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Robustness of network of networks under targeted attack

Gaogao Dong,[1, 2, *] Jianxi Gao,[3, 2] Ruijin Du,[1, 4, 2] Lixin Tian,[1] H. Eugene Stanley,[2] and Shlomo Havlin[5]

[1] *Nonlinear Scientific Research Center, Faculty of Science, Jiangsu University, Zhenjiang, 212013, China*
[2] *Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA*
[3] *Department of Automation, Shanghai Jiao Tong University, Shanghai, 200240, China*
[4] *College of Mathematics Science, Chongqing Normal University, Chongqing, 401331, China*
[5] *Department of Physics, Bar-Ilan University, 52900, Ramat-Gan, Israel*

The robustness of a network of networks (NON) under random attack has been studied recently [J. Gao et al., Phys. Rev. Lett. **107**, 195701 (2011)]. Understanding how robust a NON is to targeted attacks is a major challenge when designing resilient infrastructures. We address here the question how the robustness of a NON under targeted attack is affected by targeted attack on high or low degree nodes. We introduce a targeted attack probability function that is dependent upon node degree, and study the robustness of two types of NON under targeted attack: (i) a tree of n fully interdependent Erdős-Rényi or scale-free networks, and (ii) a starlike network of n partially interdependent Erdős-Rényi networks. For any tree of n fully interdependent Erdős-Rényi networks and scale-free networks under targeted attack, we find that the network becomes significantly more vulnerable when nodes of higher degree have higher probability to fail. When the probability that a node will fail is proportional to its degree, for a NON composed of Erdős-Rényi networks we find analytical solutions for the mutual giant component P_∞ as a function of p, where $1-p$ is the initial fraction of failed nodes in each network. We also find analytical solutions for the critical fraction p_c, which causes the fragmentation of the n interdependent networks, and for the minimum average degree $\bar{k}_{\min}$ below which the NON will collapse even if only a single node fails. For a starlike NON of n partially interdependent Erdős-Rényi networks under targeted attack, we find the critical coupling strength q_c for different n. When $q > q_c$, the attacked system undergoes an abrupt first order type transition. When $q \leq q_c$, the system displays a smooth second order percolation transition. We also evaluate how the central network becomes more vulnerable as the number of networks with the same coupling strength q increases. The limit of $q = 0$ represents no dependency, and the results are consistent with the classical percolation theory of a single network under targeted attack.

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I. INTRODUCTION

Most studies of complex networks over the last decade have focused on single networks that do not interact with other networks [1–21]. Most real-world networks, however, are actually "networks of networks" and are made up of two or more networks that interact and are interdependent. Examples include various infrastructures such as water and food supply networks, communication networks, fuel networks, financial transaction networks, and power-station networks [22–26]. Recently, Buldyrev et al. [27] developed a theoretical framework for studying the robustness of two fully interdependent networks subject to random attack. They found that due to the dependency coupling between networks, they become extremely vulnerable to random failures and the system collapse in an abrupt first order transition. Parshani et al. [28] generalized this framework and investigated networks that are only partially interdependent. They found that reducing the coupling strength could lead to a change from first order (abrupt) to a second order (continuous) percolation transition due to random attack. Recently Gao et al. [29] generalized this theory to include the "network of networks" (NON) case—to a system of n interacting networks. They developed an analytical framework for studying percolation under random failures and presented exact analytical solutions for an interdependent NON system. The results for a NON suggest that the classical percolation theory extensively studied in physics and mathematics is a limiting case of $n = 1$ of the general case of percolation in a NON. For more recent studies of a NON, see Refs. [16, 30]. In real-world scenarios, however, failures may not be random but may be the result of a targeted attack on specific central nodes [31–35]. Huang et al. [31] studied the robustness of two interdependent networks when high or low degree nodes are under targeted attack by mapping the targeted-attack problem in interdependent networks to the random-attack problem. They found that interdependent scale-free (SF) networks are difficult to defend using strategies such as protecting the high-degree nodes. In this paper by

*Electronic address: dfocus.gao@gmail.com

introducing a probability function of targeted attack, we study the robustness of a NON system of any tree of n fully interdependent networks and of a starlike NON structure of n partially interdependent networks (see Fig. 1).

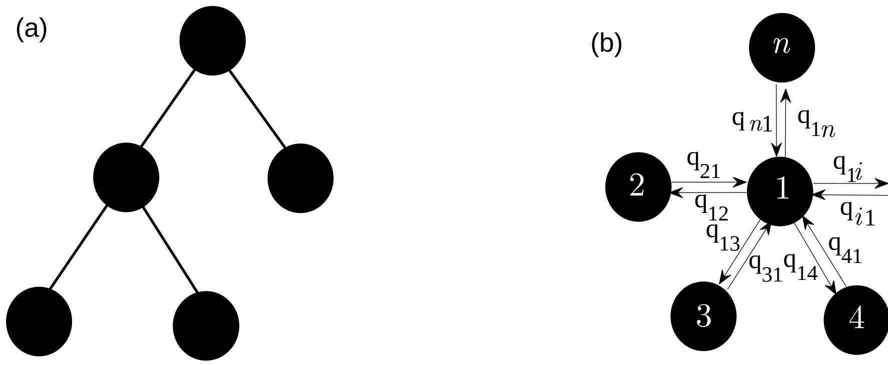

FIG. 1: Each circle represents one network. Two networks i and j form a partially interdependent pair of networks if a certain fraction $q_{ji} > 0$ of nodes of network i directly depend on nodes of network j, i.e., they cannot function if the nodes in network j on which they depend do not function. Two networks i and j are fully interdependent if $q_{ji} = q_{ji} = 1$, thereby establishing a one-to-one correspondence. (a) an example of a treelike NON composed of five fully interdependent networks. (b) A starlike NON composed of n partially dependent networks. The arrows pointing on q_{1i} and $q_{i1}(i = 1, 2, 3, ..., n)$ represents partial dependency of the central network,1, on a neighboring network i and q_{1i} represents the dependency of the neighboring network i on the central network respectively.

II. THE MODEL

Gallos et al. [35] proposed a probability function for a targeted attack on an isolated single network, $W_\alpha(k_i) = k_i^\alpha / \sum_{i=1}^{N} k_i^\alpha$, which represents the probability that a node with degree k_i will be removed. For the case $\alpha < 0$, it was assumed that $k_i \neq 0$. This function was used to study the robustness of a single scale-free network to different attack strategies [35], but in a system comprised of interdependent networks, zero-degree nodes strongly influence overall system robustness. In order to understand the robustness of a NON under targeted attack and to solve the problem analytically we must avoid singularities for $k_i = 0$ in $W_\alpha(k_i)$. We thus propose a new probability function for targeted attack that represents the probability that a node i with degree k_i has been removed,

$$W_\alpha(k_i) = \frac{(k_i + 1)^\alpha}{\sum_{i=1}^{N}(k_i + 1)^\alpha}, -\infty < \alpha < +\infty. \quad (1)$$

When $\alpha < 0$, nodes with higher k_i are protected and nodes with lower k_i have a higher probability of failure. In order to avoid singularities for $k_i = 0$ and $\alpha < 0$, the term $k_i + 1$ is taken. When $\alpha > 0$, nodes with higher k_i have a higher probability of failure and are therefore vulnerable. The case $\alpha \to \infty$, corresponds to the targeted attack case where nodes are removed in strict order, from high degree to low degree. When $\alpha = 0$, all nodes in the networks have the same probability of failure, $W_0 = \frac{1}{N}$, and the nodes are removed at random. The important special case = 1 corresponds to the acquaintance immunization strategy [13, 31, 32, 35] and can be solved analytically.

