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Controlling collective dynamics in complex, minority-game resource-allocation systems

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Resource allocation takes place in various kinds of real-world complex systems, such as the traffic systems, social services institutions or organizations, or even the ecosystems. The fundamental principle underlying complex resource-allocation dynamics is Boolean interactions associated with *minority games*, as resources are generally limited and agents tend to choose the least used resource based on available information. A common but harmful dynamical behavior in resource-allocation systems is herding, where there are time intervals during which a large majority of the agents compete for a few resources, leaving many other resources unused. Accompanying the herd behavior is thus strong fluctuations with time in the number of resources being used. In this paper, we articulate and establish that an intuitive control strategy, namely pinning control, is effective at harnessing the herding dynamics. In particular, by fixing the choices of resources for a few agents while leaving majority of the agents free, herding can be eliminated completely. Our investigation is systematic in that we consider random and targeted pinning and a variety of network topologies, and we carry out a comprehensive analysis in the framework of mean-field theory to understand the working of control. The basic philosophy is then that, when a few agents waive their freedom to choose resources by receiving sufficient incentives, majority of the agents benefit in that they will make fair, efficient, and effective use of the available resources. Our work represents a basic and general framework to address the fundamental issue of fluctuations in complex dynamical systems with significant applications to social, economical and political systems.

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I. INTRODUCTION

Resource allocation is an essential process in many kinds of real-world systems, such as traffic systems (e.g., Internet, urban traffic grids, rail and flight networks), social service institutions or organizations (e.g., schools, marts, banks, and financial markets), and ecosystems of various sizes. The underlying system typically contains a large number of interacting components or agents on a hierarchy of scales, and there are multiple resources available for each agent. As a result, complex behaviors are expected to emerge ubiquitously in the dynamical process of resource allocation. In a typical situation, agents or individuals possess similar capabilities, who share the common goal of pursuing as high payoffs as possible. To exploit the resource allocation dynamics in multi-agent systems to reduce the likelihood of or even to eliminate harmful or catastrophic behaviors is of significant interest.

A general framework to address and understand the extremely rich and complex dynamics of many real-world systems is complex adaptive systems [1–3]. Especially suitable for resource-allocation dynamics is the paradigm of minority-game (MG) dynamics [4], introduced by Challet and Zhang to address the classic El Farol bar-attendance problem conceived by Arthur [5]. In an MG system, each agent makes choice ($+1$ or -1, e.g., to attend a bar or to stay at home) based on available global information in the memory such as the winning choice in a previous round of interaction. In particular, the agents who got the minority choice are rewarded, and those belonging to the majority group lose due to limited resources. The MG dynamics has been studied extensively in the last decade or so [6–24].

There are two basic and related approaches to the MG problem. One is based on the mean-field approximation, which was mainly developed by the statistical-physics community to relate the MG problem to those associated with non-equilibrium phase transitions [25–27]. Another approach is based on Boolean-game (BG) dynamics, where for any agent, detailed information about agents that it interacts with is assumed to be available, and the agent responds accordingly [28–33]. One interesting result was that *coordination* can emerge from local interactions in BG and, as a result, the system as a whole can achieve "better than random" performance in terms of utilization of resources.

A common behavior in many social, economical and ecosystems is *herding*, where many agents take on the same action [34]. In the past, the herd behavior has been extensively studied and recognized to be one important factor contributing to the origin of complexity, which can lead to enhanced fluctuations and significant reduction in the payoff of the entire system [11, 35–37]. For the resource-allocation problem, the desired performance is that all the resources are used efficiently. When herding occurs, many agents may go after a very limited number of resources, causing crowding in the use of these resources, while many other resources are significantly under-used. The herd behavior is thus regarded as harmful for resource-distribution systems. An outstanding issue is whether effective control strategy can be developed to prevent herding in multi-agent systems with competition for multiple resources.

In this paper, we investigate a realistically feasible control strategy to harnessing herding in complex resource-allocation systems, *pinning* control in the framework of Boolean dynam-

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ics. In particular, we show that even a small amount of pinning can effectively prevent or greatly mitigate the herd behavior in resource-allocation systems. Take the urban traffic system as an example. The basic idea of pinning control is to select certain individuals and pin (or fix) their options to access resources by certain incentives, e.g., compensations or rewards. This is similar in spirit to the strategy of immunization to prevent wide spread of disease or virus in complex social or technological networked systems [38–42], where certain individuals are preferred to be immunized to the virus of concern. However, as we show analytically and demonstrate numerically in this work, the dynamical mechanism of pinning control in resource-allocation systems is quite different from that underlying the immunization problem in complex networks. In general, we anticipate pinning control to be an effective strategy to eliminate or suppress harmful herd behaviors in complex systems describable by Boolean-game dynamics.

In Sec. II, we describe our Boolean-game model under pinning control. In Sec. III, we present a conventional mean-field theory to analyze the dynamics of free systems in the absence of control. In Sec. IV, we point out the difficulties associated with the conventional mean-field theory and develop a modified mean-field theory to understand the system behavior under pinning control. Different pinning schemes and network topologies are considered. In Sec. V, we offer concluding remarks and discuss relevance of our results to real-world complex systems.

II. MODEL

A. Boolean-game dynamics

Similar to the MG dynamics, there are two alternative resources: $r = +1$ and -1 in a BG dynamic system, and only the agents belonging to the *global minority* group are rewarded by $+1$. As a result, the system profit is equal to the number of agents in the global-minority group. In particular, we consider a BG dynamic system composed of N agents competing for the two resources, both of which have accommodating capacity $N/2$. If the number of agents choosing one given r ($+1$ or -1) is smaller than $N/2$, then it is the global-minority group, and the system profit is equal to the number of agents in this group.

