}$ in system (1), here, k is the quasimomentum of the plane wave. We will use periodic boundary conditions (due to the annular geometry): thus we have $k = 2\pi l/\mathcal{N}$, where l is integer ($l = 0, ..., \mathcal{N} - 1$).

The Hamiltonian of the system (1) is

$$H = \sum\{-\frac{1}{2}(\psi_{n+1}\psi_n^* + \psi_n\psi_{n+1}^*) + \epsilon_n|\psi_n|^2 + [\frac{(a\chi + C_{DD0})}{2}|\psi_n|^2 + C_{DD1}(|\psi_{n+1}|^2 + |\psi_{n-1}|^2) + C_{DD2}(|\psi_{n+2}|^2 + |\psi_{n-2}|^2)]|\psi_n|^2\}. \quad (2)$$

III. STABILITIES OF THE SYSTEM WITHOUT DEFECTS

Let us first consider the stabilities of the system with $\epsilon_n = 0$. We employ the plane waves $\psi = \psi_0 e^{i(kn-u_0t)}$ which is the stationary solution of Eq. (1), where k is the quasimomentum of the condensate. The stability analysis of such state can be carried out by perturbing the carrier wave with small amplitude phonons: $\psi = [\psi_0 + u(t)e^{iqn} + \nu^*(t)e^{-iqn}]e^{i(kn-u_0t)}$, where q is the quasi-momentum of the excitation, the perturbation function $u(t)$ and $\nu(t)$ have the same periodicity as the lattices, then Eq. (1) becomes

$$i\frac{\partial}{\partial t}\begin{pmatrix} u \\ \nu \end{pmatrix} = \widehat{\sigma}\widehat{A}\begin{pmatrix} u \\ \nu \end{pmatrix} \quad (3)$$

where $\widehat{A} = \begin{pmatrix} L_+ & C\psi_0^2 \\ C(\psi_0^*)^2 & L_- \end{pmatrix}$ with $L\pm = \cos(k) - \cos(q \pm k) + C|\psi_0|^2$ and the effective interaction parameter $C = a\chi + C_{DD0} + 2C_{DD1}\cos(q) + 2C_{DD2}\cos(2q)$. It is important to note that this effective atom interaction depends on the quasimomentum of the excitation. This momentum dependent atom interaction is induced by the nonlocal dipolar interaction. For a non-dipolar gas (i.e. $C_{DD0} = C_{DD1} = C_{DD2} = 0$), C does not depend on q. Here $\widehat{\sigma}$ is the Pauli matrix. By straightforward calculation, the eigenvalues of $\widehat{A}$ are easily found as

$$\lambda_\pm = 2\cos(k)\sin^2(\frac{q}{2}) + C|\psi_0|^2 \pm \sqrt{P^2 + C^2|\psi_0|^4}, \quad (4)$$

and the discrete nonlinear G-P equation excitation spectrum (eigenvalues of $\widehat{\sigma}\widehat{A}$) is given by

$$\eta_{\pm} = P \pm \sqrt{Q^2 - 2CQ|\psi_0|^2}, \quad (5)$$

where $P = \sin(q)\sin(k)$, $Q = \cos(k)[\cos(q) - 1]$.

Base on Eq. (4), we can easily find that the boundary of energetic stability of Bloch waves is described by (λ should be real positive) $\cos^2(\frac{q}{2}) \leq \cos(k)[\cos(k)+C|\psi_0|^2]$. Clearly, the energetic instability can be completely excited when $\cos(k) < 0$. For long-wave length perturbing ($q \to 0$), this condition can be reduced to a critical contact scattering length a_c for maintaining the stability in dipolar condensate.

$$a \geq a_c = \frac{1}{\chi}[\frac{\sin^2(k)}{\cos(k)|\psi_0|^2} - C_{DD0} - 2C_{DD1} - 2C_{DD2}]. \quad (6)$$

In Fig. 1 we plot the energetic stability diagram of the system and the area above the critical scattering length a_c corresponds to the energetically stable region. The critical parameter a_c for both dipolar condensate and non-dipolar condensate is shown in Fig. 1. We find the critical scattering length a_c decreases with increasing the strength of optical lattices s, when fix the quasimomentum k of the plane wave. And for a fixed s, a_c increases with increasing k. Interestingly, we find the dipolar condensate is more stable than the non-dipolar system, i.e., the dipolar gas can preserve the stability with attractive local interaction (contact interaction). With the increasing of s, the critical scattering length a_c of the non-dipolar gas tends to 0, while the critical scattering length a_c of the dipolar condensate tends to $-20a_0$.