We assume that in an attack, a fraction of nodes $1 - p_i$ according to Eq. (1) in each network i is removed and only nodes which belong to the giant connected component of each network i remain functional. The failure of these nodes causes further failures in other networks and vice versa [27–29, 36], resulting in a cascade of failures. We now describe the generating function and the giant component of one network i in the NON after a fraction of nodes are removed with a probability $W_\alpha(k_i)$, which can be applied to all networks in the NON system. The generating function of the degree distribution of network i is defined

$$G_{i0}(x) \equiv \sum_k P_i(k)x^k. \quad (2)$$

where $P_i(k)$ is the degree distribution of network i. The generating function of the associated branching process is, $G_{i1}(x) = \frac{G'_{i0}(x)}{G'_{i0}(1)}$ [27–29, 31, 32, 40, 42]. The average degree of network i is $\bar{k}_i = \sum_k P_i(k)k$. We follow an approach similar to that of attacking a pair of coupled networks [31].

Applying Eq. (1), after removing a fraction of $1-p_i$ of nodes of network i but keeping the edges of the remaining nodes, the degree distribution $P_{p_i}(k)$ of the remaining nodes is

$$P_{p_i}(k) = \frac{A_{p_i}(k)}{p_i N}, \tag{3}$$

where $A_{p_i}(k)$ is the number of nodes with degree k in network i.

When one more node is removed, $A_{p_i}(k)$ changes to

$$A_{(p_i-1/N)}(k) = A_{p_i}(k) - \frac{P_{p_i}(k)(k+1)^\alpha}{\sum_k P_{p_i}(k)(k+1)^\alpha}. \tag{4}$$

In the limit $N \to \infty$, the derivative of $A_{p_i}(k)$ with respect to p_i is obtained from Eq. (4),

$$\frac{dA_{p_i}}{dp_i} = N \frac{P_{p_i}(k)(k+1)^\alpha}{\sum_k P_{p_i}(k)(k+1)^\alpha}. \tag{5}$$

By differentiating Eq. (3) with respect p_i and using Eq. (5), we obtain,

$$-p_i \frac{dP_{p_i}(k)}{dp_i} = P_{p_i}(k) - \frac{P_{p_i}(k)(k+1)^\alpha}{\sum_k P_{p_i}(k)(k+1)^\alpha}. \tag{6}$$

In order to solve Eq. (6), we define $G_\alpha(x) \equiv \sum_k P_i(k) x^{(k+1)^\alpha}$ and introduce a new variable $u_i \equiv G_\alpha^{-1}(p_i)$ [31, 32, 38]. We find the solution of Eq. (6) to be

$$P_{p_i}(k) = \frac{1}{p_i} P_i(k) u_i^{(k+1)^\alpha}, \tag{7}$$

$$\sum_k P_{p_i}(k)(k+1)^\alpha = \frac{u_i G_\alpha^{'}(u_i)}{G_\alpha(u_i)}, \tag{8}$$

which can be shown to satisfy Eq. (6). After removing a fraction $1-p_i$ of nodes from network i according to Eq. (1) and their links, the generating function of the nodes left in network i is

$$G_{ib}(x) \equiv \sum_k P_{p_i}(k) x^k = \frac{1}{p_i} \Sigma_k P_i(k) u_i^{(k+1)^\alpha} x^k. \tag{9}$$

In this step, after removing the links between removed nodes and remaining nodes, the generating function of network i composed of the remaining nodes is [42, 43],

$$G_{ic}(x) \equiv G_{ib}(1 - \tilde{p}_i + \tilde{p}_i x), \tag{10}$$

where $\tilde{p}_i$ is the ratio between the number of links of the remaining nodes and the number of original links in network i,

$$\tilde{p}_i = \frac{p_i N_i \bar{k}(p_i)}{N_i \bar{k}} = \frac{\sum_k P_i(k) k u_i^{(k+1)^\alpha}}{\sum_k P_i(k) k}, \tag{11}$$

where $\bar{k}(p_i) = \sum_k P_{p_i}(k) k$ is the average degree of the remaining nodes in network i.

If we find the generating function $\tilde{G}_{i0}(x)$ of network $\tilde{i}$ after randomly removing the $1-p_i$ fraction of nodes, the new generating function of the remaining nodes is the same as $G_{ic}(x)$. The targeted-attack problem on network i is mapped to a random-attack problem on network $\tilde{i}$, by solving the equation $\tilde{G}_{i0}(1 - p_i + p_i x) = G_{ic}(x)$ [31],

$$\tilde{G}_{i0}(x) = G_{ib}(1 - \frac{\tilde{p}_i}{p_i} + \frac{\tilde{p}_i}{p_i} x). \tag{12}$$

The generating function of the associated branching process is [42, 43],

$$\tilde{G}_{i1}(x) = \frac{\tilde{G}'_{i0}(x)}{\tilde{G}'_{i0}(1)}. \tag{13}$$

When a fraction of $1-p_i$ of nodes in the ith network is removed with the probability given by Eq. (1), the fraction of nodes that belongs to the giant component is given by

$$g_i(p_i) = 1 - \tilde{G}_{i0}[1 - p_i(1 - f_i)], \tag{14}$$

where $f_i \equiv f_i(p_i)$ satisfies a transcendental equation

$$f_i = \tilde{G}_{i1}[1 - p_i(1 - f_i)]. \tag{15}$$

In particular, for the case $\alpha = 1$, from Eqs. (9), (10), and (12), $G_{ib}(x), G_{ic}(x), \tilde{G}_{i0}(x)$ become

$$G_{ib}(x) = \frac{u_i}{p_i}\Sigma_k P_i(k) u_i^k x^k = \frac{u_i}{p_i} G_{i0}(u_i x), \tag{16}$$

$$G_{ic}(x) = \frac{u_i}{p_i} G_{i0}(u_i(1 - \tilde{p}_i + \tilde{p}_i x)), \tag{17}$$

$$\tilde{G}_{i0}(x) = \frac{u_i}{p_i} G_{i0}(u_i(1 - \frac{\tilde{p}_i}{p_i} + \frac{\tilde{p}_i}{p_i} x)), \tag{18}$$

where u_i satisfies

$$u_i \equiv G_{i0}^{-1}(p_i), \tag{19}$$

$$\tilde{p}_i \equiv \frac{u_i^2 G'_{i0}(u_i)}{G'_{i0}(1)}. \tag{20}$$

Next we apply the framework of the n equations developed in Ref. [36] to the no-feedback condition. The n unknowns $x_{t,i}$ represent the fraction of nodes that survived in network i at step t of the cascading failures after removing the failed nodes and the nodes dependent upon the failed nodes in the other networks,

$$x_{t,i} = p_i \prod_{j=1}^{K} (1 - q_{ji} + q_{ji} y_{t,ji} g_j(x_{t,j})), \tag{21}$$

where for $t = 1$, $x_{t,i} = p_i$, and

$$y_{t,ji} = \frac{x_{t,j}}{1 - q_{ij} + q_{ij} y_{t,ij} g_i(x_{t,i})}. \tag{22}$$

The terms $y_{t,ji}$ represent the fraction of nodes in network j that survive after the damage from all the networks connected to j is taken into account (except from network i at stage t of the cascading failure) and $y_{t=1,ji} = p_j$. The giant component of each network i at t stage of cascading failures can also be found from the equation $P_{t,i} = x_{t,i} g_i(x_{t,i})$, where $g_i(x_{t,i})$ is the fraction of the remaining nodes of network i at t stage of cascading failures that are part of its giant component.

