Collapse, decay, and single-|k| turbulence from a generalized nonlinear Schrödinger equation

Shaoyan Cui (□□□), M. Y. Yu (□□□), and Dian Zhao (□□)

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Collapse, decay, and single-|*k*| turbulence from a generalized nonlinear Schrödinger equation

Shaoyan Cui (崔少燕)*,[1] M. Y. Yu (郁明阳)†,[2,3] and Dian Zhao (赵典)[2]

[1] School of Mathematics and Information, Ludong University, Yantai 264025, China

[2] Institute for Fusion Theory and Simulation, Department of Physics, Zhejiang University, Hangzhou 310027, China

[3] Institut für Theoretische Physik I, Ruhr-Universität Bochum, D-44780 Bochum, Germany

*E-mail: shycui@ldu.edu.cn
†E-mail: myyu@zju.edu.cn

Abstract

Turbulence governed by a generalized nonlinear Schrödinger equation (GNSE) including viscous heating and nonlinear damping is numerically investigated. It is found that a large localized pulse can suffer modulational instability and then collapse into the shortest wavelength modes, as for the classical NSE. However, the total energy of the nonconservative GNSE can also become constant during the collapse via local balance of energy gain and loss in the phase-space. After the collapse, instead of inverse cascading into a state of strong turbulence with broad spectrum, a single-step cascade, or condensation, into modes of one predominant wavelength can occur. In fact, after attaining total-energy balance the turbulent system as a whole evolves like a closed adiabatic system.

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I. Introduction

For more than half a century the nonlinear Schrödinger equation (NSE) has been used as a paradigm equation for describing the evolution of many physical phenomena [1-6]. Novel analytical methods have also been developed for solving this nonlinear partial differential equation [4]. Application of the NSE is further broadened by its many generalizations that include taking into account higher-order and exotic nonlinearities, linear and nonlinear dissipation, external sources and forces, etc. [5-9]. When generalized to allow for complex coefficients (such as for dissipative nonequilibrium systems), the equation can also be identified to be a generalization of a nonlinear reaction-diffusion equation, which is another paradigm evolution equation with applications in many areas [6,10-16]. However, the generalized equations are usually more difficult to analyze, and except for special cases [7,8] numerical methods must be employed.

A most interesting aspect of dissipative nonequilibrium systems is that they can allow for, locally or globally, regular motion or structure via self-organization in the physical or phase space, even in the presence of turbulence or nonthermal fluctuations [17]. The generalized NSE (GNSE), where the coefficients of the group dispersion and nonlinear terms can be complex, is an ideal model for investigating such systems. The evolution of a highly localized large amplitude initial pulse governed by a two-dimensional (2D) GNSE was investigated numerically for a nonlinear dispersive-dissipative medium [13-15]. 2D systems [18-23] are of interest not only because they indeed exist, such as electrons on liquid helium surfaces, as well as waves, colloids and granules in or on fluids and plasmas, etc., but also because they often exhibit novel physical characteristics. As predicted by Zakharov for the collapse of Langmuir waves [1,9], Zhao et al. [14] found that a localized large amplitude pulse first undergoes modulational instability and then collapses into the shortest wavelength modes (as determined by the smallest scale of the physical problem or the numerical scheme) allowed by the system. The collapse is followed by inverse cascade of energy back to modes of longer wavelengths, until a stationary state of homogeneous turbulence with a spiky energy spectrum appears. However, no self-organization or condensation in the physical or phase space was found [14,15].