While, a unique feature of the BG dynamic system, in contrast to the original MG dynamic system, is that agents make use of only local information from immediate neighbor in making choice. The neighborhood of agents is determined by the connecting structure of the underlying network. Each agent receives inputs from its neighboring agents and updates its state according to the Boolean function, a function that generates either $+1$ and -1 from the inputs [30]. Here, to be concrete, we assume that, an agent i who has k_i neighbors, will choose $+1$ at time step $t+1$ with the probability,

$$P_{i\to\oplus} = n_-^t/(n_+^t + n_-^t) = n_-^t/k_i, \tag{1}$$

and -1 with the probability $P_{i\to\ominus} = 1 - P_{i\to\oplus}$. Here, n_+^t and n_-^t respectively are the numbers of $+1$ and -1 neighbors of agent i at time step t. Notably, agent in BG dynamics attempts to take on a global-minority choice without any global information (e.g., previous global-minority choice), but basing her choice on the observation of neighbors' previous behavior.

The dynamical variable of the BG system is A_t, the number of $+1$ agents in the system at time step t. Obviously, the optimal solution for the resource allocation is $A_t = N/2$. A measure of BG system's performance is the variance of A_t:

$$\sigma^2 = \frac{1}{T}\sum_{t=1}^{T}(A_t - \frac{N}{2})^2, \tag{2}$$

which characterizes the statistical deviation from the optimal resource utilization over time interval T [31]. A smaller value of σ^2 corresponds to more optimal resource allocation and thus leads to higher efficiency. A general phenomenon in BG system is that, as agents strive to join the minority group, harmful herd behavior can emerge, associated with which large oscillation in A_t takes place. Our goal is to develop an efficient control strategy to suppress or eliminate the harmful herd behavior.

B. Pinning control scheme

Our basic idea to control herd behavior is to "pin" certain agents to freeze their states so as to realize optimal resource allocation, following the general principle of pinning control of complex dynamical networks [43–49]. In our approach, the fraction of agents to be pinned (fixed) is ρ_{pin}, and the fraction of unpinned or free nodes is $\rho_{free} = 1 - \rho_{pin}$. The numbers of free agents and pinned agents are $N_f = N \cdot \rho_{free}$ and $N_p = N \cdot \rho_{pin}$, respectively. The free agents make choices according to local information, while the inputs from the pinned agents are fixed.

Our pinning scheme has two features: order of pinning and pinning pattern. First, the order of pinning denotes the way how certain agents are chosen for pinning. We consider two methods: random pinning (RP), where a number of agents are randomly chosen to be pinned, and degree-preferential pinning (DPP) in which agents are selected for pinning according to their connectivity or degree in the underlying network. In particular, agents with higher degree are more likely to be pinned. These two methods thus correspond to random error and intentional attack in the literature on robustness of network systems [50–53]. The second feature, pinning pattern, defines the particular states that the selected agents are pinned to. Here we define "All $+1$" (or "All -1") as the pattern where all the pinned agents are forced to choose $+1$ (or -1), and "Half ±1" as the situation where the agents are to be pinned at $+1$ and -1 alternately. The effect of pinning also depends on the network topology. We consider four representative network topologies: all-to-all coupling, random [54], scale-free [55], and assortatively mixed scale-free networks [56].

To facilitate a comparative analysis between the free and the pinned systems, we define a modified cumulative variance as,

$$\sigma^2 = \frac{1}{T}\frac{\sum_{t=1}^{T}(A_t - \frac{N}{2})^2}{1-\rho_{pin}}, \tag{3}$$

so that the fluctuations of the systems are comparable with respect to $\rho_{pin} \in [0, 1)$.

C. Simulation results

Simulations are carried out for resource-allocation dynamics on the following types of networks: fully connected networks (FCN), ER random networks [54], scale-free networks (SFN) [55], and under the two pinning schemes (RP or DPP). The states of the pinned agents are set according to "Half ±1", "All $+1$", and "All -1." For all the free agents, $+1$ and -1 are uniformly distributed initially. The evolutionary time is set to be $T = 10^4$. As an example, Fig. 1 shows, for FCN and SFN, time series of A_t for different pinning fraction ρ_{pin}, where the pinning scheme is DPP under the rule "Half ±1." We observe that, in the absence of pinning control ($\rho_{pin} = 0$), herd behavior prevails in the *free* system, associated with which there are oscillations with extremely large variances σ^2. Such a fluctuation state in which A_t oscillates between 0 and N is in fact an *absorbing state* of the system, in which the resource allocation is extremely unreasonable and inefficient. As pinning control is turned on, even when only a few agents are pinned, e.g., $\rho_{pin} = 0.01$, the fluctuations are weakened considerably and harmful absorbing state no long exists. Figure 1 also shows the corresponding distributions of A_t for different cases. The general numerical observation is that pinning control is highly effective in suppressing or even eliminating herd behavior.

III. MEAN-FIELD THEORY OF FREE SYSTEMS

We aim to develop a comprehensive theoretical understanding of the pinning control method with respect to herd behavior. To gain insight, we first derive an analytic theory for free systems.

