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Collision of microswimmers in viscous fluid

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We investigate effects of boundary conditions on the surface of self-propelled spherical swimmers moving in a viscous fluid with low Reynolds number. We first show that collisions between the swimmers are impossible under the commonly used no-slip conditions. Next we demonstrate that collisions do occur if the more general Navier boundary conditions, allowing for a finite slip on the surface that produces drag, are imposed on the boundary of swimmers. The presence of a small inertia for each swimmer does not influence whether collisions occur between swimmers.

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I. INTRODUCTION

Suspensions of autonomously moving microscopic swimmers, both biological and synthetic, continue to attract enormous attention of broad scientific community. A number of intriguing phenomena was reported in these suspensions, from the onset of collective motion to viscosity reduction and increase of self-diffusivity etc, see e.g. [1–10]. An accurate representation of swimmer-swimmer interactions has proved to be essential for reproducing physically observed phenomena [11–17]. One aspect of swimmer-swimmer interactions that is clearly observed in experiments are collisions between bacteria or bacteria and walls [9, 18, 19]. The primary objective of this study is to identify what features of the model allow for the collisions between the swimmers in finite time. On the other hand, no collisions occur in finite time in the commonly accepted models of passive smooth bodies in a Stokesian fluid, where no-slip boundary conditions are imposed on the surface of the body. This is the well-known *no-collision paradox*, see [20].

Here we show that the no-collision paradox is also valid for self-propelled spheres. However, collisions between self-propelled swimmers occur in finite time if instead of no-slip conditions we apply a more general Navier boundary conditions admitting finite slip on the surface of the swimmer. In the same manner the results for the collision time, both for no-slip boundary conditions and Navier boundary conditions, remain true, if the inertia of the spheres is taken into account. The corresponding slip length β in the Navier condition is associated with surface roughness, which is on nanometer scale for smooth objects, and, therefore, does not play a significant role in the dynamics of macroscopic objects. However, even very small surface roughness become critical when interactions of micron-size objects, such as bacteria are considered: body of many bacteria can be considered rough on submicron scale. For example, many bacteria are covered by multiple non-flagellar thin protein filaments (pili), having the diameter of 2-10 nm. The pili play important role in the process of bacterial conjugation and DNA transfer, may extend from the bacterial body to 10 μm [21, 22]. The pili length varies from bacteria to bacteria, and may also vary from colony to colony of the same bacteria depending on the growth conditions. For *Bacillus subtilis* used in experiments in [2–4] no long pili typically are observed, suggesting that a rough estimate for β could be of the order of 100-200 nm. Another factor contributing to β is that *Bacillus subtilis* has multiple flagella distributed over the body. Even when the flagella are bundled, they make the surface of the bacterial body somewhat non-smooth, giving roughly the same estimate for β. The goal of this paper is to illustrate that natural (Navier) boundary conditions in the fluid/solid model properly describe collisions of swimmers observed in experiments. Therefore, we consider, for simplicity, a perfect coaxial configuration of swimmers where collisions occur with the highest probability. While this is an idealized case, the qualitative conclusions apply to swimmers in close proximity where the result would be essentially independent from the angle.

We briefly survey facts related to collisions of passive spheres and the no-collision paradox. First, consider two spheres moving along the same axis due to a constant external force of magnitude $f_{\rm ext}$ pushing them toward each other. Let $h(t)$ be the half-distance between the spheres. In the following we neglect the inertia of the spheres. The force balance is

$$-F_{\rm drag} + f_{\rm ext} = 0, \tag{1}$$

where $F_{\rm drag}$ is a magnitude of the drag force of the fluid. For low Reynolds number the drag force $F_{\rm drag}$ depends linearly on the speed of the spheres $-h'(t)$: $F_{\rm drag} = -\kappa_{\rm pass} h'(t)$, where $\kappa_{\rm pass}$ is the drag coefficient. If the boundary conditions on the sphere's surface are no-slip, which are the most common for modeling rigid impenetrable body motion in fluid, then

$$\kappa_{\rm pass} \sim 1/h \tag{2}$$

for small h, see [23]. This relation, in particular, implies that $f_{\rm ext} = F_{\rm drag} < -Ch'(t)/h(t)$, $C > 0$, resulting in

$h(t) > h(0)\exp(-Cf_{\mathrm{ext}}t)$. The positive constant C does not depend on h. Thus, the half-distance $h(t)$ tends to zero in the limit of $t \to \infty$ but never vanishes completely. Therefore no collisions are possible in finite time. This is referred to as the *no-collision paradox*.

The formal absence of collisions in the model shows that it is inadequate for suspensions where experimental evidence often shows collisions between the particles. It turns out, that if one takes into account the tangent slip on the surface of a sphere, i.e., replaces the no-slip boundary conditions by the Navier boundary conditions with a finite slip length β, then the no-collision paradox does not hold. The motion of rigid bodies with a rough no-slip surface is effectively described by the same Navier boundary conditions (see, e.g., [24]), where β plays the role of the mean surface roughness. The Navier boundary conditions allow for modeling collisions matching experiments without taking into account non-hydrodynamic short-range interactions such as Van der Waals forces or electrostatic forces due to charging of the bacterial body. In good electrolytes, like bacterial growth medium, these forces are screened on the scale of the Debye length which is of the order of a few nanometers [25].

In Ref. [26] for the case of two passive spheres subject to Navier boundary conditions the authors obtained

$$\kappa_{\mathrm{pass}} \sim \ln(1/h) \tag{3}$$

for small separation distances h. In this case Eq. (1) reduces to the inequality:

$$Ch(t)(\ln h(t) - 1) - f_{\mathrm{ext}}t > Ch(0)(\ln h(0) - 1),$$

which predicts a collision, $h(T) = 0$, for some finite time T. The slip length β is small (typically from a few nanometers to a few hundreds of nanometers, depending on material properties of the sphere's surface).