In this paper, we reconsider the evolution of a localized pulse governed by a 2D GNSE with an anisotropic external potential, as well as viscous heating and nonlinear damping. The coefficients of the GNSE are chosen such that total energy gain and loss by the system can achieve balance during the evolution. As expected [1,14], at the start of the evolution the isolated initial pulse first suffers modulational instability, followed by collapse into the shortest-wavelength modes. The shortest wavelength is zero in an ideal (nondissipative) system [1], but here it is determined by a competition between the wavelength-dependent growth and intensity-dependent damping of the modes. Our result shows that the stepwise inverse-cascade process that is expected to follow the collapse is replaced by rapid (but still slow compared to the collapse) decay into modes of a specific wavelength, but with broad amplitude and phase-angle distributions. This occurs after the total energy of the system becomes constant through balance of the total energy gain and loss. In the real space an inhomogeneous turbulence structure, or pattern of oscillations, appears and evolves slowly. The energy spectrum becomes more and more localized around eight angles in the phase space, and the spatial wave pattern becomes more and more stationary (averaged over the mode oscillations). Its total energy remains the same as that attained shortly after the collapse. In fact, after achieving total-energy balance, the system effectively evolves adiabatically. The phase space condensation here is somewhat similar to that of light amplification by simulated emission of radiation (laser) and Bose-Einstein condensation (BEC). To our knowledge, self-organized turbulent condensation after a collapse has not been reported earlier.

II. Formulation

The 2D GNSE investigated here is

$$iE_t + p\nabla^2 E + \left(V(x,y) + q|E|^2\right)E = 0\,, \qquad (1)$$

where the coefficients $p = p_r + ip_i$ and $q = q_r + iq_i$ are complex. For convenience of discussion, we shall consider that (1) describes the evolution of the envelope of the fields of electron plasma waves [1,9]. When applied to other systems, some terms used in our descriptions and discussions may have different physical meanings. Accordingly, besides the usual group dispersion and nonlinear frequency shift that are characteristic of the standard NSE, Eq. (1) also includes viscous or diffusive heating if $p_i > 0$ and nonlinear damping or cooling if $q_i > 0$. The external potential is given by

$$V(x,y) = V_0\left\{1 - \mathrm{sech}^2\left[(x/a)^2 + (y/b)^2\right]\right\}, \qquad (2)$$

which can be anisotropic. The isolated initial pulse is taken to be Gaussian, or

$$E(0,x,y) = E_0 \exp\left[-\left(x^2 + y^2\right)/c^2\right], \qquad (3)$$

so that even if *a*, *b*, and *c* are of the same order, the initial pulse is spatially much narrower than the external potential.

As for the standard NSE, the evolution of the system depends strongly on group dispersion $\left(p_r\nabla^2 E\right)$ and nonlinear interaction $\left(q_r|E|^2 E\right)$. However, here it also depends on diffusion $\left(p_i\nabla^2 E\right)$ and amplitude-dependent dissipation or growth $\left(q_i|E|^2 E\right)$. We can easily obtain from (1) for the evolution of the total "energy" of the system the relation

$$\partial_t \int |E|^2 dA = 2\int \left[p_i|\nabla E|^2 - q_i|E|^4\right] dA, \qquad (4)$$

where $dA = dxdy$. Eq. (4) shows that positive values of p_i and q_i also correspond to total energy gain and loss, respectively. In this case the right-hand side of (4) can vanish during the evolution and the total energy can become constant. The overall system then becomes effectively adiabatic, so that stationary structure/pattern or turbulence can be expected.

III. Numerical results

Accordingly, we investigate Eq. (1) numerically for $p = 0.5 + 0.05i$, $q = 0.6 + 0.5i$, $V_0 = 6$, $E_0 = 0.1$, $a = 4$, $b = 0.4$, and $c = 1.5$. That is, we consider a system with positive group dispersion ($p_r > 0$), viscous heating ($p_i > 0$), nonlinear frequency up-shift ($q_r |E|^2 > 0$), and nonlinear damping ($q_i |E|^2 > 0$), as well as an external potential $V(x, y)$. The latter is taken to be highly (symmetry-breaking) anisotropic ($a \gg b$) in order to speed up the initial modulational instability, which can otherwise be relatively slow. It turns out that the final state does not depend much on $V(x, y)$ because in the present problem a large gain of the total energy via the (k^2-dependent, where ***k*** is the mode wave vector) viscous heating takes place as soon as collapse into the smallest-scale (largest ***k***) modes occurs. Periodic boundary conditions are used on all sides of the 256×256 simulation box.