In the mean-field framework, agents in different states are well mixed. At time step t, An individual i of degree k_i has on average $n_+^t = k_i\rho_\oplus^t$ neighbors that adopt $+1$, where $\rho_\oplus^t = A_t/N$ is the density of $+1$ agents in the whole system. According to the updating rule Eq. (1), i will choose $+1$ at the next time step $t+1$ with the probability

$$P_{i\to\oplus} = k_i(1-\rho_\oplus^t)/k_i = 1-\rho_\oplus^t. \tag{4}$$

The probability for an agent to choose -1 is $P_{i\to\ominus} = \rho_\oplus^t$. The conditional transition probability for A_{t+1} agents to select $+1$ at the next time step $t+1$ obeys the binomial distribution given by

$$P(A_{t+1}|A_t) = \binom{N}{A_{t+1}} \cdot (P_{i\to\oplus})^{A_{t+1}} \cdot (1-P_{i\to\oplus})^{N-A_{t+1}}. \tag{5}$$

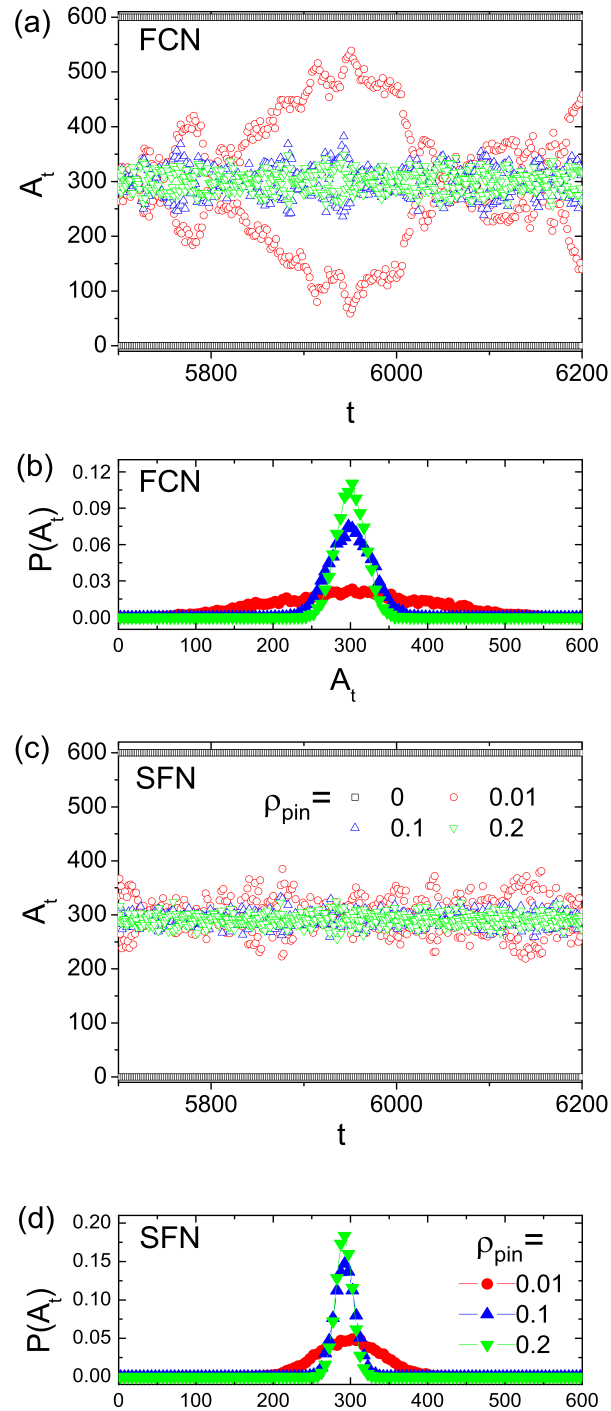

FIG. 1: (Color online). For FCN (a) and SFN network (c), time series A_t for $\rho_{pin} = 0, 0.01, 0.1$, and 0.2. The network size is $N = 600$ and average degree is $\langle k \rangle = 6$. The pinning rule is DPP with "Half ±1." The probability density distributions $P(A_t)$ for $\rho_{pin} = 0.01$, 0.1, and 0.2 (b,d) are calculated from time series A_t of length $T = 10^4$.

The expectation value of A_{t+1} is $E(A_{t+1}) = N \cdot P_{i\to\oplus}$, and the variance of A_{t+1} about $E(A_{t+1})$ can be explicitly written as $\delta^2 = N \cdot P_{i\to\oplus} \cdot (1 - P_{i\to\oplus})$. From Eq. (4), we have,

$$E(A_{t+1}) = N \cdot (1 - \rho_{\oplus}^{t}) = N - A_t. \qquad (6)$$

The expected difference of A_{t+1} from the *optimal solution* $N/2$ is

$$\Delta A_{t+1} \quad = E(A_{t+1}) - N/2 = N - A_t - N/2 = -\Delta A_t.$$

The relation of the expected departures from the optimal state for two successive time steps is thus

$$|\Delta A_{t+1}| = |\Delta A_t|. \qquad (7)$$

If a large event takes place initially in the system (e.g., $A_{t=0} \gg N/2$, or $A_{t=0} \ll N/2$), the departure from $N/2$ will not decrease, so large oscillations will persist with the state of the winning side reversing at each time step. In fact, A_t is a Markov-chain process with successive random number drawn from Eq. (5). As soon as A_t reaches zero or N in the stochastic process, A_t will oscillate between 0 and N continuously, landing the free system in an absorbing state. Herd behavior is thus prevalent in the free system, a hallmark of which is large oscillations in A_t.

A key quantity in the stochastic description of the resource-allocation process is the distribution $P(A_t)$, the probability that A_t agents adopt $+1$ at time t. Since A_t fluctuates about $N/2$, the choice $+1$ acts as the global majority and minority choice alternately. The stable distribution thus obeys $P(A_{t+2l}) = P(A_t)$ and $P(A_{t+2l+1}) = P(A_{t+1})$, for $l \in \mathbf{N}$.