II. MODEL

Here we consider two swimmers, each modeled by a sphere (representing the body) and the propulsion force (representing the action of flagella). We write the model in a non-dimensional form given by the choice of the typical length a being half a diameter of a swimmer's body, typical swimming velocity V and viscosity of water μ. The force and pressure scales are taken to be $\mu V a$ and $\mu V a^{-1}$, respectively.

In the nondimensional form each swimmer is modeled by a sphere of unit radius with a propulsion force $f_p\mathbf{d}^i$ applied to fluid at the point $\mathbf{x}_p^i$ distance $(1+\lambda)$ behind the center $\mathbf{x}_c^i$ of the swimmer (see e.g. Fig. 1). Here $\mathbf{d}^i = (\mathbf{x}_c^i - \mathbf{x}_p^i)/|\mathbf{x}_c^i - \mathbf{x}_p^i|$ is the orientation of the swimmer.

Let $h(t)$ be the half-distance between the swimmers. Then the velocity of the ith swimmer is $\mathbf{v}^i = -h'(t)\mathbf{d}^i$ and the acceleration of the ith swimmer is $\mathbf{a}^i = -h''(t)\mathbf{d}^i$.

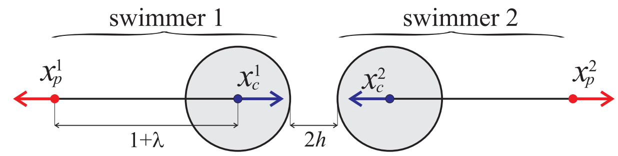

FIG. 1: Schematic illustration of two swimmers.

Denote $U = \mathbf{v}^i \cdot \mathbf{d}^i = -h'(t)$, $i = 1, 2$. Due to reflection symmetry U does not depend on i.

The motion of the fluid at low Reynolds number (with fluid viscosity $\mu = 1$) is described by the Stokes equation

$$\begin{cases} -\triangle \mathbf{u} + \nabla p &= \sum_i f_p \mathbf{d}^i \delta(\mathbf{x} - \mathbf{x}_p^i) \\ \mathrm{div}(\mathbf{u}) &= 0 \end{cases} \quad \text{in } \Omega_F \tag{4}$$

subject to the Navier boundary conditions

$$\begin{cases} (\mathbf{u} - U\mathbf{d}^i) \cdot \mathbf{n} &= 0 \\ (\mathbf{u} - U\mathbf{d}^i) \times \mathbf{n} &= -\beta\sigma(\mathbf{u}, p)\mathbf{n} \times \mathbf{n}, \end{cases} \quad \text{on } \partial B^i, \tag{5}$$

where $\sigma(\mathbf{u}, p) = \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) - pI$ is the stress tensor. The slip length β is non-negative. Taking $\beta = 0$, one recovers the no-slip boundary conditions. The self-propulsion is enforced through the force balance

$$m\mathbf{a}^i = \mathbf{F}^i_{\mathrm{drag}} + f_p \mathbf{d}^i, \ i = 1, 2, \tag{6}$$

It is known that due to the torque from a counter-clockwise rotation of the flagella the body of bacterium rotates clockwise. However, we ignore this effects here.

The force balance (6) can be written in the scalar form:

$$mh''(t) - F_{\mathrm{drag}} + f_p = 0, \tag{7}$$

where $F_{\mathrm{drag}} = -\mathbf{F}^i_{\mathrm{drag}} \cdot \mathbf{d}^i$ does not depend on $i = 1, 2$. The inertial forces are typically neglected:

$$F_{\mathrm{drag}} = f_p. \tag{8}$$

To analyze (7) or (8) we need to evaluate F_{drag}. Due to the linearity of the problem (4-6) we have

$$F_{\mathrm{drag}} = \kappa_{\mathrm{pass}} U + \kappa_{\mathrm{prop}} f_p, \tag{9}$$

where the coefficient κ_{pass} is equal to the drag force of the fluid onto spheres moving toward each other with unit speed and the coefficient κ_{prop} is equal to the drag force exerted by the fluid on stationary swimmers with unit propulsion force. This means that $\kappa_{\mathrm{pass}} = F_{\mathrm{drag}}$ evaluated for the solution of the problem (4-6) with $U = 1$ and $f_p = 0$, and $\kappa_{\mathrm{prop}} = F_{\mathrm{drag}}$ evaluated for the solution of the problem (4-6) with $U = 0$ and $f_p = 1$.

The speed of a massless ($m = 0$) swimmer is a function of the distance and, following (9), can be written as:

$$U = f_p(1 - \kappa_{\mathrm{prop}})/\kappa_{\mathrm{pass}}. \tag{10}$$

To solve (4) and (5), we need to evaluate the coefficients κ_{pass} and κ_{prop}. We will do this first in the case of a single swimmer. We have two reasons to do so. Firstly, the comparison between the velocities of a single swimmer and the a swimmer in the presence of another swimmers will characterize the role of interactions. Secondly, this will allow us to use the coefficient κ_{prop} computed for a single swimmer as an estimate for the same coefficient for two interacting swimmers.