For a non-dipolar condensate with purely contact interaction, the system with attractive contact interaction (attractive local interaction) is fundamentally unstable against collapse, while the system with repulsive contact interaction prevents the collapse and is stable. For a dipolar gas, the nonlocal repulsive dipolar interaction can compensate the local attractive contact interaction and the effective interaction of the system can be repulsive. So the system with attractive contact interaction could be stable as long as the effective interaction of the system is repulsive (i.e. $C > 0$). That is, dipoles can stabilizing the condensate due to the effectively nonlocal repulsive dipolar interaction.

Furthermore, the modulational instability (dynamical instability) can be induced when the eigenfrequency η in Eq. (5) becomes imaginary, i.e., $C|\psi_0|^2 \geq -\cos(k)\sin^2(\frac{q}{2})$. Therefore, when the effective atomic interaction is repulsive ($C > 0$), the system suffers an exponential growth of perturbations with $\cos(k) < 0$. For long-wave length perturbing ($q \to 0$), this condition reduces to a critical scattering length a_c for preserving the modulational stability of the system.

$$a \geq a_c = -\frac{1}{\chi}(C_{DD0} + 2C_{DD1} + 2C_{DD2}). \quad (7)$$

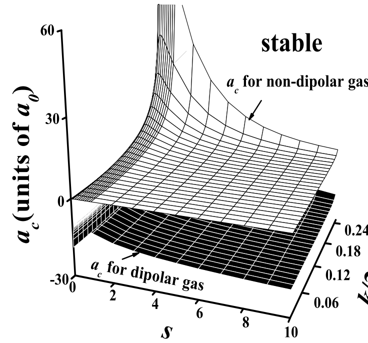

FIG. 1: Energetic stability diagram of the system. a_0 is the Bohr radius.

a_c given by Eq. (7) is plotted in Fig. 2. The area above the line corresponds to the stable region. When s is small, we can find the critical scattering length a_c rapidly grows to a maximum which is due to the influence of the inter-site dipolar interaction. And then a_c decreases gradually with increasing of lattice depth s until close to $-20a_0$. Also, we can observe that a_c for different s is less than zero. However, it is well known that the dynamical stability in the non-dipolar gas can be induced when the scattering length a is positive. So the dipolar condensate is more stable in dynamics than the system without dipolar interaction. On the other hand, dipoles can suppress both energetic and dynamical instabilities because of the effective nonlocal repulsive dipolar interaction. Our results are in good agreement with the recent experiments [38, 39].

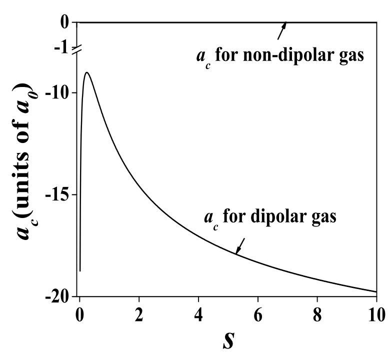

FIG. 2: Modulational stability diagram of the system. The critical scattering length a_c as a function of the lattice depth s. a_0 is the Bohr radius.