According to Eq. (5), the conditional transition probability for two successive time steps t and $t+1$, we have the following two-step conditional probability, or the transition probability:

$$\begin{aligned} T(A_{t+2}, A_t) &\equiv P(A_{t+2}|A_t) \qquad (8) \\ &= \sum_{A_{t+1}} P(A_{t+2}|A_{t+1}) \cdot P(A_{t+1}|A_t). \end{aligned}$$

To simplify notation, we set $A_t = i$, $A_{t+1} = k$, and $A_{t+2} = j$, with $i, k, j \in [0, N]$. The conditional transition probability of the free system is thus given by

$$\begin{aligned} T(j,i) &\equiv P(j, t+2|i, t) \\ &= \sum_k P(j, t+2|k, t+1) \cdot P(k, t+1|i, t) \\ &= \sum_k [\binom{N}{j} \cdot (1 - \frac{k}{N})^j \cdot (\frac{k}{N})^{N-j}] \cdot [\binom{N}{k} \cdot \\ &\quad (1 - \frac{i}{N})^k \cdot (\frac{i}{N})^{N-k}]. \end{aligned}$$

The resulting *balance equation* governing the dynamics of the Markov chain reads,

$$P(j) = \sum_i P(j, t+2|i, t)P(i) = \sum_i T(j,i)P(i), \qquad (9)$$

which is in fact a *discrete-time master equation*. For large t, the system evolves into the stable state defined by $P(i) = P(j)$. Equation (9) can be written in the matrix form as

$$P(\mathbf{A}) = \mathbf{T}P(\mathbf{A}), \qquad (10)$$

where $\mathbf{T}$ is an $N \times N$ matrix with element $\mathbf{T}_{ji} = T(j,i)$. The stable distribution of A_t is then $P_1(\mathbf{A})$, the eigenvector of matrix $\mathbf{T}$ associated with eigenvalue $\lambda = 1$. For the *free* system, we thus obtain $P(A) = \delta_{A,0}$ or $\delta_{A,N}$ with equal probability on average, where the exact value of $P(A)$ depends on the initial condition and the number of time steps (even or odd). This explains the simulation results in Fig. 1 for the case of $\rho_{pin} = 0$, where $A_t = 0, N, 0, ...$ is an absorbing state and thus is the stable state of the free system. Once we obtain the stable distribution $P(\mathbf{A})$ analytically from Eq. (10), we can calculate the cumulative variance [Eq. (3)] of the system as,

$$\sigma^2 = \frac{\overline{(A_t - N/2)^2}}{1 - \rho_{pin}} = \frac{\sum_{A=0}^{N} P(A)(A - N/2)^2}{1 - \rho_{pin}}. \qquad (11)$$

The fluctuation of the *free* system is thus given by $\sigma^2 = N^2/4$.

While we have considered the resource-allocation dynamics in networked systems in which agents interact with each other without any restriction, the discrete-time master equation Eq. (10) can be used to analyze and understand oscillatory dynamics in general complex adaptive systems.

IV. MEAN-FIELD ANALYSIS OF SYSTEMS UNDER PINNING CONTROL

We now develop a theory to understand the working of pinning control in suppressing/eliminating herd behavior. The setting is a networked system of N agents in which a fraction ρ_{pin} of the agents are not allowed to choose resources freely. Without loss of generality, we focus on the "Half ± 1" pinning rule.

A. Mean-field analysis for well-mixed free and pinned agents

We first consider the case of random pinning. Under the assumption that the dynamical properties of pinned and free nodes are identical, the interactions among them are well-mixed. Consequently, the probability for the neighbor of one given *free* agent to be pinned is

$$P_p = N_p/N = \rho_{pin}, \qquad (12)$$

where N_p is the number of pinned agents in the system. For a free agent i with degree k_i, the average numbers of pinned and free neighbors, denoted by n_p and n_f, respectively, are

$$\begin{aligned} n_f &= (1 - P_p)k_i = (1 - \rho_{pin})k_i, \\ n_p &= P_p k_i = \rho_{pin} k_i, \qquad (13) \end{aligned}$$

where half of a pinned neighbor adopt $+1$, and the other half adopt -1. According to the updating rule Eq. (1), the probability for i to choose $+1$ at the next time step $t+1$ is

$$P_{i\to\oplus} = \frac{n_f(1-\rho_{\oplus}^{t,f}) + n_p/2}{k_i} = (1-\rho_{pin})(1-\rho_{\oplus}^{t,f}) + \frac{\rho_{pin}}{2}, \tag{14}$$

where $\rho_{\oplus}^{t,f}$ stands for the density of free agents who choose $+1$, and $\rho_{\oplus}^{t}$ is the density of $+1$ agents in the whole system. When $\rho_{pin} = 0$, $\rho_{\oplus}^{t,f}$ reduced to $\rho_{\oplus}^{t}$, we have $P_{i\to\oplus} = 1 - \rho_{\oplus}^{t}$, which is reduced to the result for the free system, that is, Eq. (4).

Using a similar reasoning that leads to the conditional transition probability of A_{t+1} for the free system as in Eq. (5), we obtain the corresponding result for the pinning system:

$$\begin{aligned} &P(A_{t+1}|A_t) = P(A_{t+1}^f|A_t^f) \\ &= \binom{N_f}{A_{t+1}^f} \cdot (P_{i\to\oplus})^{A_{t+1}^f} \cdot (1-P_{i\to\oplus})^{N_f - A_{t+1}^f}, \end{aligned} \tag{15}$$

where $A_{t'}$ and $A_{t'}^f$ are related by $A_{t'} = A_{t'}^f + N_p/2$, $A_{t'}^f$ and $N_p/2$ are the numbers of free $+1$ agents and pinned $+1$ agents in the system at time t', respectively. The deviation of A_t from the optimal state $N/2$ is mainly due to the fluctuation of the free agents. From the binomial distribution, we get the expectation number of the free $+1$ agents at time $t+1$, and the variance about the expectation number as,

$$\begin{aligned} E(A_{t+1}^f) &= N_f \cdot P_{i\to\oplus}, \\ \delta_f^2 &= N_f \cdot P_{i\to\oplus} \cdot (1-P_{i\to\oplus}). \end{aligned} \tag{16}$$