III. ONE SWIMMER

Let $\mathbf{w}$ and $p_{\mathbf{w}}$ be, respectively, the flow velocity and the pressure generated by a sphere moving in the direction of the unit vector $\mathbf{d}$ with unit speed. To evaluate κ_{pass} we use the exact expression for the function $\mathbf{w}$ (see, e.g., [27]): $\mathbf{w}(\mathbf{x}) = (-B + 1/2\, A\Delta)\, \mathcal{G}(\mathbf{x})\mathbf{d}$, where $\mathcal{G}$ is a fundamental solution of the Stokes equation,

$$A = \frac{1}{4} \cdot \frac{1}{1+3\beta}, \quad \text{and} \quad B = -\frac{3}{4} \cdot \frac{1+2\beta}{1+3\beta}.$$

Hence, we can calculate κ_{pass} using the Gauss formula:

$$\kappa_{\text{pass}} \;=\; \int\limits_{B} [\nabla \cdot \sigma(\mathbf{w}, p_{\mathbf{w}})]\, d\mathbf{x} \cdot \mathbf{d} = -8\pi B = 6\pi \frac{1+2\beta}{1+3\beta}.$$

To evaluate κ_{prop} we use:

$$\kappa_{\text{prop}} = \mathbf{w}|_{\mathbf{x}=-(1+\lambda)\mathbf{d}} \cdot \mathbf{d}. \tag{11}$$

One can derive this formula by using the Lorentz reciprocal theorem (see Appendix A). Thus,

$$\kappa_{\text{prop}} = \frac{3(1+2\beta)(1+\lambda)^2 - 1}{2(1+3\beta)(1+\lambda)^3}. \tag{12}$$

The coefficient κ_{prop} depends on the “passive” parameter β and the “active” parameters f_p and λ, while κ_{pass} depends only on the “passive” parameter β. In the case of a single swimmer, (10) takes the form:

$$\frac{U}{f_p} = \frac{(1-\kappa_{\text{prop}})}{\kappa_{\text{pass}}} = \frac{6\beta\lambda(1+\lambda)^2 + \lambda^2(3+2\lambda)}{12\pi(1+2\beta)(1+\lambda)^3}. \tag{13}$$

Sending $\lambda \to \infty$, *i.e.,* taking isolated sphere, we obtain the relation between drag force and velocity for a moving ball

$$\lim_{\lambda\to\infty} U(\lambda, \beta) = \frac{1}{6\pi} \frac{(1+3\beta)}{(1+2\beta)} f_p,$$

which gives the Stokes drag law, $U = f_p/6\pi$, for $\beta = 0$.

IV. TWO COAXIAL SWIMMERS

Let $\mathbf{w}$ and $p_{\mathbf{w}}$ be the flow and the pressure, respectively, generated by two spheres moving toward each other with unit speed and without rotation. The expression for $\mathbf{w}$ can be found in [28]. The expression for κ_{pass} is of the form:

$$\kappa_{\text{pass}} = \frac{2\sqrt{2}\pi}{c} \sum_{n=1}^{+\infty} (b_n + d_n), \tag{14}$$

where b_n and d_n are defined in [28] (see also Appendix B). For no-slip conditions (i.e. $\beta = 0$) b_n and d_n can be computed explicitly. For the Navier conditions ($\beta > 0$) b_n and d_n are defined through a recurrence relation subject to $b_n, d_n \to 0$ as $n \to +\infty$, and need to be evaluated numerically.

To evaluate κ_{prop} we use the Lorentz reciprocal theorem (see Appendix A) and obtain (similar to (11)):

$$\kappa_{\text{prop}} = \mathbf{w}|_{\mathbf{x}=-(1+\lambda+h)\mathbf{d}} \cdot \mathbf{d}. \tag{15}$$

This yields the following expression for κ_{prop}:

$$\frac{\sinh\frac{\zeta}{2}}{c^2} \sum_{n=1}^{+\infty} \left[b_n \sinh\left(n - \tfrac{1}{2}\right)\zeta + d_n \cosh\left(n + \tfrac{3}{2}\right)\zeta \right], \tag{16}$$

where $c = \sinh\alpha$, $\alpha = \ln(1 + h + \sqrt{h}\sqrt{2+h})$ and

$$\zeta = \ln \frac{2 + \lambda + h + \sqrt{h}\sqrt{2+h}}{2 + \lambda + h - \sqrt{h}\sqrt{2+h}}.$$

In contrast with the case of a single swimmer the coefficients κ_{pass} and κ_{prop} also depend on h in the case of two interacting swimmers. To obtain the expression for U one needs to substitute (14) and (16) into (10).

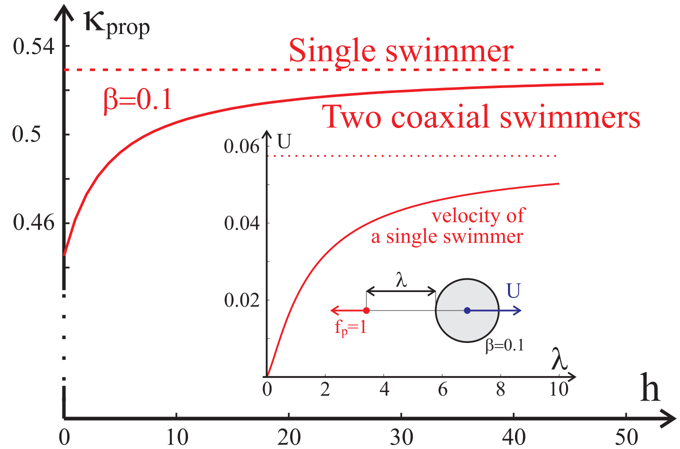

FIG. 2: Dependence of the drag coefficient κ_{prop} of the fluid on a stationary sphere from the propulsion point-force as a function of semi-distance h between the swimmers. The parameter κ_{prop} for a single swimmer (dashed line) does not depend on h and is always above the same parameter for two coaxial swimmers (solid line). (Inset) Swimming speed U vs tail length λ.

Now we demonstrate *finite-time collisions between two swimmers*. The velocity of the i-th swimmer is $U\mathbf{d}^i = -h'(t)\mathbf{d}^i$. Rewrite the force balance (8)

$$F_{\text{drag}} - f_p = -\kappa_{\text{pass}}h'(t) - f_p(1-\kappa_{\text{prop}}) = 0. \qquad (17)$$

In both situations, $\beta = 0$ and $\beta > 0$, the parameter κ_{prop} is a bounded quantity with respect to $h \to 0$, i.e., $\kappa_{\text{prop}} = O(1)$. Thus, if $\beta = 0$, then the balance equation (17) reduces to $\ln(h(t)/h(0)) > -Ct$, where C does not depend on $h(t) > 0$. Here we integrated (17) with respect to the time variable over interval $[0,t]$. The obtained inequality implies that $h(t) > h(0)e^{-Ct}$, which prevents collisions (see Appendix D for rigorous proof).