When $t \to \infty$, the cascading failures have ended and the final giant component of network i is $P_{\infty,i} = x_i g_i(x_i)$, where

$$x_i = p_i \prod_{j=1}^{K} (1 - q_{ji} + q_{ji} y_{ji} g_j(x_j)), \tag{23}$$

$$y_{ji} = \frac{x_j}{1 - q_{ij} + q_{ij} y_{ij} g_i(x_i)}, \tag{24}$$

where x_i is calculated over the K networks interlinked with network i by partial dependency links q_{ji}. The terms y_{ji} represent the fraction of nodes in network j surviving after the damage from all the connected networks to network j except from network i is taken into account [29, 36, 40]. When the dependency has a feedback between two interdependent networks, Eq. (23) becomes simpler as $y_{ji} = x_j$ [40]. We next address two cases of n coupled networks: (i) a treelike NON with fully interdependent networks, and (ii) a starlike NON with partially interdependent networks.

III. TREELIKE FULLY INTERDEPENDENT NETWORK OF NETWORKS

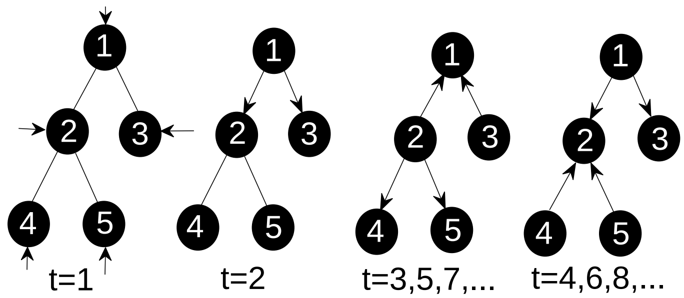

FIG. 2: The dynamic of cascading failures of treelike NON system composed of five fully interdependent networks. Each node represents an ER network or SF network. The arrow (on the link) illustrates the direction of the damage spreading from one network to another within the treelike NON, at step t. At $t = 1$, arrows point to five networks, which means that a fraction $1 - p_i$ of nodes in the five networks are removed according to Eq. (1). Furthermore, we remove nonfunctional nodes, which become disconnected from the largest component of each network. At $t = 2$, arrows from network 1 point to networks $2, 3$, which illustrating that all nodes that depend on the removed nodes in network 1 are removed and nonfunctional nodes are also removed in networks $2, 3$. In this step (t=2), network 1 spread the damage to networks $2, 3$. At $t = 3, 5, 7, ...$, arrows from networks $2, 3$ point to networks $1, 4, 5$. This reflects that networks $2, 3$ spread damage to their neighboring networks $1, 4, 5$. At $t = 4, 6, 8, ...$, networks $1, 4, 5$ spread damage to their neighboring networks $2, 3$. Thus, the damage is spread between networks, back and forth, until they arrive to a mutually stable giant components or collapse.

We now analyze the specific classes of a treelike NON formed of Erdős-Rényi (ER) or scale-free (SF) networks [2, 3, 10, 11, 27, 31]. By using the dynamics of cascading failures shown in Fig. 2, we first compare our theoretical results of Eqs. (16)-(20) with simulation results shown in Fig. 3 for a NON formed of five ER networks and SF networks. The theory is in excellent agreement with simulations.

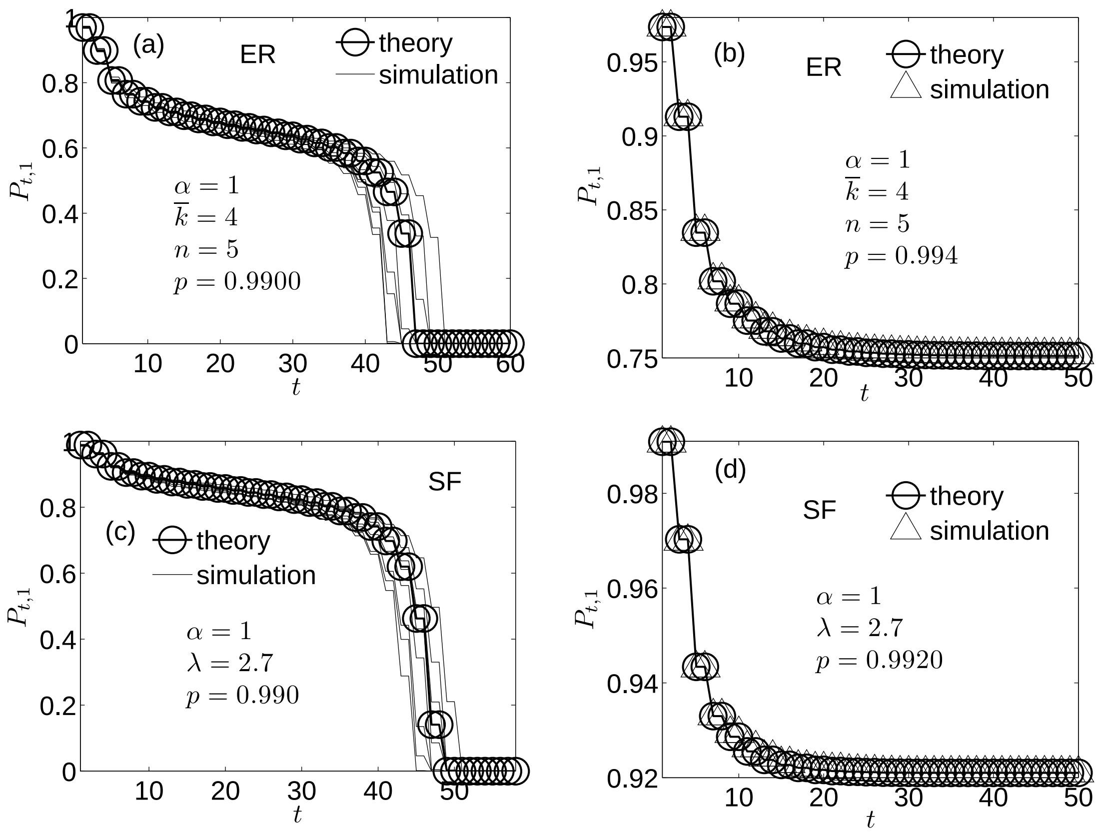

FIG. 3: For the case of $\alpha = 1$ which corresponds to the solvable "acquaintance immunization" strategy, we present the giant component (theory and simulation) of network 1, $P_{t,1}$ during the t cascading failures for fully interdependent treelike NON composed of five (a,b) ER and (c,d) SF networks. (a) For each ER network in the NON, we chose $\bar{k}_i = \bar{k} = 4$, $N_i = N = 10^4$ and $p_i = p = 0.990 < p_c = 0.9920$. (b) The giant component of network 1 (Fig. 2) with the same parameters as in Fig. 3(a) but for $p_i = p = 0.9940 > p_c = 0.9920$. (c) For each SF network in the NON, we chose $\alpha = 1$, $\lambda = 2.7$, $N_i = N = 10^4$ and $p_i = p = 0.9900 < p_c = 0.9910$. (d) The giant component with the same parameters as in Fig. 3(c) but for $p_i = p = 0.9920 > p_c = 0.9910$. The results of simulations are obtained by averaging over 50 realizations.