The expectation number of $+1$ agents (including the pinned $+1$ agents) is

$$E(A_{t+1}) = E(A_{t+1}^f) + N_p/2, \tag{17}$$

which can be written as a function of ρ_{pin} and A_t:

$$E(A_{t+1}) = \underline{(1-\rho_{pin})} \cdot (N - A_t) + \underline{\rho_{pin}} \cdot \frac{N}{2}, \tag{18}$$

$$= (N - A_t) + \rho_{pin} \cdot (A_t - \frac{N}{2}). \tag{19}$$

From Eq. (19), we obtain the following expected difference from $N/2$:

$$\begin{aligned} \Delta A_{t+1} &= E(A_{t+1}) - N/2 = -(1-\rho_{pin}) \cdot (A_t - N/2) \\ &= -(1-\rho_{pin}) \cdot \Delta A_t \end{aligned} \tag{20}$$

The relation between the expected deviations from the optimal state $N/2$ for two successive time steps is then given by

$$|\Delta A_{t+1}| = (1-\rho_{pin}) \cdot |\Delta A_t|. \tag{21}$$

Compared with the expected departure obtained in the free system, as given by Eq. (7), we see that, once pinning is implemented, the deviation from the optimal state decays by the factor $(1-\rho_{pin})$ at each time step and, consequently, the oscillation of the system is suppressed. In case of large events, pinning will make A_t to approach the equilibrium value $N/2$.

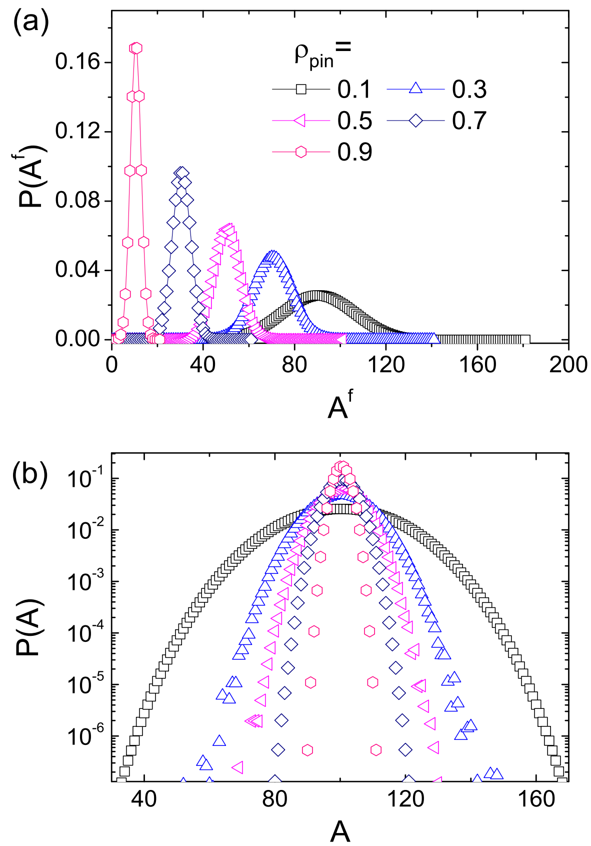

FIG. 2: (Color online). Analytical results of the stable distribution of A^f (a) and A (b) for a system of $N = 201$ agents, from the mean-field analysis under the assumption of well-mixed interactions between the pinned and free agents, as given by Eq. (22). The graph $P(A)$ in (b) is on a logarithmic-normal plot.

A stochastic analysis similar to that for the free system can then be carried out for a pinning system. From Eqs. (8) and (15), we can get the conditional transition probability from time step t to $t+2$ as,

$$\begin{aligned} &T(j,i) \equiv P(j,t+2|i,t) \\ &= \sum_k P(j,t+2|k,t+1) \cdot P(k,t+1|i,t) \\ = \quad &\sum_k \{[\binom{N_f}{j-\frac{1}{2}N_p} \cdot (1-\frac{k}{N})^{j-\frac{1}{2}N_p} \cdot (\frac{k}{N})^{N_f-(j-\frac{1}{2}N_p)}] \\ &\cdot[\binom{N_f}{k-\frac{1}{2}N_p} \cdot (1-\frac{i}{N})^{k-\frac{1}{2}N_p} \cdot (\frac{i}{N})^{N_f-(k-\frac{1}{2}N_p)}]\}, \end{aligned} \tag{22}$$

where $i,k,j \in [N_p/2, N-N_p/2]$, which denote A_t, A_{t+1}, and A_{t+2}, respectively. Following the steps from Eq. (9) to Eq. (11) for a free system, we can derive formulas of $P(A^f)$ and $P(A)$ for the pinning system, based on the assumption Eq. (12) that the pinned and free nodes are identical with well-mixed interactions. Figure 2 shows the corresponding results for different values of ρ_{pin}. We see that the stable distribution $P(A)$ has a Gaussian profile, with the expectation value of $E(A) = N/2$, which should be compared with the value $P(A) = \delta_{A,0}$ or $\delta_{A,N}$ for the free system. This result indicates that the harmful absorbing state associated with a free system has essentially been eliminated even when only a few agents are pinned. Representative numerical evidence supporting this result is shown in Fig. 1 for FCN and SFN with only 1% of the agents pinned.

From Eq. (11), we can calculate the variance σ^2 of the system for different values of ρ_{pin} analytically, as shown by the open circle marked "MF1" in Fig. 3. Simulation results for FCN (open black square), SFN (solid triangle) and ER random network (open triangle) under DPP (up triangle) or RP (down triangle) and the "Half ± 1" rule are also shown. We observe that the variance σ^2 of the system decreases dramatically in a power-law manner as pinning control is turned on, and the agreement between theoretical prediction and numerical simulations for FCN is good. However, for ER random network and SFN, there is marked difference between the theoretical and numerical results, especially for the DPP scheme, indicating that the approach of mean-field, stochastic type of analysis may not be adequate to account for the behavior of the system under pinning control. In the following, we shall develop a modified mean-field analysis to overcome this difficulty.