We now consider the case of Navier boundary conditions. Note that *the parameter κ_{prop} is smaller than the same parameter κ_{prop} calculated for a single swimmer*, see (12). This statement is illustrated by the numerical plot in Figure 2 and it can be explained as follows. Recall that κ_{prop} is the drag force exerted on a swimmer whose velocity is zero and generated by unit propulsion. If there are two swimmers in the fluid moving toward each other, the propulsion of the second swimmer has the opposite direction with respect to the direction of the first swimmer and, therefore, decreases the total drag force exerted on the first swimmer. Thus, using (12) we obtain

$$\kappa_{\text{prop}} < \frac{3(1+2\beta)(1+\lambda)^2 - 1}{2(1+3\beta)(1+\lambda)^3}.$$

Hence, for any $\lambda > 0$ there exists $\delta > 0$, such that $\kappa_{\text{prop}} < 1-\delta$. Therefore, collision happens in finite time following the arguments used for passive spheres after (3) with the force $f_{\text{ext}} = f_p$. The time before collision is of the order $\ln(1/\beta)$, also see Appendix C. A naive way of estimating the collision time is the time it takes a single free-moving swimmer to cover the half-distance between two swimmers. We define the *collision delay time* (due to interactions) as the discrepancy between this naive collision time, where swimmer-swimmer interactions are neglected, and the true collision time where interactions are accounted for in the limit of large initial separations between swimmers. The collision delay time can be seen on Fig. 3 as the difference in times graphs intersect the t-axis. For $\beta = 0$, the solid blue line gets infinitely close to the t-axis, but never intersects it. Therefore, the collisions delay time is infinite.

For the microswimmers with the parameters of *Bacillus subtillis* the collision delay time is on the order of one decisecond. Here we assumed the radius to be 1μm, the slip parameter $\beta = 0.2\mu$m, the flagella length $\lambda = 10\mu$m and the typical free-swimming speed of 20μm/s.

It is also useful to estimate the dependence of the collision delay time on the slip parameter β. For two swimmers of unit radius each, unit free-swimming speed and tail length $\lambda = 10$ the collision delay time was obtained numerically for $\beta = \{1,\dots,100\} \times 10^{-4}$. We also used Eureqa program [29] to find an interpolation to the numerical data in a simple analytical form: $T_{\text{col.del.}} \approx \frac{5.8-\ln(\beta)}{3.5}$, see Fig. 4. In the dimensional scales the collision time due to interactions has the form $T_{\text{col.del.}} \approx \frac{a}{U_0}\frac{5.8-\ln(\beta/a)}{3.5} \approx 0.11$ sec.

V. CONCLUSIONS

We considered two features involved in modeling of swimmers: the boundary conditions and body inertia. Our analysis showed that the key feature influencing the possibility of collisions between swimmers is the type of conditions imposed on the boundary of the swimmers. For the no-slip conditions, typically used to model swimming micro-organisms, no collisions are possible in finite time. This result was proved analytically and is analogous to the no-collision paradox for passive bodies. On the other hand, if Navier conditions (slip with friction) are imposed on the surface of the bodies of the swimmers, the collisions are possible in finite time. Our estimate of the collision time for microswimmers with parameters resembling those of *Bacillus subtilis* shows that with Navier boundary conditions the delay due to the interactions in the collision time is small (fraction of a second) and does not qualitatively change the course of collision. This is observed in experiments.

Unlike the boundary conditions, including the body inertia into the model does not influence the possibility of collisions (see Appendix E). For no-slip boundary conditions collisions are neither possible with nor without inertia. For Navier boundary conditions collisions are possible both with and without body inertia.

The results of this paper can be extended to the cases where Navier and/or no-slip conditions are prescribed only on part of the boundary of microswimmer and to

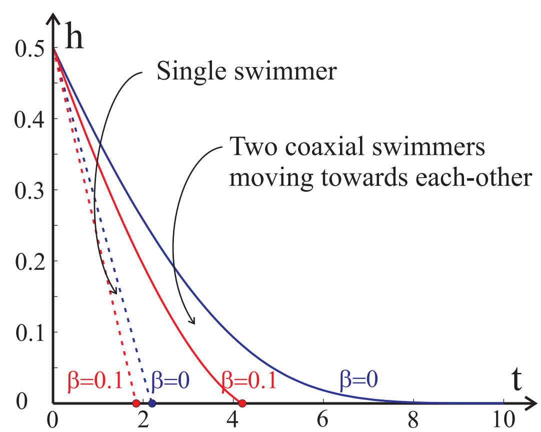

FIG. 3: Distance to the origin, h, as a function of time, t, for a single swimmer and for two coaxial swimmers for no-slip ($\beta = 0$, blue) and Navier ($\beta = 0.1$, red) boundary conditions. Swimmers with Navier conditions are moving faster than swimmers with no-slip conditions, in the presence or absence of another swimmer.

the case of a "squirmer" [30]. When separation distance $2h$ is small, the drag force is concentrated on the front edge of a bacterium. Therefore if Navier conditions are applied here, then the logarithm estimate on $\kappa_{\rm pass}$ (3) remains valid and collisions are possible. On the other hand, if no-slip conditions are applied on the front part of the boundary, collisions are no longer possible. We can also extend the results of the paper to the case of a swimmer approaching a wall. If either the swimmer or the wall does not allow for slip then collision cannot happen. If both the swimmer's and the wall's surface allows for slip, then collision happens in finite time.