For the case of treelike NON formed of n ER networks, the generating function of the ER network is $G_{i0}(x) = e^{\bar{k}_i(x-1)}$. From Eqs. (13), (18)–(20), we obtain $\tilde{G}_{i0}(x) = \tilde{G}_{i1}(x) = e^{\bar{k}_i u_i^2 (x-1)}$ for $\alpha = 1$, where u_i can be expressed in term of the Lambert function $u_i = \frac{W(p_i \bar{k}_i e^{\bar{k}_i})}{\bar{k}_i}$. From Eqs. (14)-(15), we can get $g_i(x_i) = 1 - \tilde{G}_{i0}[1 - x_i(1 - f_i)]$, $f_i = \tilde{G}_{i1}[1 - x_i(1 - f_i)]$ and $g_i(x_i) = 1 - f_i$. For a treelike NON with fully interdependent, $q_{ij} = q_{ji} = 1$, we get

$$x_i = p_i \prod_{j=1}^{n} y_{ji}(1 - f_j), \tag{25}$$

$$y_{ji} = \frac{x_j}{y_{ij}(1 - f_i)}, \tag{26}$$

$$P_\infty = P_{\infty,i} = \prod_{i=1}^{n} p_i g_i(x_i) = \prod_{i=1}^{n} p_i (1 - f_i), \tag{27}$$

$$x_i = \frac{P_{\infty,i}}{g_i(x_i)} = \frac{\prod_{j=1}^{n} p_j (1 - f_j)}{1 - f_i}, \tag{28}$$

where $f_i = \tilde{G}_{i1}(1 - x_i(1 - f_i)) = e^{-\bar{k}_i u_i^2 \prod_{j=1}^{n} p_j (1 - f_j)}, i = 1, 2, \ldots, n$.

When $p_i = p$ and $\bar{k}_i = \bar{k}$, we have $u_i = u$, and $f_i = f$ satisfies

$$f = e^{-\bar{k}u^2 p^n (1-f)^n}. \tag{29}$$

Then, from Eq. (27), we obtain

$$P_\infty = \prod_{i=1}^{n} p_i g_i(x_i) = p^n (1-f)^n. \tag{30}$$

From Eqs. (29) and (30), we can get

$$f = e^{-\bar{k}u^2 P_\infty}. \tag{31}$$

The size of the mutual giant component for all value of p, $\bar{k}$, u and n becomes

$$P_\infty = p^n (1 - e^{-\bar{k}u^2 P_\infty})^n, \tag{32}$$

where

$$u = \frac{W(p\bar{k}e^{\bar{k}})}{\bar{k}}. \tag{33}$$

Let $r = e^{-p^n u^2 (1-f)^n}$, then $r = f^{\frac{1}{\bar{k}}}, f = r^{\bar{k}}$, and Eq. (29) becomes $r = e^{-p^n u^2 (1-r^{\bar{k}})^n}$. Then the critical case corresponds to the tangential condition: $1 = \frac{d}{dr} e^{-p^n u^2 (1-r^{\bar{k}})^n}$, and the critical value of $p = p_c$ can be found,

$$p_c^n = \frac{1}{nf(1-f)^{n-1}\bar{k}u^2}. \tag{34}$$

Substituting Eq. (34) into Eq. (29), we can get the critical value of $f = f_c$

$$f_c = e^{\frac{f_c - 1}{n f_c}}, \tag{35}$$

where the solution can be expressed in terms of the Lambert function $W(x)$, $f_c = -[nW(-\frac{1}{n}e^{-\frac{1}{n}})]^{-1}$.

We support our analytical results, Eqs. (32)–(33) and Eqs. (9)–(15) and (23)–(24) by simulating the giant component of the networks as a function of p as shown in Figs. 4(a) and 4(b). The results of the theory agree well with the simulations. The average convergence stage $\langle \tau \rangle$ as a function of p for several $\bar{k}$ are simulated and shown in Figs. 4(c) and 4(d). From Figs. 4(c) and 4(d) the first order transition threshold can be easily identified by the sharp peak, which represents the divergence of the number of cascading stages [44].