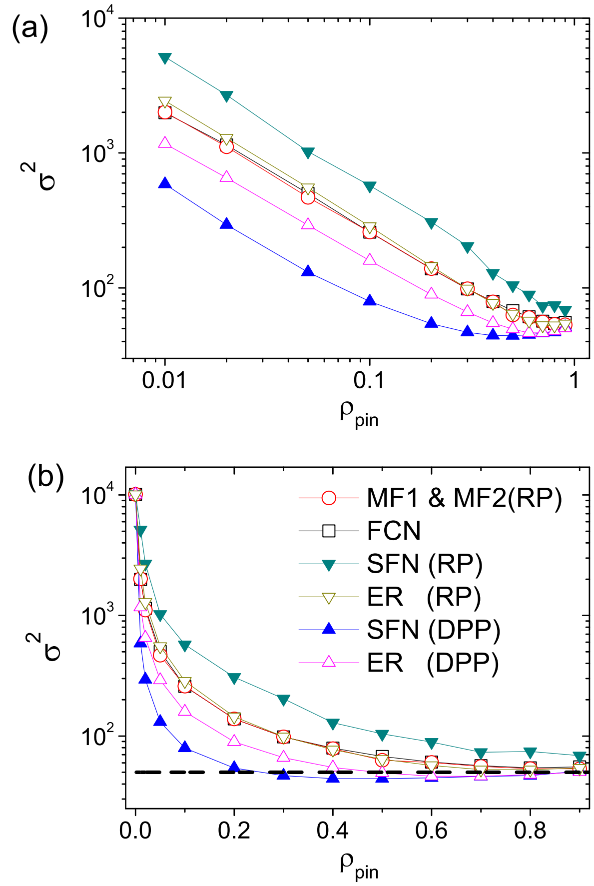

FIG. 3: (Color online). Modified cumulative variance σ^2 [Eq. (3)] as a function of ρ_{pin} from mean-field analysis (red circle marked by "MF1"), where pinned and free nodes are well mixed as in Eq. (22). Simulation results for FCN, SFN, and ER random network are also shown, where all network size is $N = 201$, the pinning schemes are RP and DPP under the "Half ± 1" rule, the average degrees of SFN and ER random networks is $\langle k \rangle = 6$, and the number of network realizations is 200. The graphs in (a) and (b) are on a logarithmic and a logarithmic-linear scale, respectively.

B. Modified mean-field analysis

The assumption Eq. (12) in which free and pinned agents are identical and well mixed may not be valid in general, especially when the underlying network is heterogeneous, such as SFNs. In such a case, the probability for a free node to contact with a pinned node will deviate from ρ_{pin}, requiring modifications of the conventional mean-field analysis.

1. *Analysis of degree-preferential pinning on scale-free networks*

We first discuss the DPP scheme on SFNs generated by the classic preferential-attachment rule [55], with the degree distribution given by $P(k) = 2m^2/k^3$, where m is the number of edges each new node brings in as the system grows. The average degree of the network is $\langle k \rangle = 2m$, and the minimum degree is $k_{min} = m$. For DPP scheme from large to small degree, the density of pinned agents ρ_{pin} and the minimum degree of pinned agents (denoted by k') are related to each other as

$$\rho_{pin} = \int_{k'}^{\infty} P(k)dk, \tag{23}$$

giving

$$k' = \sqrt{\frac{m^2}{\rho_{pin}}}, \tag{24}$$

which can be used to distinguish pinned and free agents in terms of their degrees, i.e., an agent with $k \geq k'$ (or $k < k'$) is pinned (or free). The total number of links in the whole network, denoted by L, is

$$L = \frac{1}{2}\int_{k_{min}}^{\infty} kNP(k)dk. \tag{25}$$

The number of the so-called pinning-affected links L_{pin} and that of free-related links L_{free} can be defined, respectively, as

$$L_{pin} = \frac{1}{2}\int_{k'}^{\infty} kNP(k)dk, \tag{26}$$

$$L_{free} = \frac{1}{2}\int_{k_{min}}^{k'} kNP(k)dk, \tag{27}$$

where $L_{pin} + L_{free} = L$. From the fraction of pinning-affected links, we have the following probability for one neighbor of a given free agent to be a pinned agent:

$$P_p = L_{pin}/L. \tag{28}$$

For a SFN under DPP, we have $L = mN$, $L_{pin} = mN\sqrt{\rho_{pin}}$, and $L_{free} = mN(1 - \sqrt{\rho_{pin}})$ and, consequently, $P_p = \sqrt{\rho_{pin}}$. It should be noted that Eq. (28) differs from Eq. (12) in that the former is expressed in terms of the pinning-affected links and the latter is with respect to the fraction of pinned agents. This difference underlies our modified mean-field analysis.