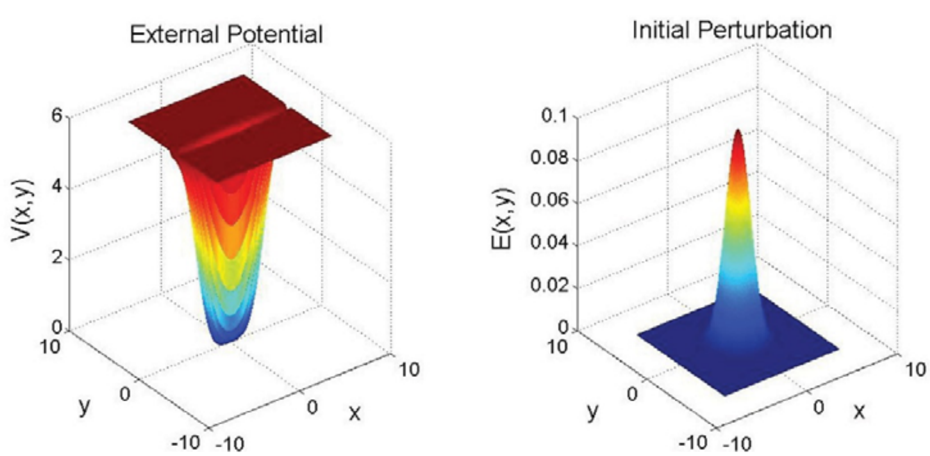

Fig. 1. The external potential $V(x, y)$ and the initial pulse $E(0, x, y)$.

Figure 1 shows the external potential $V(x, y)$ and the initial pulse $E(0, x, y)$. Figure 2 shows the evolution of the total energy $\int |E(t,x,y)|^2 dA$. The left and right columns in Figs. 3 to 5 show the full physical space of $|E(t,x,y)|^2$ and a quadrant of the spectrum $|E(t,k_x,k_y)|^2$ (in arbitrary units), respectively, at various stages of the evolution. The initial pulse represented by the longest-wavelength modes (in the smallest ***k*** corner) of the spectrum first suffers modulational instability, here asymmetric because of the anisotropic external potential that helps to speed up the instability. Collapse begins at around $t \sim 0.95$, as can be seen by the appearance of very small scale periodic structures in $|E(t,x,y)|^2$ (clearly visible on the iso-energy contours) as well as modes in the far (largest ***k***) corner of the spectrum $|E(t,k_x,k_y)|^2$. In Fig. 3 we can also see that within a very short time much of the energy of the system is transferred from the longest (smallest ***k***) wavelength modes making up the modulated initial pulse to the shortest (largest ***k***) wavelength modes. As the long wavelength modes vanish, a rapid and complete change of the structure, or pattern, of the $|E(t,x,y)|^2$ distribution, takes place, which is characteristic for collapse [1,11,18]. However, here the collapse is accompanied by rapid viscous heating ($\propto p_i k^2$), as can be noted in Fig. 2 from the large jump of the total energy $\int |E|^2 dA$ at $t \sim 0.11$ and in Fig. 3 from the large jump of the energy scale in the spectra between $t \sim 0.1$ and $t \sim 0.15$. (Recall that the total energy of the classical NSE is conserved until the collapse, when the system becomes singular [1].)