The above analysis indicates that the Navier boundary conditions are more appropriate for modeling swimming micro-organisms than the no-slip conditions, while the later are simply the limiting case case of the former.

ACKNOWLEDGEMENTS

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Appendix A: Formula for $\kappa_{\rm prop}$

In this appendix we derive the relation (15). The relation (11) can be derived in the same manner. We use the Lorentz reciprocal theorem (or the 2nd Green formula for Stokes equation; see [23]) which is formulated in the next paragraph.

Let the equations $-\nabla\cdot\sigma(u^1,p^1)=F_1$, $-\nabla\cdot\sigma(u^2,p^2)=F_2$ and $\nabla\cdot u^1=0$, $\nabla\cdot u^2=0$ hold in a domain Ω. Here

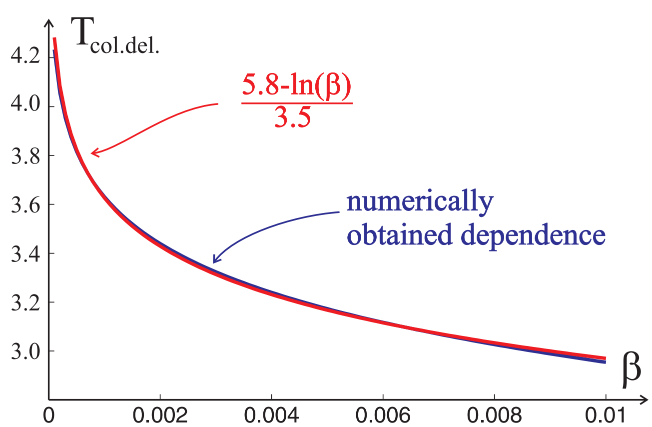

FIG. 4: Dependence of the collision delay time due to interactions $T_{\rm col.del.}$ on the slip parameter β, $\beta\ll 1$, $\lambda=10$ and sphere radius 1.

$\sigma(u,p)=1/2(\nabla u+(\nabla u)^T)-p\mathrm{I}$ is a stress tensor. Then

$$\int_\Omega F_1\cdot u^2+\int_{\partial\Omega}\sigma(u^1,p^1)\mathbf{n}\cdot u^2 = \int_\Omega F_2\cdot u^1+\int_{\partial\Omega}\sigma(u^2,p^2)\mathbf{n}\cdot u^1. \quad (18)$$

We use the formula (18) with $\Omega=\mathbb{R}^3\backslash(B^1\cup B^2)$, $u^1=\mathbf{v}$ and $u^2=\mathbf{w}$:

$$-\int_\Omega\sum_{i=1,2}\mathbf{d}^i\cdot\delta(\mathbf{x}-\mathbf{x}_p^i)\mathbf{w}+\int_{\partial\Omega}\sigma(\mathbf{v},p_\mathbf{v})\mathbf{n}\cdot\mathbf{w} = \int_{\partial\Omega}\sigma(\mathbf{w},p_\mathbf{w})\mathbf{n}\cdot\mathbf{v} \quad (19)$$

Rewrite the first term:

$$-\int_\Omega\sum_{i=1,2}\mathbf{d}^i\cdot\delta(\mathbf{x}-\mathbf{x}_p^i)\mathbf{w}=-\sum_{i=1,2}\mathbf{w}|_{\mathbf{x}=\mathbf{x}_p^i}\mathbf{d}^i. \quad (20)$$

To analyze the boundary terms we need the following equality:

$$\sigma(\mathbf{v},p_\mathbf{v})\mathbf{n}\cdot\mathbf{w}=\sigma(\mathbf{v},p_\mathbf{v})\mathbf{n}\cdot\mathbf{d}^i -\beta(\sigma(\mathbf{v},p_\mathbf{v})\mathbf{n}\times\mathbf{n})\cdot(\sigma(\mathbf{w},p_\mathbf{w})\mathbf{n}\times\mathbf{n}). \quad (21)$$

Analogously,

$$\sigma(\mathbf{w},p_\mathbf{w})\mathbf{n}\cdot\mathbf{v}=-\beta(\sigma(\mathbf{v},p_\mathbf{v})\mathbf{n}\times\mathbf{n})\cdot(\sigma(\mathbf{w},p_\mathbf{w})\mathbf{n}\times\mathbf{n}). \quad (22)$$

Plugging (20),(21),(22) into (19) and using the definition of $\kappa_{\rm prop}$ we get (15).

Appendix B: The exact solution for two spheres with Navier BC

In this appendix we present coefficients b_n and d_n used in (16). The formula (16) and the exact solution for the problem of two spheres with Navier boundary conditions were obtained in [28].

Coefficients d_n solve the following infinite system of linear algebraic equations:

$$4\beta A_n d_{n-1}+(\overline{C}_n+4\beta\tilde{C}_n)d_n+4\beta B_n d_{n+1}=\overline{F}_n+4\beta\tilde{F}_n.$$

The system is supplemented with the following restriction:

$$d_n\to 0.$$

Coefficients A_n, $\overline{C}_n$, $\tilde{C}_n$, B_n, $\overline{F}_n$ and $\tilde{F}_n$ are determined below ($m=n+1/2$):

$$\begin{aligned}A_n &= -(m+1/2)\sinh m\alpha,\\ B_n &= -(m-1/2)\sinh(m+2)\alpha,\\ \tilde{C}_n &= 2m\sinh(m+1)\alpha\cosh\alpha.\end{aligned}$$

$$\overline{C}_n = \frac{2}{\sinh(m-1)\alpha}\left[\sinh 2m\alpha - m\sinh 2\alpha\right].$$

$$\overline{F}_n = -\frac{c^2}{\sqrt{2}}\cdot\frac{m^2-1/4}{m+1}\cdot\frac{2}{\sinh(m-1)\alpha}\times$$
$$\times\left\{me^{-\alpha}\sinh\alpha + e^{-m\alpha}\cosh m\alpha\right\}.$$

$$\tilde{F}_n = \frac{e^{-(m+1)\alpha}(m+3)(m\sinh\alpha + e^{\alpha})}{2m(m+1)(m+2)}$$

Coefficients b_n are evaluated according to the following formula:

$$b_n = G_m - d_m\frac{\sinh(m+1)\alpha}{\sinh(m-1)\alpha},$$

where

$$G_m = \frac{c^2 e^{-m\alpha}}{2\sqrt{2}}\frac{m^2-1/4}{m^2-1}\left[\frac{(m+1)e^{\alpha}-(m-1)e^{-\alpha}}{\sinh(m-1)\alpha}\right].$$

Appendix C: The collision time.