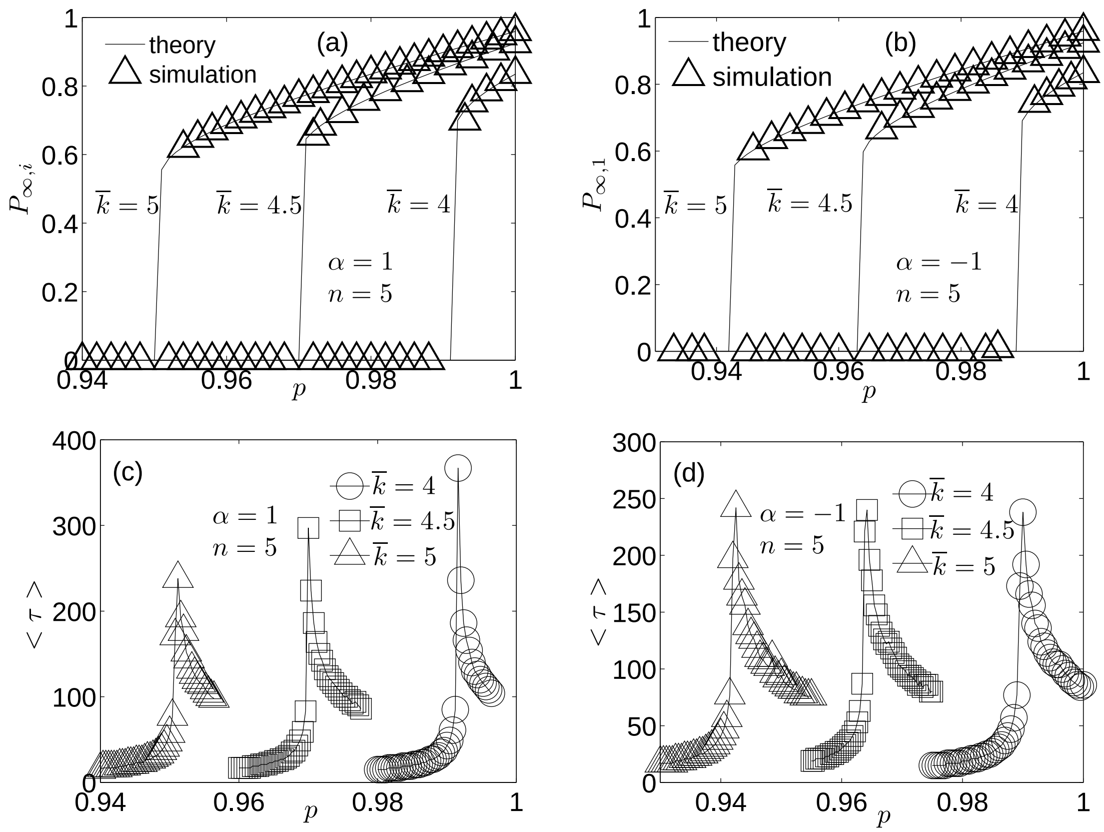

FIG. 4: (a)-(b) Comparison between theory and simulations of the giant component of network i, $P_{\infty,i}$ of treelike NON system formed of five ER networks as function of p. (c)-(d) Simulations of number of cascading stages $<\tau>$ as a function of p for different $\bar{k}$. In the simulations, $N_i = N = 10^4$ and the results are obtained by averaging over 50 realizations. The sharp peaks represent the location of p_c.

As an important characteristic of percolation theory, the critical threshold p_c is analyzed to study the robustness of a fully interdependent treelike NON in Figs. 5(a)–5(f). The threshold p_c as a function of α obtained from Eqs. (9)–(15) and (23)–(24) is shown in Fig. 5(a). Figure 5(a) shows that the critical fraction p_c continuously increases with α for a different number n of networks, which means that the NON becomes more vulnerable when the nodes with higher k have a higher probability of failure. Figures 5(b) and 5(c) show that the NON becomes less robust with increasing n. Moreover, for fixed n, p_c decreases when $\bar{k}$ increases as seen in Figs. 5(b)–5(f). Furthermore, for a fixed n, when $\bar{k}$ is smaller than the minimum average degree $\bar{k}_{\min}(n)$, $p_c = 1$. In this case ($\bar{k} \le \bar{k}_{\min}(n)$) the NON will collapse even if a single node fails. Using Eqs. (34), for $p_c = 1$, we find

$$\bar{k}_{\min}(n) = \frac{1}{n f_c (1-f_c)^{n-1}}, \tag{36}$$

which is independent of α (see Ref. [36] for $\alpha = 0$). In fact, when all n interdependent ER networks have a minimum average degree $\bar{k}_{\min}$, the ER networks are at critical condition, and the NON collapses even if a single node fails irrespective of whether it is high degree or low degree. This is an inherent property of interdependent networks and is not influenced by the probability that nodes are attacked.

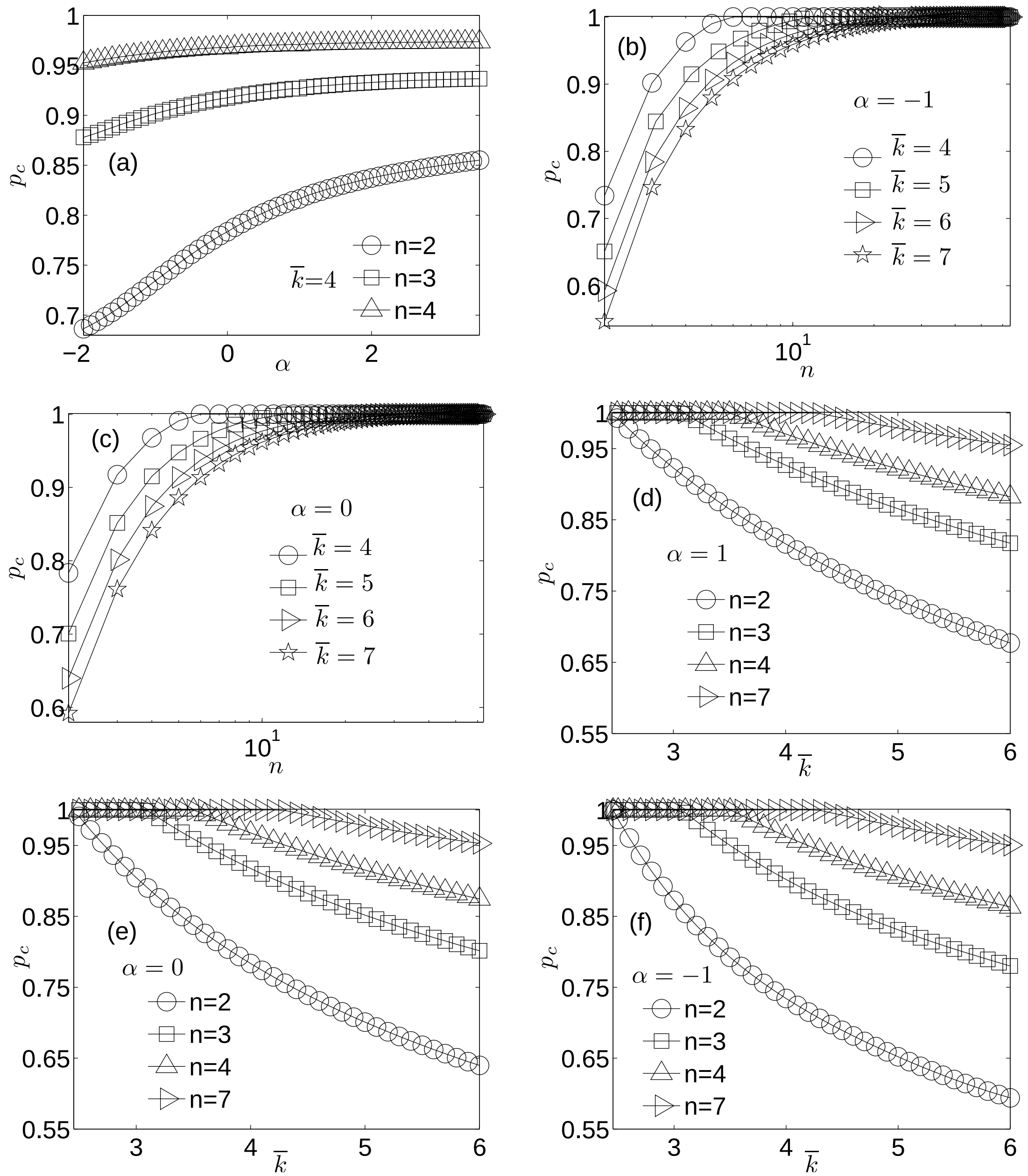

FIG. 5: (a) The critical threshold p_c as a function of α for different n. The critical threshold p_c as a function of n for (b) $\alpha = -1$ and (c) $\alpha = 0$ for several values of $\bar{k}$. (d)-(f) For $\alpha = 1, 0, -1$, p_c as a function of $\bar{k}$ for different n. Note that for $\bar{k} \leq \bar{k}_{\min}$, $p_c = 1$, i.e., the network collapse even if a single node is removed.