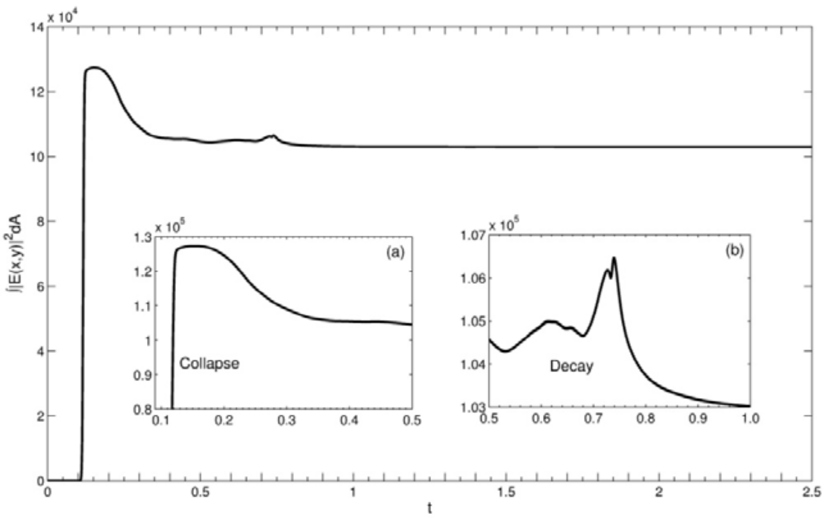

Fig. 2. Evolution of the total energy $\int |E(t,x,y)|^2 dA$. The insets (a) and (b) show enlargements of collapse-stop and single-step cascade stages, respectively. (Note that for the classical NSE the total energy would be constant.)

As an immediate consequence of the collapse and the resulting large increase in $E(t,x,y)$, the effect of the external trapping potential $V(x,y)$ becomes negligible. The boundary conditions then replace its role, as is evident from the $|E(t,x,y)|^2$ patterns at $t=0.095$ and 0.1. Fig. 2 shows that at $t \sim 0.13$ the rapid growth of the total (in the phase space also local) energy is braked by the nonlinear damping ($\propto q_i |E(t,x,y)|^2$), which becomes important as the mode amplitudes rapidly increase. The shortest-wavelength modes interact with each other as well as the remaining long-wavelength modes until all the latter vanish at around $t \sim 0.3$. During that time the total energy also decreases slightly as a result of competition between nonlinear damping and viscous heating, i.e., the system is slightly cooled.

From the spectrum at $t=0.5$ in Fig. 4, we can see that the shortest-wavelength modes begin to decay, or condensate, into modes around a specific wavelength (corresponding to $|\boldsymbol{k}|=110$) but with large phase angle spread. This process is accompanied by a small bump in the evolution of the total energy (see Fig. 2(b)) arising from competition between the energy gain and loss mechanisms at this longer wavelength. A slowly evolving turbulent pattern for $|E(t,x,y)|^2$ gradually emerges. Fig. 5 shows that at still longer times the modes slowly become more and more concentrated around two (eight in the complete phase space) phase angles in the spectrum.