In this appendix we provide arguments how to obtain estimate on the collision time.

The result for k_{pass} from [26] may be rewritten in the following form:

$$h<\beta: \quad \kappa_{\text{pass}} = \frac{1}{c_0\beta}\ln 1/h + O(1/\beta) + O(\ln 1/h),$$
$$h>\beta: \quad \kappa_{\text{pass}} \le c/h$$

where c_0 and c from $O(1)$, bounded uniformly with repect to both β and h.

Define the collision time:

$$t_{coll} = \int_0^h \frac{dh}{U(h)},$$

where $U(h) = -h'(t)$. Assume that the balance equaton (1) holds. Then

$$t_{coll} = \int_0^{\beta}\frac{dh}{U(h)} + \int_{\beta}^{1}\frac{dh}{U(h)} < c_1\ln(1/\beta) + C_1.$$

where c_1 and C_1 are some positive constant independent from h and β.

Appendix D: Rigorous proof of no collisions for no-slip BC

In this appendix we give the rigorous proof that no-slip boundary conditions on the swimmers' surface imply no collision for finite time.

The proof is based on the Lorentz reciprocal theorem (see Appendix A) and the estimate for κ_{pass} (2), which implies the same result if the swimmers are replaced by passive spheres.

First, we formulate the problem and the result.

The problem for two swimmers approaching each other with speed U is considered:

$$\begin{cases} \Delta u - \nabla p = \sum\limits_{i=1,2}\delta(\mathbf{x}-\mathbf{x}_i^p)f_p\mathbf{d}^i, \\ \nabla\cdot u = 0, \\ u = U\mathbf{d}^i, \\ \mathcal{F}_i(u,p) = f_p\mathbf{d}^i. \end{cases} \tag{23}$$

Here we used the following notation:

$$\mathcal{F}_i(u,p) = \int_{\partial B^i}\sigma(u,p)\mathbf{n}.$$

The result is written in the following paragraph.

There exist positive constants $C_1, C_2 > 0$ such that

$$h(t) > C_1 e^{-C_2 t} \tag{24}$$

for all $t > 0$.

The rest of the section is devoted to the proof of this statement. First, we will show that the result follows from (29). Then we prove (29).

Let u satisfy (23). Then $u = v + w$ where v and w satisfy the following systems of equations:

$$\text{I.:}\begin{cases} \Delta v - \nabla q = 0, \\ \nabla\cdot v = 0, \\ v = V\mathbf{d}^i, \\ \mathcal{F}_i(v,q) = f_p\mathbf{d}^i, \end{cases} \quad \text{II.:}\begin{cases} \Delta w - \nabla\pi = \sum\limits_{i=1,2}\delta(\mathbf{x}-\mathbf{x}_i^p)f_p\mathbf{d}^i, \\ \nabla\cdot w = 0, \\ w = W\mathbf{d}^i, \\ \mathcal{F}_i(w,\pi) = 0. \end{cases}$$

In both systems, I and II, equalities in the first and the second line are satisfied in $\Omega_h = \mathbb{R}^3\backslash(B^1\cap B^2)$, equalities in the third line are satisfied on ∂B^i for all $i = 1,2$, equalities in the fourth line are satisfied for all $i = 1,2$. Note that if f_p is given we can find v and q from the first, the second and the fourth equalities of system I. After that we can find $V \in \mathbb{R}$ from the third equality. Similarly, for system II. We note that

$$U = V + W. \tag{25}$$

One can see that $\mathcal{F}_i(v,q) = V\kappa_{\text{pass}}$. Then in view of (2) we conclude that there exists a positive constant $C > 0$ such that

$$V < Cf_p h. \tag{26}$$

Assume that

$$W < 0. \tag{27}$$

In view of (26) and (27) we obtain that $-h'(t) = V + W < Cf_p h(t)$. Substituting it to (25) we have

$$h'(t) > -Cf_p h(t),$$

which implies that $h(t) > h(0)e^{-Cf_p t}$ and, thus, separation distance $2h(t)$ will never vanish.

Therefore, the result follows if prove that $W<0$.

We use the Lorentz reciprocal theorem. Take $\Omega=\Omega\backslash(B^1\cap B^2)$, $u^2=w$, $p^2=\pi$ and $F_2=-\sum_{i=1,2}\delta(\mathbf{x}-\mathbf{x}_p^i)f_p\mathbf{d}^i$. Let u^1 be such function that there holds $\Delta u^1-\nabla p^1=0$ (in other words, $F_1=0$), and u^1 satisfies the following boundary conditions:

$$u^1=W\mathbf{d}^i \text{ on } \partial B_i,\ \ i=1,2.$$

From the Lorentz reciprocal theorem we get:

$$W\sum_{i=1,2}\mathbf{d}^i\cdot\int_{\partial B_i}\sigma(u^1,p^1)\mathbf{n}=f_p\sum_{i=1,2}\mathbf{d}^i\cdot u^1(\mathbf{x}_p^i).$$

We get the following equality:

$$W=-\frac{f_p}{F_1(u^1,p^1)}u_z^1(\mathbf{x}_p^1). \tag{28}$$

Thus, we need to prove that

$$u_z^1(\mathbf{x}_p^1)>0. \tag{29}$$

We recall that u^1 satisfies the following system

$$\begin{cases}\Delta u^1-\nabla p^1=0,\ \text{in } \Omega_h\\ \nabla\cdot u^1=0,\ \text{in } \Omega_h\end{cases}$$
$$u^1=W\mathbf{d}^i,\ \text{on } \partial B_i,\ \ i=1,2.$$

The proof of (29) is divided by three parts. First, the result from [31, p.247, §3] is used to write the exact solution of the system above. Second, we calculate u^1 at point $\mathbf{x}_p^1$ and (29) is obtained in the third part.