For the case of a treelike NON formed of n SF networks, $P_i(k) = \frac{(k+1)^{1-\lambda} - k^{1-\lambda}}{(M+1)^{1-\lambda} - m^{1-\lambda}}$ is the degree distribution of nodes of network i, where k, M and m are the degree, maximum, and minimum degree of network i respectively [4, 27, 31]. The critical thresholds p_c of the NON are obtained from Eqs. (9)–(15) and (23)–(24), and are presented in Fig. 6. Figure 6(a) shows that p_c for interdependent SF networks is nonzero for the entire range of α, which differs from that of a single SF network for which $p_c = 0$ [4, 5, 31, 35]. For fixed n, p_c increases when α increases since the NON becomes less robust and the high degree nodes are removed with a higher probability. Also, with the same probability of removing nodes (fixed α) the NON system becomes more vulnerable when the number of networks increases, as seen in Fig. 6. When lower degree nodes are removed with a higher probability, Fig. 6(b) shows that the NON becomes less robust with increasing λ. This is also illustrated for $\alpha = 0$ and 1 in Figs. 6(c) and 6(d), where p_c becomes larger with larger λ.

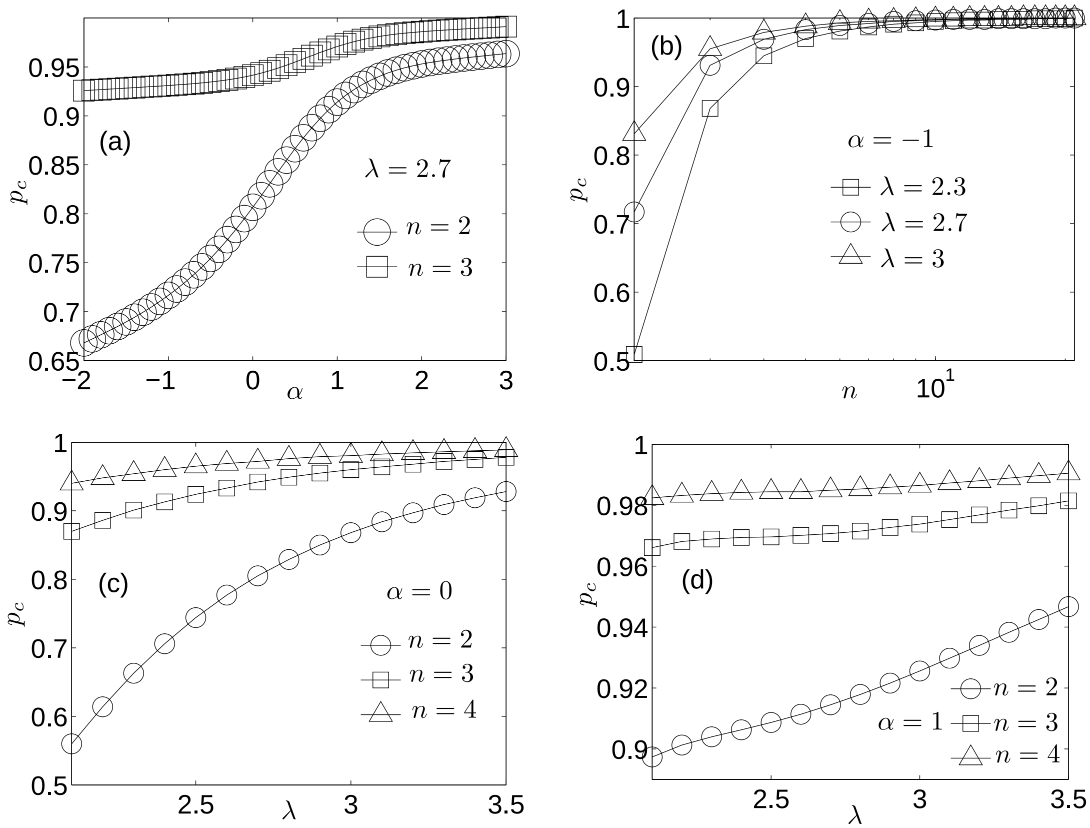

FIG. 6: (a) The critical threshold p_c as a function of α for NON composed of $n = 2$ and 3 SF networks with $\lambda = 2.7$. (b) The critical threshold p_c as a function of n for $\alpha = -1$ and different λ. (c)-(d) p_c as a function of λ for $\alpha = 0, 1$ and for different n respectively. The lowerest degree of the SF networks is taken to be $m = 2$.

IV. THE PARTIALLY INTERDEPENDENT STARLIKE NON

We now study the robustness of a starlike partially interdependent NON (see Fig. 7) using the analytical framework provided in Eqs. (23-24). Using the dynamics of cascading failures shown in Fig. 7, we compare the theoretical results of Eqs. (9)–(15) and (21)–(22) with the simulation results in Fig. 8 for a starlike NON formed of five ER networks.

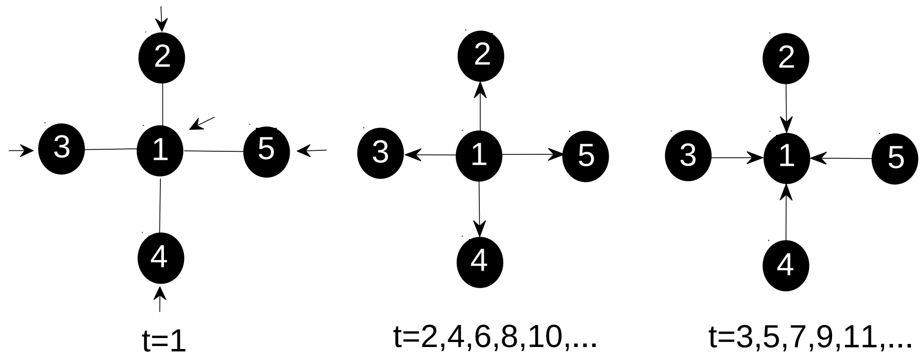

FIG. 7: The dynamic of cascading failures for starlike NON with five networks. Each node represents a network and the arrow (on the link) illustrates the damage spreading from one network to another network in the NON. At $t = 1$, we remove a fraction $1 - p_i$ of nodes in the five networks according to Eq. (1). Furthermore, non connected nodes in the networks are also removed. At $t = 2, 4, 6, ...$, the central network 1 spreads damage to the neighboring networks $2, 3, 4, 5$. At $t = 3, 5, 7, ...$, networks $2, 3, 4, 5$ spread damage to central networks 1. At last, the mutual giant component of the NON becomes stable and no further removal of nodes occurs.