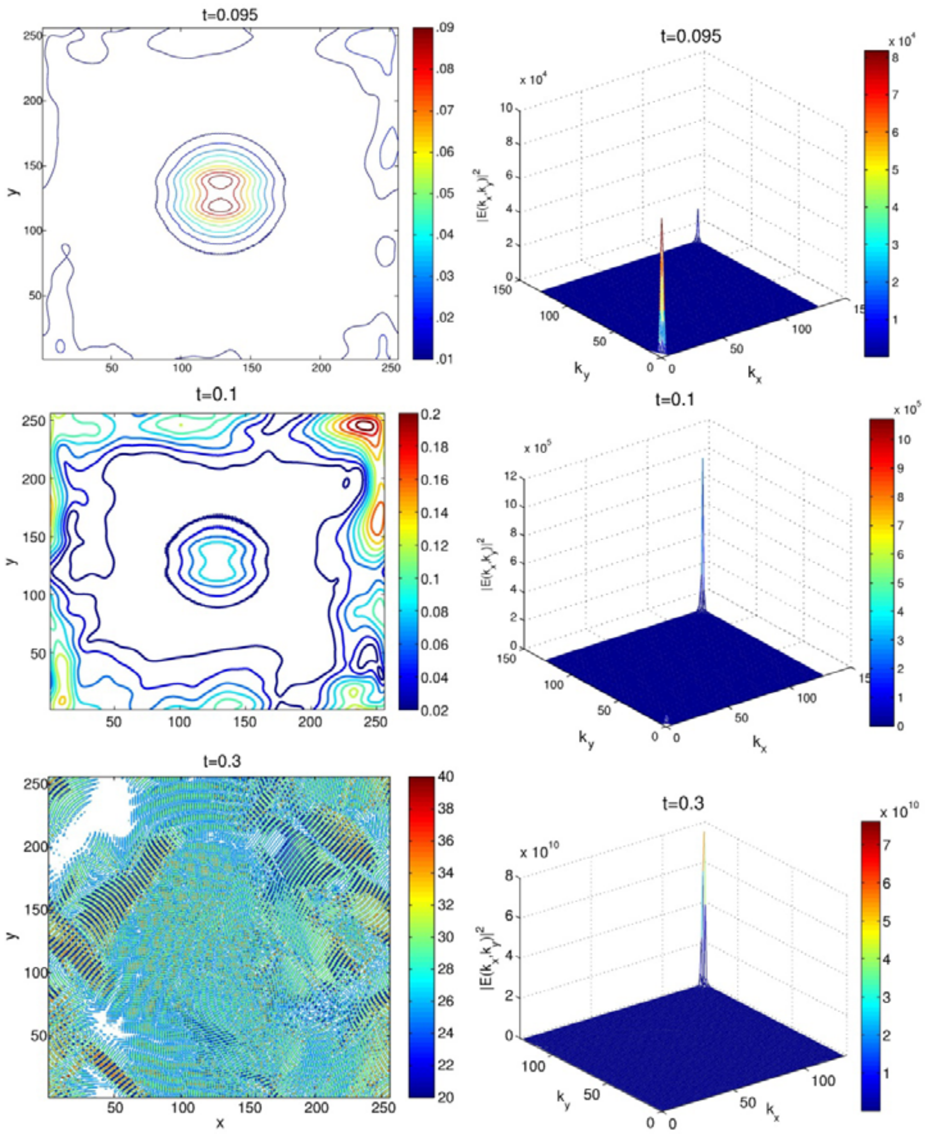

Fig. 3. (Color online) The $\left|E\left(t,x,y\right)\right|^{2}$ pattern in the real space (left column) and a quadrant of the spectrum $\left|E\left(t,k_{x},k_{y}\right)\right|^{2}$ in the phase space (right column, in arbitrary units) during the collapse stage. For convenience of computation, we have redefined the real-space axes scales (from that in Fig. 1). Note the large jump in the amplitude scales between $t=0.1$ and $t=0.3$ arising from viscous heating at large $\boldsymbol{k}$.