Exact solution. Introduce cylindrical coordinates $\mathbf{x}=(x,y,z)=(\rho\cos\phi,\rho\sin\phi,z)$. Denote $e_\rho=(\cos\phi,\sin\phi,0)$ and $e_z=(0,0,1)$. Due to axial symmetry of the problem we have $u^1=u_\rho e_r+u_z e_z$ where scalar functions u_ρ and u_z dont depend on ϕ. Following [31],

$$u_z(z,\rho)=-\frac{1}{\rho}\frac{\partial}{\partial\rho}\psi(z,\rho),\quad u_\rho(z,\rho)=\frac{1}{\rho}\frac{\partial}{\partial z}\psi(z,\rho).$$

In order to write function ψ we introduce the bipolar coordinates $\zeta\in[0,+\infty)$, $\eta\in[0,\pi]$ (we need the case $z>0$ only)

$$z=c\frac{\sinh\zeta}{\cosh\zeta-\cos\eta},\quad \rho=c\frac{\sin\eta}{\cosh\zeta-\cos\eta}, \tag{30}$$

where $c=\sinh\alpha$ and α is such positive number that $\cosh\alpha=1+h$. Note that due to the equality

$$(z-c\coth\zeta)^2+\rho^2=(c\,\mathrm{cosech}\zeta)^2$$

one can easily verify that surface $\{\zeta=\alpha\}$ is sphere ∂B_1. Also we will need the formula:

$$\zeta+i\eta=\ln\left(\frac{\rho+i(z+c)}{\rho+i(z-c)}\right). \tag{31}$$

The function ψ is defined as follows

$$\psi(\zeta,\eta)=(\cosh\zeta-\cos\eta)^{-3/2}\sum_{n=0}^{\infty}U_n(\zeta)C_{n+1}^{-1/2}(\cos\eta), \tag{32}$$

where

$$\begin{aligned}U_n(\zeta)&=b_n\sinh(n-\frac{1}{2})\zeta+d_n\cosh(n+\frac{3}{2})\zeta,\\ C_{n+1}^{-1/2}(x)&=\frac{P_{n-1}(x)-P_{n+1}(x)}{2n+1}.\end{aligned}$$

Here $P_n(x)$ are Legendre polynomials

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

and the coefficients b_n and d_n are given by the following formulas

$$\begin{aligned}b_n&=WR^2\sinh^2\alpha\frac{n(n+1)}{\sqrt{2}(2n-1)}\times\\ &\times\left[\frac{4\cosh^2(n+\frac{1}{2})\alpha+2(2n+1)\sinh^2\alpha}{2\sinh(2n+1)\alpha-(2n+1)\sinh 2\alpha}-1\right],\\ d_n&=WR^2\sinh^2\alpha\frac{n(n+1)}{\sqrt{2}(2n+3)}\times\\ &\times\left[1-\frac{4\cosh^2(n+\frac{1}{2})\alpha-2(2n+1)\sinh^2\alpha}{2\sinh(2n+1)\alpha-(2n+1)\sinh 2\alpha}\right].\end{aligned}$$

The evaluation of $u^1(\mathbf{x}_p^1)$. We need to calculate u^1 when $z_0=1+h+\xi$ and $\rho\to 0+$. In bipolar coordinates it corresponds to

$$\zeta_0=\ln\frac{z_0+c}{z_0-c}\quad\text{and}\quad \eta_0=0.$$

Then using (31) (to obtain $\frac{\partial\zeta}{\partial\rho}$ and $\frac{\partial\eta}{\partial\rho}$), $P_n(1)=1$, $P_n'(1)=n(n+1)/2$ and the equality

$$\frac{\partial\psi}{\rho\partial\rho}=\frac{\partial\zeta}{\rho\partial\rho}\frac{\partial\psi}{\partial\zeta}+\frac{\partial\eta}{\partial\rho}\frac{\partial\psi}{\rho\partial\eta}$$

we obtain

$$\begin{aligned}u_z(z_0,0)&=-\lim_{\rho\to 0+}\frac{1}{\rho}\frac{\partial}{\partial\rho}\psi(z_0,\rho)\\ &=\frac{(\cosh\zeta_0-1)^{1/2}}{c^2}\sum_{n=0}^{\infty}U_n(\zeta_0).\end{aligned} \tag{33}$$

The inequality (29) follows, if we prove that $U_n(\zeta_0)>0$ for all $n\in\mathbb{N}$.