IV. SUPERFLUIDITY WITH DEFECTS

We now consider the dynamical properties of Eq. (2) with defects. As discussed above, when $\cos(k) < 0$, the system becomes unstable, so we consider the case in which $\cos(k) > 0$. The angular momentum of this system is defined as

$$L(\tau) = i\sum(\psi_n\psi^*_{n+1} - \psi^*_n\psi_{n+1}). \tag{8}$$

$L(\tau)$ oscillates between the initial value L_0 to $-L_0$, corresponding, respectively, to plane waves with wave vector k and $-k$ in a liner system. Moreover, rotational state with opposite wave vectors k and $-k$ are degenerate in clear optical lattices. However, the defects split the degeneracy by coupling the two k and $-k$ waves, very much as the tunnelling barrier in a double well potential between the left and right localized state. For this reason, the relative population of the two waves oscillates according to an effective Josephson Hamiltonian [3, 43]. In this limit, one can employ a two-mode ansatz for the dynamical evolution of the wave function:

$$\psi_n(\tau) = A(\tau)e^{ikn} + B(\tau)e^{-ikn}. \tag{9}$$

We set $A, B = \sqrt{n_{A,B}(\tau)}e^{i\phi_{A,B(\tau)}}$, $z = n_A - n_B$, and $\phi = \phi_A - \phi_B$.

To understand the dynamics of the system, we discuss the variation of the angular momentum with some related parameters. So using ansatz (9) in Eq. (8), we get

$$L = 2\mathcal{N}z\sin(k). \tag{10}$$

Therefore, we observe that the angular momentum is proportional to z. $\langle L\rangle = 0$ implies that the wave is completely reflected, which means that the system is in a normal state with defects induced damping. $\langle L\rangle \neq 0$ implies that the wave is only partially reflected by the defects, that is, the system is in a superfluid state. Here, the $< ... >$ stands for a time average. The latter case corresponds to a self-trapping of the angular momentum. The incident wave can not be reflected completely and coherence is preserved. The observation of a persistent current is associated with a superfluid regime of the system (1).

A. a single defect

Let us consider, first, the case of a single defect

$$\epsilon_n = \epsilon\delta_{n,\overline{n}}. \tag{11}$$

Defining the effective Lagrangian as $\pounds = \sum \frac{i}{2}(\dot{\psi_n}\psi^*_n - \psi_n\dot{\psi^*_n}) - H$, both H and the morn $\sum|\psi_n|^2 = N$ are conserved and using ansatz (9) in this effective Lagrangian, we have

$$\begin{aligned}\frac{\pounds}{\mathcal{N}} =& - n_A\dot{\phi}_A - n_B\dot{\phi}_B - \frac{2\epsilon}{\mathcal{N}}\sqrt{n_An_B}\cos(\phi_A - \phi_B + 2k\overline{n}) \\ &- Cn_An_B,\end{aligned} \tag{12}$$

with the relation $\sum e^{2kn} = 0$. Using the Euler-Lagrange equations $\frac{d}{dt}\frac{\partial\pounds}{\partial\dot{q}_i} = \frac{\partial\pounds}{\partial q_i}$ for the variational parameters $q_i(\tau) = n_{A,B}, \phi_{A,B}$ in Eq. (12) and with the replacement $\phi + 2k\overline{n} \to \phi$, we obtain

$$\dot{z} = -\frac{2\epsilon}{\mathcal{N}}\sqrt{1-z^2}\sin(\phi), \tag{13}$$

$$\dot{\phi} = \frac{2\epsilon}{\mathcal{N}}\frac{z}{\sqrt{1-z^2}}\cos(\phi) + Cz. \tag{14}$$

where $C = a\chi + C_{DD0} + 4C_{DD1}\cos(2k) + 4C_{DD2}\cos(4k)$ is the effective atom interaction, which, interestingly, depends on the quasi-momentum k (induced by the nonlocal dipolar interaction). That is, within a two-mode ansatz in Fourier space, the dynamics of the system map on a nonrigid pendulum with quasi-momentum depended nonlinearity. The effective Hamiltonian (i.e. the total conserved energy) becomes

$$H = -\frac{2\epsilon}{\mathcal{N}}\sqrt{1-z^2}\cos(\phi) + \frac{Cz^2}{2}. \tag{15}$$