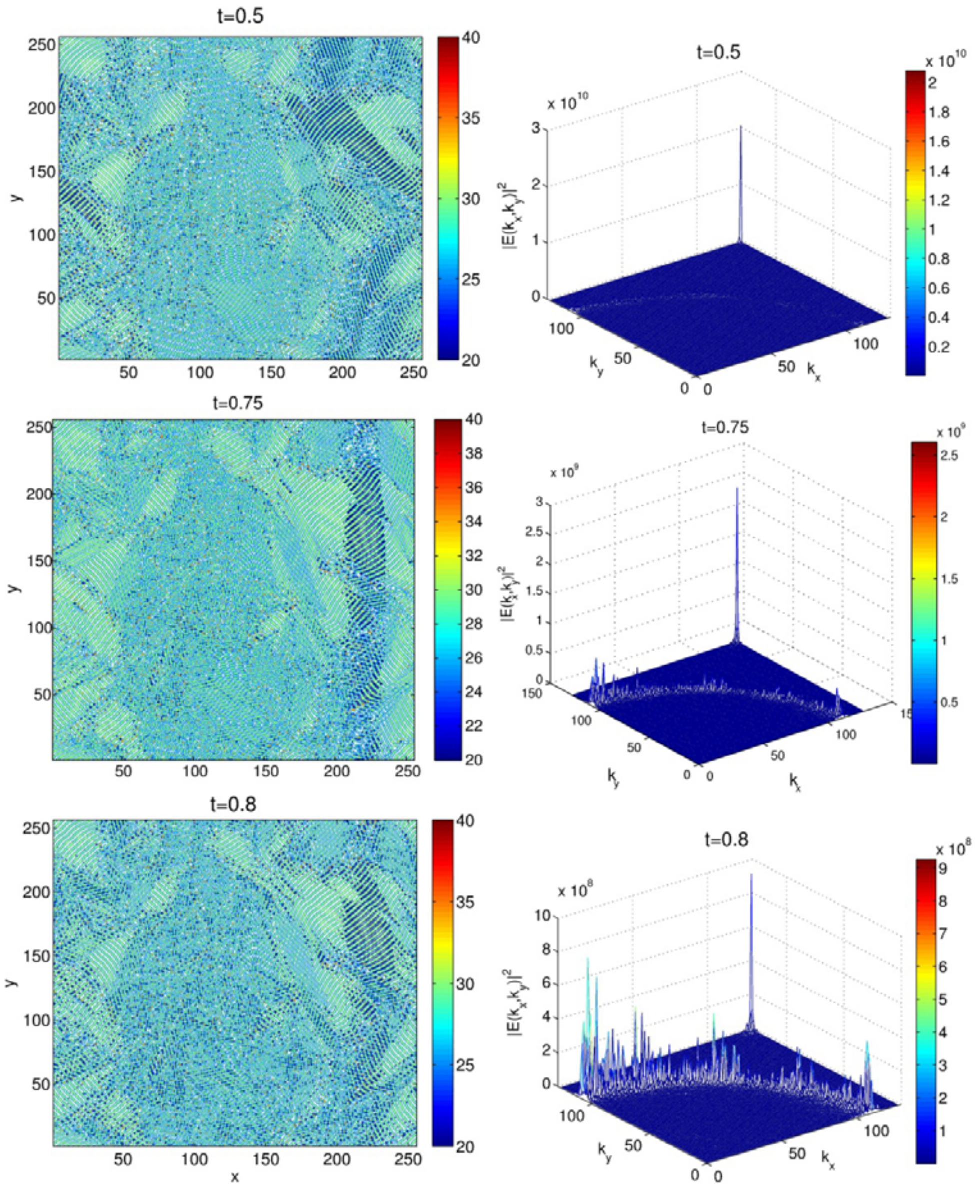

Fig. 4. (Color online) Same as in Fig. 3, at later times. Decay, or one-step-cascade, to modes of predominantly one $|k|$.