Positivity of $U_n(\zeta_0)$. The relation $U_n(\zeta_0)>0$ is equivalent to

$$\frac{(2n+3)\sinh(n-\frac{1}{2})\zeta_0}{(2n-1)\sinh(n+\frac{3}{2})\zeta_0}>\frac{K_0-K_1}{K_0-K_2} \tag{34}$$

where

$$\begin{aligned} K_0 &= 2\sinh(2n+1)\alpha - (2n+1)\sinh 2\alpha \\ K_1 &= 4\cosh^2(n+\frac{1}{2})\alpha - 2(2n+1)\sinh^2\alpha \\ K_2 &= 4\cosh^2(n+\frac{1}{2})\alpha + 2(2n+1)\sinh^2\alpha \end{aligned}$$

Observe that the left hand side of (34) is a decreasing function of ζ_0 and we may assume that $\zeta_0 < \alpha$ (since $\zeta_0 \to 0$ as $h \to 0$). Thus, the left hand side in (34) can be changed by

$$\frac{(2n+3)\sinh(n-\frac{1}{2})\alpha}{(2n-1)\sinh(n+\frac{3}{2})\alpha}.$$

Inequality (34) follows if one appies the following inequality with $m = n + \frac{1}{2}$:

$$\frac{(m+1)\sinh(m-1)\alpha}{(m-1)\sinh(m+1)\alpha} > \frac{(\sinh 2m\alpha - m\sinh 2\alpha) - 2\left(\cosh^2 m\alpha - m\sinh^2\alpha\right)}{(\sinh 2m\alpha - m\sinh 2\alpha) - 2\left(\cosh^2 m\alpha + m\sinh^2\alpha\right)}.$$

Rewrite the inequality above:

$$\frac{(m+1)\sinh(m-1)\alpha}{(m-1)\sinh(m+1)\alpha} > \frac{m\sinh 2\alpha + 2\cosh^2 m\alpha - \sinh 2m\alpha - 2m\sinh^2\alpha}{m\sinh 2\alpha + 2\cosh^2 m\alpha - \sinh 2m\alpha + 2m\sinh^2\alpha}. \tag{35}$$

First, let us simplify the right hand side:

$$m\sinh 2\alpha + 2\cosh^2 m\alpha - \sinh 2m\alpha \pm 2m\sinh^2\alpha = 2[m\sinh\alpha \cdot e^{\pm\alpha} + \cosh m\alpha \cdot e^{-m\alpha}].$$

Thus, the right hand side of (35) can be rewritten as

$$\frac{m\sinh\alpha \cdot e^{-\alpha} + \cosh m\alpha \cdot e^{-m\alpha}}{m\sinh\alpha \cdot e^{\alpha} + \cosh m\alpha \cdot e^{-m\alpha}}.$$

Next, let us note that (35) is equivalent to

$$I_{\text{num}} - I_{\text{den}} < 0, \tag{36}$$

where

$$I_{\text{num}} = (m-1)\sinh(m+1)\alpha \cdot (m\sinh\alpha \cdot e^{-\alpha} + \cosh m\alpha \cdot e^{-m\alpha})$$

and

$$I_{\text{den}} = (m+1)\sinh(m-1)\alpha \cdot (m\sinh\alpha \cdot e^{\alpha} + \cosh m\alpha \cdot e^{-m\alpha}).$$

One can check that

$$4(I_{\text{num}} - I_{\text{den}}) = = 4e^{-m\alpha}(\cosh\alpha + m\sinh\alpha)(-\sinh 2m\alpha + m\sinh 2\alpha).$$

Due to the estimate

$$-\sinh 2m\alpha + m\sinh 2\alpha < 0,$$

(36) holds.

Thus, (29) holds and the proof of the no collision result is complete.

Appendix E: Collisions with inertia

In this appendix we prove the result about collisions, if inertia of swimmers is taken into account. Namely, collisions do not happen for finite time, if no-slip BC are imposed. Collisions do happen, if Navier BC are imposed, and the collision time is of the order $\ln(1/\beta)$.

Consider a problem for two swimmer with inertia taken into the account, i.e. the equation of motion given by

$$mh''(t) + \kappa_{\text{pass}} h'(t) + f_p(1 - \kappa_{\text{prop}}) = 0, \tag{37}$$

with initial conditons $h(0) = h_0$ and $h'(0) = h_1$. Recall that κ_{pass} and κ_{prop} admit the following relations

$$\kappa_{\text{pass}} \sim \begin{cases} 1/h, & \text{for no-slip BC,} \\ 1/\beta \ln 1/h, & \text{for Navier BC,} \end{cases} \quad 0 < \kappa_{\text{prop}} < 1 - \delta.$$

No-slip BC. The equation (37) implies that

$$mh''(t) + Ch'(t)/h(t) + f_p > 0.$$

Here we used that $\kappa_{\text{pass}} \sim 1/h$ and $\kappa_{\text{prop}} > 0$. This inequality and $h'(t) < 0$ imply that $C\ln h(t) + f_p t > mh_1 + C\ln h_0$. Thus,

$$C\ln\frac{1}{h(t)} < f_p t - mh_1 - C\ln h_0. \tag{38}$$

The RHS of (38) can not be infinite in a finite time. Therefore, no collisions are possible.

Navier BC. We assume $h_0 < \beta$ and $h_1 < 0$, since it takes time of the order 1 to cover distance to point $h = \beta$. The equation (37) implies that

$$mh''(t) + \frac{h'(t)}{\beta}\ln\frac{1}{h(t)} + f_p\delta < 0.$$

Here we used that $\kappa_{\text{pass}} \sim 1/\beta \ln 1/h$ and $\kappa_{\text{prop}} < 1 - \delta$. Hence, after integration of the inequality we obtain

$$-mh'(t) > \frac{h(t)}{\beta}\left(1 + \ln\frac{1}{h(t)}\right) + f_p\delta t - mh_1 - \frac{h_0}{\beta}\left(1 + \ln\frac{1}{h_0}\right) \tag{39}$$

Note that $h/\beta\ln(1/h) < 2\ln 1/\beta$ for $h < \beta$. Thus, RHS of (39) is estimated from above by $-C\ln(1/\beta) + f_p\delta t$. Using (39), $h(T) = 0$ and $mh(0) - mh(T) = -\int_0^T mh'(t)dt$, we get $mh_0 > f_p\delta T^2 - C\ln 1/\beta \cdot T$. Thus, the collision happens in finite time T, and $T < c_1 \ln 1/\beta$.

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