Let us derive the critical condition for supporting a superfluid flow. That is the occurrence of transition between the regimes with $\langle L\rangle = 0$ and the regime with $\langle L\rangle \neq 0$. Eq. (10) indicates that the angular momentum L is proportional to z. Therefore $\langle z\rangle = 0$ (i.e., z oscillates around 0 and $\langle L\rangle = 0$) implies that the wave is completely reflected by the defects. $\langle z\rangle \neq 0$ (i.e., z oscillates around a non zero value and $\langle L\rangle \neq 0$) implies that the wave is only partially reflected by the defects, and the system is in a superfluid regime. Hence, if z can not reach the value 0, then we can have $\langle z\rangle \neq 0$ and the system is in a superfluid regime. To avoid the system reach the state $z = 0$, the initial energy of the system H_0 should be larger than the energy of this state, i.e., $H_0 > H(z = 0)$. Initially, we set $z(0) = 1$ and $\phi(0) = 0$, so the conserved initial energy is $H_0 = C/2$. Because $H(z = 0) = -\frac{2\epsilon}{\mathcal{N}}\cos(\phi)$. We clearly see that the maximum value of $H(z = 0)$ is $2\epsilon/\mathcal{N}$, i.e., $H(z = 0) = -\frac{2\epsilon}{\mathcal{N}}\cos(\phi) \leq \frac{2\epsilon}{\mathcal{N}}$. Hence, if $H_0 = C/2 \geq 2\epsilon/\mathcal{N}$, i.e., $C \geq 4\epsilon/\mathcal{N}$, then $H_0 > H(z = 0)$ should be satisfied for all value of ϕ. That is, when $C \geq 4\epsilon/\mathcal{N}$, z can not reach the value 0 and the system will be in a superfluid regime. So, we find a critical condition for supporting the superfluid flow $a\chi + C_{DD0} + 4C_{DD1}\cos(2k) + 4C_{DD2}\cos(4k) = 4\epsilon/\mathcal{N}$. From this condition we can obtain a critical atomic scattering length a_c for maintaining the superfluidity

$$a_c = \frac{1}{\chi}[\frac{4\epsilon}{\mathcal{N}} - C_{DD0} - 4C_{DD1}\cos(2k) - 4C_{DD2}\cos(4k)]. \tag{16}$$

The system can be divided into two regimes by the critical condition: a normal regime when $a < a_c$ and a superfluid regime when $a > a_c$. When $a < a_c$, L oscillates around 0; and when $a = a_c$, asymptotically $L \to 0$, with $a > a_c$, $\langle L\rangle \neq 0$. In a normal state a plane wave reflected by the defect, while a plane wave travels coherently through the defects in a superfluid state.

Importantly, Eq. (16) indicates that the superfluidity of the system strongly depends on the defect ϵ, dipolar interaction and the quasi-momentum k of the gas. The competition between defect, dipolar interaction and quasi-momentum of the gas provides a critical scattering length a_c for maintaining the superfluidity. For fixed defect, the presence of nonlocal dipolar interaction can reduce a_c, even results in a negative a_c and enhances the superfluidity of the system.

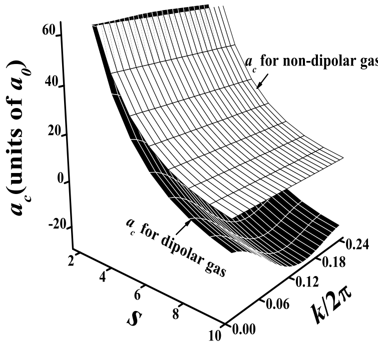

FIG. 3: The critical scattering length a_c against lattice depth s and the quasimomentum k of the gas associated with a single defect. a_0 is the Bohr radius. $\epsilon = 0.05$ and $\mathcal{N} = 100$.