Utilizing Eq. (28), we can write the average numbers of the free and pinned neighbors for a free agent i of degree k_i as

$$n_f = (1 - P_p)k_i = (1 - \sqrt{\rho_{pin}})k_i, \tag{29}$$
$$n_p = P_p k_i = \sqrt{\rho_{pin}} \tag{30}$$

Similar to the analysis procedure from Eq. (14) to Eq. (22), we can obtain the corresponding results for SFNs with DPP under the "Half ± 1" rule. The probability for agent i to choose $+1$ is

$$\begin{aligned} P_{i\to\oplus} &= \frac{n_f(1-\rho_\oplus^{t,f}) + n_p/2}{k_i} \\ &= (1-P_p)(1-\rho_\oplus^{t,f}) + P_p/2 \\ &= \frac{1-P_p}{1-\rho_{pin}}(1-\rho_\oplus^t) + \frac{P_p - \rho_{pin}}{2(1-\rho_{pin})} \\ &\equiv a(1-\rho_\oplus^t) + b. \end{aligned} \tag{31}$$

The expectation numbers of free $+1$ agents and all $+1$ agents [$E(A_{t+1}^f)$ and $E(A_{t+1})$, respectively] can then be obtained from Eqs. (16) and (17). The expected deviation of A_t from the optimal state for two successive time steps is given by

$$|\Delta A_{t+1}| = (1 - \sqrt{\rho_{pin}}) \cdot |\Delta A_t|. \tag{32}$$

Furthermore, from Eqs. (15) and (31), we can get the conditional transition probability from time step t to $t+2$ as

$$\begin{aligned} &T(j,i) \equiv P(j,t+2|i,t) \\ &= \sum_k P(j,t+2|k,t+1)\cdot P(k,t+1|i,t) \\ &= \sum_k \{[\binom{N_f}{j-\frac{1}{2}N_p}\cdot(\frac{P_p}{2} + P_f\cdot\frac{N_f - k + \frac{N_p}{2}}{N_f})^{j-\frac{1}{2}N_p}\cdot \\ &(\frac{P_p}{2} + P_f\frac{k-\frac{N_p}{2}}{N_f})^{N_f-(j-\frac{1}{2}N_p)}] \\ &\cdot[\binom{N_f}{k-\frac{1}{2}N_p}\cdot(\frac{P_p}{2} + P_f\cdot\frac{N_f - i + \frac{N_p}{2}}{N_f})^{k-\frac{1}{2}N_p}\cdot \\ &(\frac{P_p}{2} + P_f\frac{k-\frac{N_p}{2}}{N_f})^{N_f-(i-\frac{1}{2}N_p)}]\}, \end{aligned} \tag{33}$$

where $i, k, j \in [N_p/2, N - N_p/2]$ are associated with A_t, A_{t+1}, and A_{t+2}, respectively, and $P_f = 1 - P_p$. Similar to analysis of free systems [Eqs. (10) and (11)], we obtain the analytic results of $P(\mathbf{A})$ and σ^2 of the pinning system in terms of the fraction of pinning-affected links, as shown in Fig. 4 [marked by "MF2 (DPP) on SFN"], together with the corresponding simulation results [marked by "Simu.(DPP) on SFN"]. For comparison, result from the original mean-field analysis (MF1) is also included. We see that our modified mean-field analysis yields results that match more closely those from simulations.

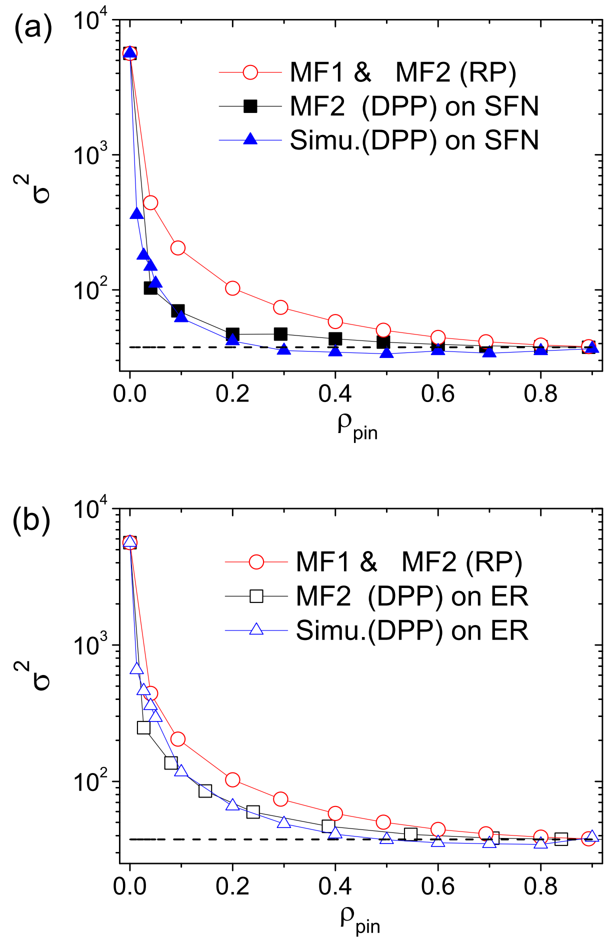

FIG. 4: (Color online). Modified cumulative variance σ^2 as a function of ρ_{pin} as predicted by the mean-field analysis (red circle marked by "MF1"), by our modified mean-field analysis (black square marked by "MF2"), in comparison with the simulation results (blue triangle). Note that "MF2" under RP scheme is the same as "MF1" (see Sec. IV B 3). Simulations are for SFNs (a) and ER random networks (b), all of average degree 6. Network size is $N = 150$ and the pinning scheme is DPP under the "Half ± 1" rule. The simulation results are averaged over 1000 realizations.