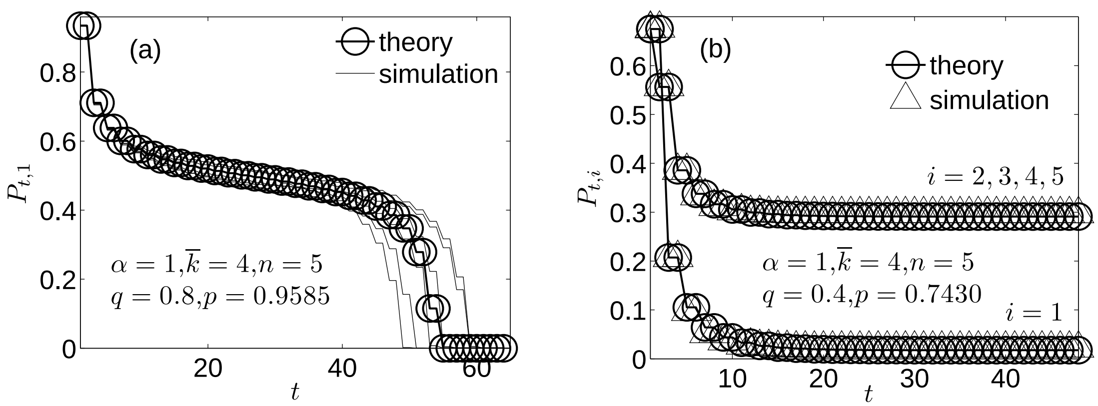

FIG. 8: For the case of $\alpha = 1$ which corresponds to the solvable "acquaintance immunization" strategy, we present theory and simulated results of the giant component of networks $P_{t,i}$, $i = 1, 2, 3, 4, 5$ during the t cascading failure for partially interdependent starlike NON composed of five ER networks are shown. (a) For each network in the NON, $\bar{k}_i = \bar{k} = 4$, $N_i = N = 10^4$, $q_{1i} = q_{i1} = 0.8$ and $p_i = p = 0.9585 < p_c = 0.9600$. (b) Theory and simulations of the giant component $P_{t,i}$ of networks as function of t with the same parameters as in Fig. 8(a) but for $p_i = p = 0.7430 > p_c = 0.7420$ and $q_{1i} = q_{i1} = 0.4$. The results of simulation are obtained by averaging over 50 realizations.

For $\bar{k}_i = \bar{k}, q_{i1} = q_{1i} = q, p_1 = p_2 = \cdots = p_n = p$ and $\alpha = 1$, by using the generating function of ER networks and Eqs. (16)–(20) and (23)–(24), we obtain

$$x_1 = p(1 - q + pqg_2(x_2))^{n-1} = p(1 - q + pq(1 - f_2))^{n-1}, \tag{37}$$

$$x_2 = p(1 - q(1 - y_{12}g_1(x_1))), \tag{38}$$

where

$$y_{12} = \frac{x_1}{1 - q(1 - y_{21}g_2(x_2))}. \tag{39}$$

From Eqs. (37)–(39), we obtain

$$x_2 = p[1 - q + pq(1 - f_1)(1 - q + pq(1 - f_2))^{n-2}], \tag{40}$$

where

$$f_1 = e^{\bar{k}u^2(f_1-1)x_1}, \tag{41}$$

$$f_2 = e^{\bar{k}u^2(f_2-1)x_2}, \tag{42}$$

$$u_i = u = \frac{W(p\bar{k}e^{\bar{k}})}{\bar{k}}. \tag{43}$$

Substituting Eqs. (37) and (40) into Eqs. (41) and (42), we obtain

$$f_1 = e^{\bar{k}pu^2(f_1-1)[1-q+pq(1-f_2)]^{n-1}}, \tag{44}$$

$$f_2 = e^{\bar{k}pu^2(f_2-1)[1-q+pq(1-f_1)(1-q+pq(1-f_2))^{n-2}]}. \tag{45}$$

Furthermore, the giant components of networks are

$$P_{\infty,1} = x_1 g_1(x_1) = p(1-f_1)[1-q+pq(1-f_2)]^{n-1} = -\frac{\ln f_1}{\bar{k}u^2}, \tag{46}$$

$$P_{\infty,2} = x_2 g_2(x_2) = p(1-f_2)[1-q+pq(1-f_1)(1-q+pq(1-f_2))^{n-2}] = -\frac{\ln f_2}{\bar{k}u^2}. \tag{47}$$

where f_1 and f_2 are obtained from Eqs. (44) and (45).

Figures 9(a) and 9(b) show an excellent agreement between the simulations of the giant components and the results in Eqs. (44) and (46). Furthermore, the central network, 1, shows a first and a second order percolation transition by regulating the coupling strength q as shown in Figs. 9(a) and 9(b). Figures 9(c) and 9(d) present simulation results of $\langle\tau\rangle$ as a function of p. The first order transition point p_c^I can be easily identified by the sharp peaks of $\langle\tau\rangle$.

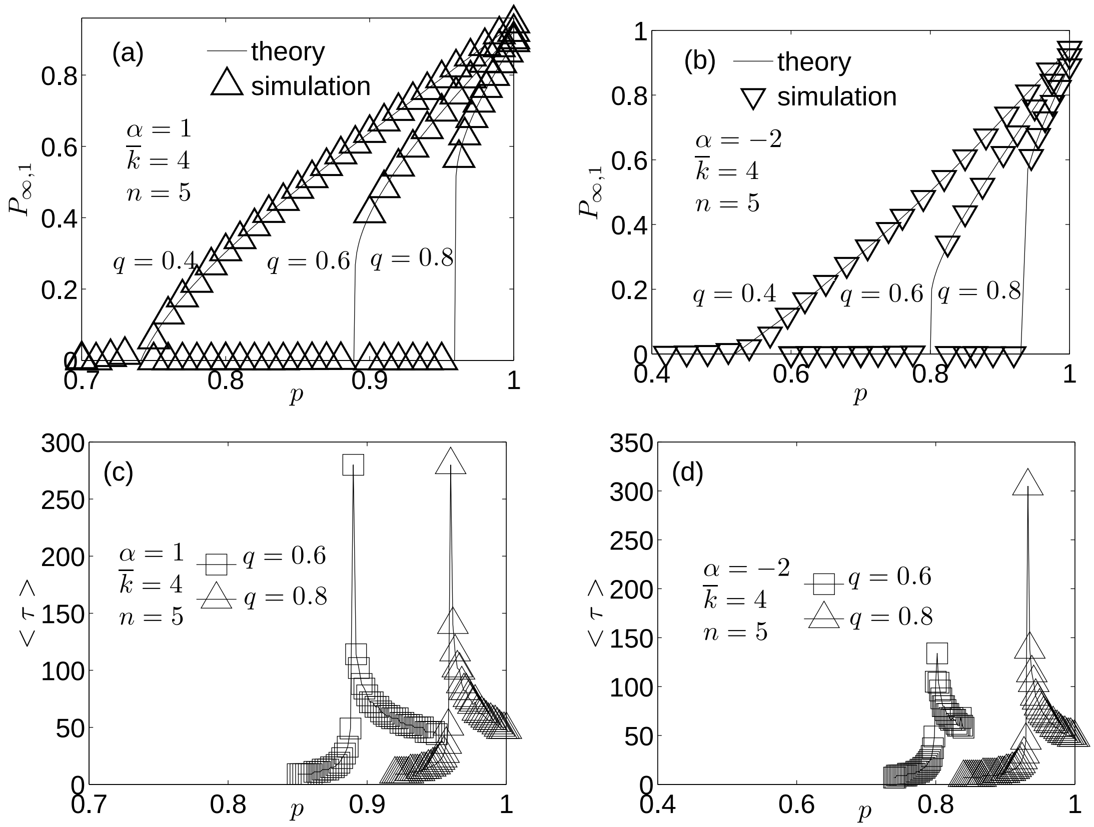

FIG. 9: (a) Comparison between simulation and theory for the giant component $P_{\infty,1}$ as a function of p for different q for starlike NON. For weak coupling strength $q = 0.4$ and $\alpha = 1$, the central network shows a second order phase transition, and for strong coupling strength $q = 0.6, 0.8$, the central network shows a first order phase transition. (b) Comparison between simulation and theory for the giant component $P_{\infty,1}$ with the same parameters in Fig. 9(a) but for $\alpha = -2$. (c)-(d) The average time at the convergence stage, $< \tau >$, as a function of p for different α and q. The results are obtained by averaging over 50 realizations for $N = 10^4$.