Figure 3 shows the critical a_c against lattice depth s and the quasimomentum k with a single defect given by Eq. (16). Clearly, we find the critical scattering length a_c for dipolar gas decreases from a positive value to $a_c = -30a_0$ with increasing lattice depth, when fix the quasimomentum k. Thus, in the deep lattice regime, the superfluid can more easily preserve due to the relatively small critical scattering length a_c. Furthermore, the critical scattering length a_c for non-dipolar gas is positive and tends gradually to zero in the deep lattice regime. That is, the dipolar condensate with a single defect can more easily support superfluid state than the system without dipolar interaction. The dipoles can enhance the superfluidity of the condensate with a single defect. In order to clearly show the relationship between a_c and the quasimomentum k, the critical scattering length a_c of dipolar condensate vs the quasimomentum k for different lattice depths of $s = 5$ and $s = 9$ is plotted in Fig. 4. It is clear that, the values of a_c for $s = 9$ are smaller than that for $s = 5$. Interestingly, there is a critical k_c, when $k < k_c$, a_c decreases with increasing k, while $k > k_c$, a_c increases with k. This nonmonotonic behavior of a_c against k is induced by nonlocal dipolar interaction (note C depends on k). To confirm the analytical results, numerical results obtained by direct numerical integrations of Eq. (1) with a fourth-order Runge-Kutta method are also shown in Fig. 4. We find that the analytical results qualitatively agree with the numerical results.

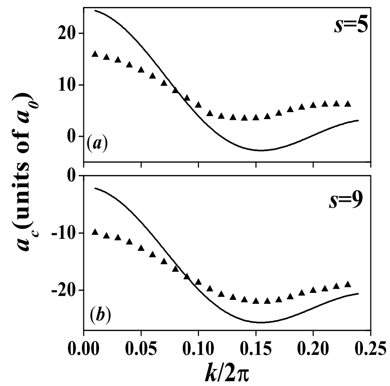

FIG. 4: The critical scattering length a_c as a function of the quasimomentum k for different lattice depth of $s = 5$ (a) and $s = 9$ (b). a_0 is the Bohr radius. The points are numerical simulations of Eq. (1) and the solid lines are the analytical results of Eq. (16). $\epsilon = 0.05$ and $\mathcal{N} = 100$.

B. the Gaussian defect

Furthermore, we now consider a Gaussian defect with width σ centered on the site $\overline{n}$:

$$\epsilon_n = \frac{\epsilon}{\sqrt{\pi}\sigma} e^{\frac{-(n-\overline{n})^2}{\sigma^2}}, \tag{17}$$

For sufficiently large $\mathcal{N}$ and $\sigma \gtrsim 1$, we can set $\sum \epsilon_n \approx \int dn \epsilon_n = \epsilon$. Using the same way as we used in the case of a single defect and setting $\phi + 2k\overline{n} \to \phi$, the effective Hamiltonian reduces to

$$H \approx -\frac{2\varepsilon e^{-k^2\sigma^2}}{\mathcal{N}} \sqrt{1-z^2}\cos(\phi) + \frac{Cz^2}{2}. \tag{18}$$

We can clearly see that the system is equal to that of a single defect with an effective defect $\epsilon_{eff} = \epsilon e^{-k^2\sigma^2}$. So, the critical a_c for supporting the superfluid is

$$a_c = \frac{1}{\chi}[\frac{4\epsilon_{eff}}{\mathcal{N}} - C_{DD0} - 4C_{DD1}\cos(2k) - 4C_{DD2}\cos(4k)]. \tag{19}$$

For fixed defect, the critical scattering length a_c against the lattice depth s and the quasimomentum k is shown in Fig. 5. Just like the case of a single defect, we can see that the critical scattering length a_c decreases with the increasing of the lattice depth s, when fix the quasimomentum k. The dipolar gas with the Gaussian defect also requires relatively smaller critical scattering length a_c to preserve the superfluidity than that in the system of the non-dipolar gas. In Fig. 6, we plot the critical scattering length a_c as a function of the quasimomentum k respectives to the lattice depth $s = 5$ and $s = 9$. We can see that the critical scattering length a_c decreases with the quasimomentum k. We note that, when $k\sigma \gg 1$, $\epsilon_{eff} \to 0$. This means that the dipolar system with Gaussian defect and large quasimomentum will always pass through the defect. This is different from

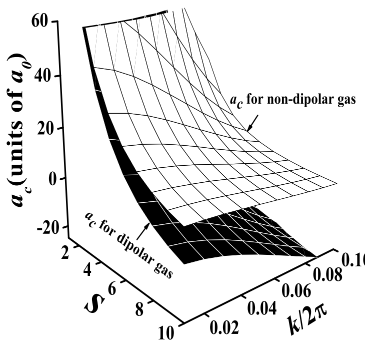

FIG. 5: The critical scattering length a_c against the lattice depth s and the quasimomentum k with the Gaussian defect. a_0 is the Bohr radius. $\epsilon = 0.05$, $\mathcal{N} = 100$, $\sigma = 2$.

the case of single defect, where a_c varies nonmonotonic with k (see Fig. 4). We also find that, for Gaussian defect, the analytical result is in good agreement with the numerical result.