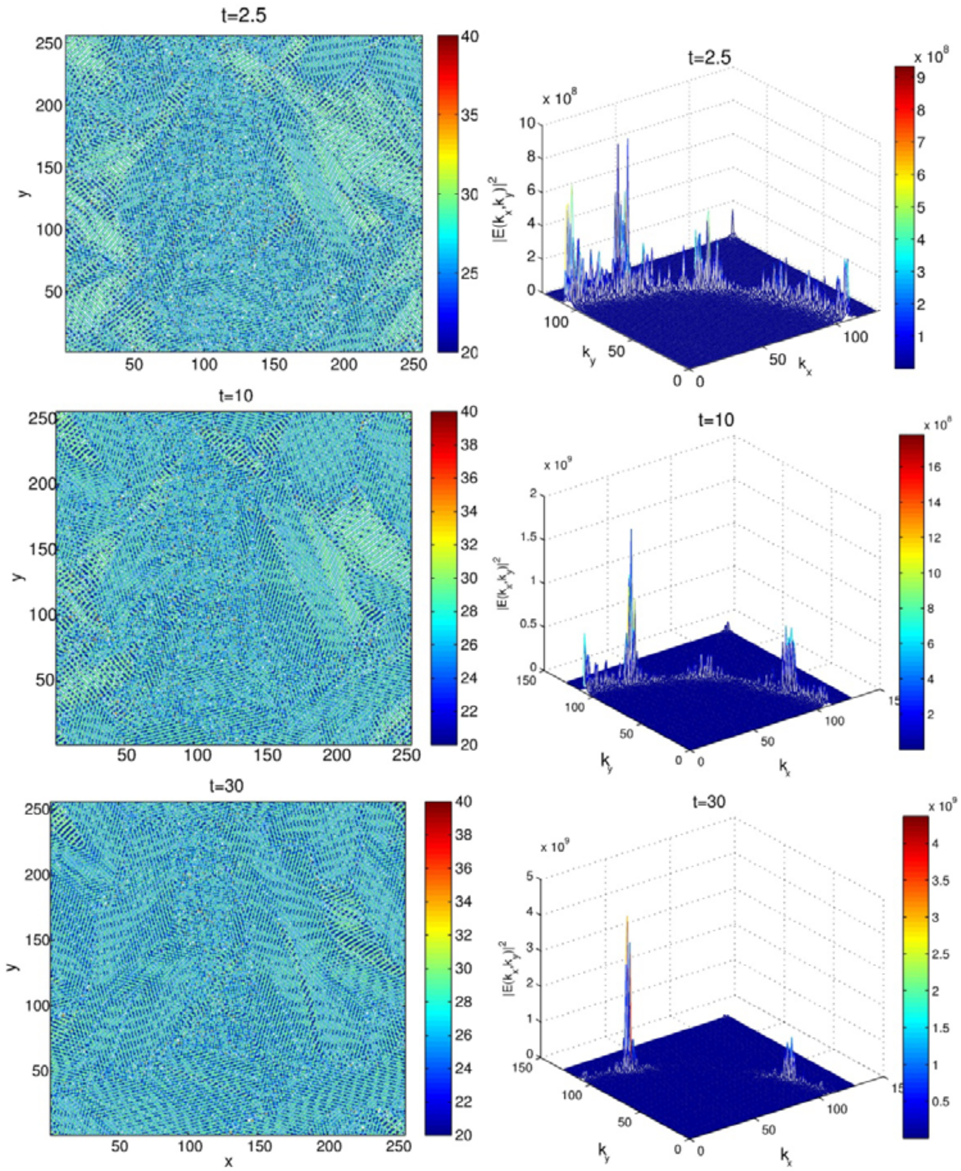

Fig. 5. (Color online) Same as in Fig. 4, at later times. Phase angle redistribution among the modes of nearly the same wavelength. Note the large difference in the time intervals between the subpanels in Figs. 3-5. The long-time evolution is very slow compared to that of the collapse and cascade stages.

For completeness, in Fig. 6 we show a three-dimensional representation of $\left|E\left(30,x,y\right)\right|^2$, where the fine structure of the "single-wavelength turbulence" can be more clearly visualized. For still longer times, the $\left|E\left(t,x,y\right)\right|^2$ distribution, still consisting of modes of one wavelength, appears to become more and more homogeneous. If that happens, energy balance would take place within each mode and the integrand in (4) would vanish. We then get $\left|E_k\right|^2 = k^2 p_i / q_i$, so that all the modes would eventually have not only the same wavelength but also the same amplitude. However, although the pattern structure of $\left|E(t,x,y)\right|^2$ indeed appears to evolve in that direction, the process is extremely slow and we could not attain such a state in our numerical simulation. On the other hand, the flatness of the total energy curve for $t \geq 1$ (Fig. 2) indicates that the energy balance is taking place locally within each mode, so that no transport in the physical space, which can leads to fluctuations in the total energy curve, is necessary.