2. Degree-preferential pinning on random networks

The degree of ER random network [54] obeys Poisson distribution:

$$P(k) = \frac{e^{-\langle k \rangle} \langle k \rangle^k}{k!}. \tag{34}$$

The relation between k' and ρ_{pin} can then be written as

$$\rho_{pin} = \sum_{k=k'+1}^{k_{max}} P(k), \tag{35}$$

where the maximum degree k_{max} for a network of size N can be calculated by $P(k_{max}) \approx 1/N$. The degree k' for a given ρ_{pin} can be calculated numerically. The quantities L, L_{pin}, and L_{free} are, respectively, given by

$$L = \frac{1}{2} \sum_{k=1}^{\infty} kNP(k) = \frac{1}{2} \sum_{k=1}^{k_{max}} \frac{kNe^{-\langle k \rangle} \langle k \rangle^k}{k!} \tag{36}$$

$$L_{pin} = \frac{1}{2} \sum_{k=k'+1}^{\infty} kNP(k) = \frac{1}{2} \sum_{k=k'+1}^{k_{max}} \frac{kNe^{-\langle k \rangle} \langle k \rangle^k}{k!} \tag{37}$$

$$L_{free} = \frac{1}{2} \sum_{k=1}^{k'} kNP(k) = \frac{1}{2} \sum_{k=1}^{k'} \frac{kNe^{-\langle k \rangle} \langle k \rangle^k}{k!} \tag{38}$$

Following a similar modified mean-field analysis for SFNs, we can calculate k' for a given value of ρ_{pin}. The quantities P_p, $P_{i \to \oplus}$, $T(j, i)$, the stable distribution $P(\mathbf{A})$, and finally σ^2 can then be obtained as a function of ρ_{pin}, as shown in Fig. 4(b). We see that the modified mean-field analysis [marked by "MF2 (DPP) on ER"] gives more accurate prediction about the system behaviors.

3. Random pinning

For random pinning on a network of a given degree distribution $P(k)$, the number of pinning-affected links and free links are

$$L_{pin} = \frac{1}{2} \int_{k_{min}}^{\infty} kN\rho_{pin} P(k) dk = \rho_{pin} L, \tag{39}$$

$$L_{free} = \frac{1}{2} \int_{k_{min}}^{\infty} kN(1 - \rho_{pin}) P(k) dk = (1 - \rho_{pin}) L, \tag{40}$$

where the number L of total links is given by Eq. (25). In the RP process, the value of L_{pin} and L_{free} are independent of the degree distribution $P(k)$. We thus have

$$P_p = L_{pin}/L = \rho_{pin}. \tag{41}$$

Similar to the analysis of DPP on heterogeneous networks, we can obtain n_f and n_p and substitute them into Eq. (31) to get $P_{i \to \oplus} = 1 - \rho_{\oplus}^t$. We see that the quantities P_p and $P_{i \to \oplus}$ for RP are the same as those given by Eqs. (14) and (31) from the idealized mean-field analysis, regardless of the network structure. The reason is that for RP, the pinned and free nodes tend to mix well on the network, satisfying the basic mean-field assumption. A consequence is then that the relation between ΔA_{t+1} and ΔA_t, the conditional transition probability $T(j, i)$, and the stable distribution $P(\mathbf{A})$ are identical to those given by the idealized mean-field analysis [Eq. (12) to Eq. (22)]. As a consequence, the analytical results for random pinning from the modified mean-field analysis [marked by "MF2 (RP)"] are the same as those from the original mean-field analysis [marked by "MF1"], as shown in Figs. 3 and 4.

V. CONCLUSION AND DISCUSSION

The collective behavior of herding can occur commonly in complex resource-distribution systems, the hallmark of which is strong and even extreme fluctuations in the usage of available resources. In particular, for a free system without any external intervention, typically the resources are accessed and utilized in a highly non-uniform manner: there are time intervals in which almost all resources are used, followed by those in which most agents in the system focus on only a few resources. Such an uneven utilization of resources makes the system inefficient and is generally harmful. What we have shown in this paper is that, implementing a simple pinning control scheme can effectively eliminate herding. While the idea of pinning control has been used widely to control complex networked systems [43–49], our contribution is to introduce it to complex resource-allocation systems. More importantly, we have developed a solid physical theory based on the mean-field approach and its variant to establish the theoretical foundation of the pinning control in such systems. Specifically, we have analyzed the approaches of random and degree-preferential pinning on networks of distinct topologies, and demonstrated that a non-random type of control strategy can be more effective than a random one [cf. Figs. 3 and 4]. The basic philosophy underlying our control scheme is "to pin a few to benefit the majority." That is, fixing a few agents' choice of resource utilization can reduce significantly the fluctuations in the whole system, resulting in remarkable improvement in its efficiency.

Our theory suggests that the best strategy to reduce fluctuations through pinning is to choose the agents of high degrees. However, one difficulty associated with the degree preferential pinning scheme is that it requires fairly complete knowledge about the degree of each agent in the network. This is especially challenging for real-world networks, where global information about the network may not be available to every agent. In addition, the interactions among the agents when competing for resources may not be readily quantified. An important issue concerns thus how herd behavior can be controlled when information about the network structure and interactions among the agents is lacking. Immunization method developed in controlling virus spreading on complex networks [38], which requires no detailed knowledge about the network and its interacting dynamics, may provide a viable approach. For example, one can consider the scheme of acquaintance pinning, in which random acquaintances of random nodes are pinned in their selection of resources.

Real-world systems for which Boolean game model and pinning scheme may be applicable include the financial market systems, urban traffic systems, computer network systems, and so on. In these systems, individuals' choice can be "pinned" by means of certain incentive policies with compensations or rewards. The incentive policy for pinning can be modeled as random fields in the dynamics and may introduce a cost linear to the number of pinned agents. We see that the system welfare, i.e., the performance of the resource allocation system measured by the variance of the number of agents choosing a resource, improves rapidly as soon as very few pinnings take place. Take the financial market system as an example, where the policies of the Market Makers are the strategy to intervene the game dynamics in the market by certain regulations or incentives so as to make the capital allocation more efficient, i.e., to realize the goal of achieving efficient markets. Our study of pinning control is directly relevant to these real-world examples. In addition to its real significance, our work represents a basic and general mathematical framework to address the role of pinning in complex resource-allocation dynamics in social, economical and political systems.

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