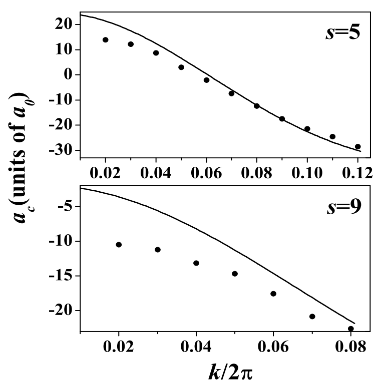

FIG. 6: The critical scattering length a_c vs the quasimomentum k respective to the shallow lattice depth $s = 5$ (a) and the deep lattice depth $s = 9$ (b). Points: numerical solutions of Eq. (1); lines: analytical results of Eq. (19). a_0 is the Bohr radius. $\epsilon = 0.05$, $\mathcal{N} = 100$, $\sigma = 2$.

Critical scattering length a_c against the defect ϵ for different lattice strength s is shown in Fig. 7. One can find that a_c increases with increasing ϵ and decreases with increasing s. Particularly, a_c for the system with Gaussian defect is much lower than that for the system with single defect. On the other hand, system with a Gaussian defect can more easily support superfluid state than the system with single defect.

V. CONCLUSION

In this work we have investigated the stability and the superfluidity of a dipolar $^{52}C_r$ condensate in a deep one-dimensional lattice. By using the perturbative and tight-binging approximation, we analyze energetic stability and modulationally stability (dynamical stability) of

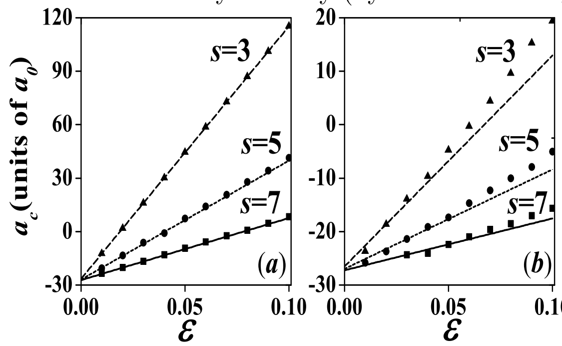

FIG. 7: The critical scattering length a_c vs defects ϵ for different lattice depths s. (a): with single defect (b): with Gaussian defect. The points are numerical simulations of Eq. (1) and the solid lines are the analytical results of Eqs. (16) and (19). $k/2\pi = 0.09$, $\mathcal{N} = 100$ and $\sigma = 2$. a_0 is the Bohr radius.

a dipolar condensate in a clean lattice. There is a critical scattering length, and the system is stable when $a > a_c$. Due to the critical scattering length decreases with the lattice depth, we show that the system is more stable in the deep lattice regime. Through the comparison with the non-dipolar gas, a dipolar gas is easier to maintain the stabilities because of the nonlocal dipolar interactions. Furthermore, the superfluidity of dipolar condensate in a deep annular lattice with defect is discussed both analytically and numerically. Within a two-mode approximation, the dynamics of the system can be considered as a single nonrigid Hamiltonian with quasimomentum dependent nonlinearity (induced by the nonlocal interaction). We find that the superfluid state can exist beyond a critical scattering length. The analytical expression of the critical scattering length for supporting a superfluid flow is obtained and we find it is determined with the competition between defect, quasimomentum of the gas and nonlocal dipolar interaction. The system can easily support superfluid state in deep lattices. Especially, the dipolar system can easily support superfluid state with low scattering length relative to the non-dipolar gas. The present results give a deep insight into the dynamics of a dipolar condensate in disordered optical lattice.

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