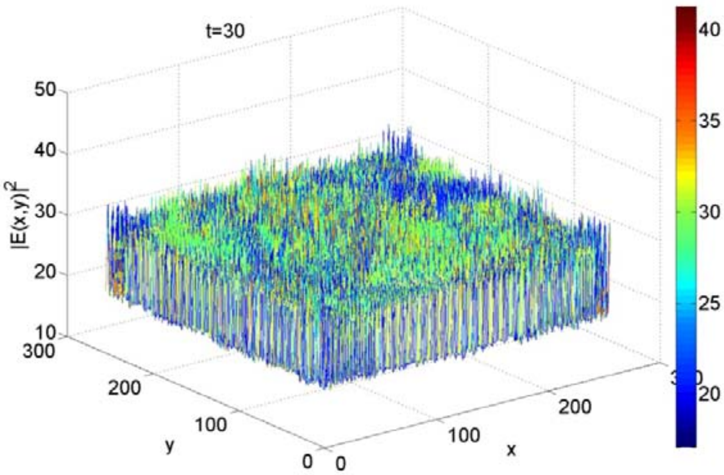

Fig. 6. (Color online) Three-dimensional view of $\left|E(30,x,y)\right|^2$, exhibiting the single-wavelength turbulence structure.

IV. Discussion

In this paper, we found that in a dispersive, diffusive, and dissipative nonlinear system governed by a GNSE an initial large amplitude localized pulse can suffer modulational instability and collapse into the smallest-scale modes. If the total gain and loss of the energy can become balanced, the system can attain a nearly homogeneous turbulent state in the physical space by decaying into modes of predominantly the same wavelength (instead of inverse-cascading to a full spectrum, as normally expected of collapse). The one-step cascade is followed by very slow secondary condensation of the same-$|k|$ modes to that of eight phase angles. That is, the long-time turbulence state is a phase-space condensate consisting of modes of a predominant wavelength concentrated at the eight phase angles, instead of a full and homogeneous spectrum. This phenomenon can be partly attributed to the fact that at long times the energy gain and loss by each mode can balance locally, although the latter does not necessarily lead to it (as in the cases studied in Refs. 14 and 15). The mode decay and redistribution scenario after the collapse resembles that of what might take place from three-mode coupling in the wave kinetic theory for weak turbulence [24]. For example, the final eight dominant modes at $\pi/12$, $5\pi/12$ and their counterparts in the other three quadrants of the ***k*** space can be attributed to three-wave decays from the four largest ***k*** modes (at the four corners of the phase space) produced by the collapse. This is then followed by couplings of three modes having the same wavelength, which eventually redistributes the energy into 8 dominant modes (thus the $5\pi/12-\pi/12=\pi/3$ angle between each pair of modes). However, the wave kinetic theory is based on small amplitude oscillations, energy and momentum conservation, and nearly homogeneous turbulence spectrum, and the rough explanation here does not account for the intermediate stages found in the simulations. Accordingly, a more rigorous theory of the condensation phenomenon here still remains to be found. On the other hand, our results can be directly applied to considering condensation in Langmuir wave turbulence [1-4,9], BEC, and laser filamentation [25], as well as other phenomena governed by the GNSE and the generalized nonlinear reaction-diffusion equations in material science, fluid dynamics, atomic and plasma physics, biology, and other areas [4-6, 10-12, 18-20, 23].

It should be emphasized that the evolution (such as the path, speed, and long-time behavior) of the initial pulse depends sensitively on the parameters of the GNSE, the boundary conditions, and the initial pulse (or other perturbations). On the other hand, for the problem considered, namely eventually the fluctuation energy becomes much larger than that of the external potential, the profile of the latter is not crucial. Furthermore, since the total energy here can become constant after the collapse, one can in principle [17] also expect the appearance of self-organized regular (nonturbulent) or coexisting turbulent/regular structures in the real space $E(t,x,y)$, a state still awaiting discovery [26].

Acknowledgments

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[25] C. Sun, S. Jia, C. Barsi, S. Rica, A. Picozzi, and J. W. Fleischer, Nature Phys. **8**, 470 (2012), especially Fig. 2 therein.

[26] It may be of interest to mention that if we interchange the signs of p_i and q_i of the present problem (so that there is now viscous damping and nonlinear growth), the same initial pulse can attain after a short time a stationary state that is completely regular, without collapse and turbulence. However, that is because the pulse becomes modulationally stable after some self adjustment.