From Eqs. (44) and (45), we obtain

$$f_1 = 1 - \frac{\frac{\ln f_2}{\bar{k}pu^2(f_2-1)} + q - 1}{pq(1-q+pq(1-f_2))^{n-2}}, \tag{48}$$

$$f_2 = 1 - \frac{1}{pq}((\frac{\ln f_1}{\bar{k}pu^2(f_1-1)})^{\frac{1}{n-1}} + q - 1), \tag{49}$$

where $u = \frac{W(p\bar{k}e^{\bar{k}})}{\bar{k}}$.

From Eqs. (46) and (48)–(49), the critical threshold p_c as a function of n for different q is obtained and shown in Fig. 10(a). We see that for $\alpha = 1$ the central network becomes significantly more vulnerable with increasing n for the same coupling strength q. In addition, for the same n the central network becomes more robust when the coupling strength q is reduced. Figure 10(b) shows that using Eqs. (9)–(15) and (23)–(24) these conclusions can also be demonstrated for $\alpha = -1$. Figures 10(a) and 10(b) also show that the central network becomes more robust for the same coupling strength q and n when nodes of lower degree have a higher probability of failure. We also analyze p_c as a function of the average network degree for different α as shown in Figs. 10(c) and 10(d). Figure 10(c) shows that the central network becomes more fragile and finally collapses as the average degree is decreased for the same coupling strength. Comparing Figs. 10(c) and 10(d) we see that when nodes of lower degree have a higher failure probability ($\alpha = -1$), the central network is more robust for the same average degree and fixed coupling strength. Therefore, in the partially starlike NON, if we decrease the number of networks and the coupling strength and increase the average network degree, it will become more robust.

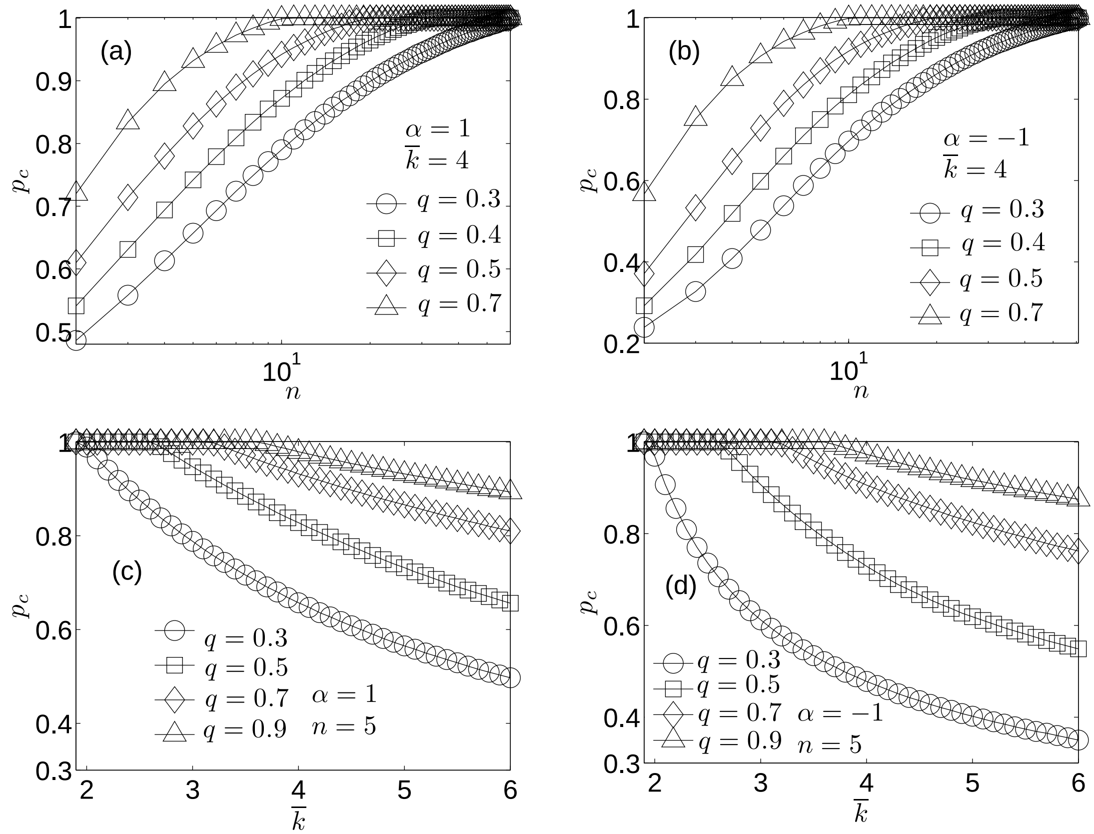

FIG. 10: Critical threshold p_c as a function of n for several q values and average degree $\bar{k}$. (a-b) p_c as a function of n for $\alpha = 1, -1$ and for different coupling strengths q. (c-d) p_c as a function of $\bar{k}$ for different coupling strengths, for $\alpha = 1, -1$.

From Eqs. (48) and (49), the second order transition threshold p_c^{II} can be obtained by taken the limit $f_1 \to 1$,

$$\frac{\ln f_2}{f_2 - 1} = \bar{k} p_c^{II} u^2 (1 - q), \tag{50}$$

$$f_2 = 1 - \frac{1}{p_c^{II} q}\left(\left(\frac{1}{\bar{k} p_c^{II} u^2}\right)^{\frac{1}{n-1}} + q - 1\right). \tag{51}$$

From Eq. (50), we get

$$f_2 = \frac{1}{-\bar{k} p_c^{II} u^2 (1 - q)} W\left(\frac{-\bar{k} p_c^{II} u^2 (1 - q)}{e^{(\bar{k} p_c^{II} u^2 (1-q))}}\right), \tag{52}$$

and by substituting Eq. (52) into Eq. (51), we obtain p_c^{II}.

Figures 9(a) and 9(b) show that changing the coupling strength can change the phase transition from first-order to second-order. Thus we examine the relation between p_c and the fraction q of interdependent nodes when, using Eq. (1), $1 - p$ nodes are removed from the central network. Figure 11 shows that for fixed average degree $\bar{k}$ and α, there exists a critical threshold q^c of coupling strength that changes for different n. When the coupling is strong ($q > q^c$), the NON exhibits a first order phase transition at a transition threshold p_c^I at which the giant component abruptly approaches zero. When the coupling is weak ($q < q^c$), the NON shows a second order phase transition at transition threshold p_c^{II} at which the giant component smoothly changes from a finite value to zero. Note that the phase transition lines of both p_c^I and p_c^{II} converge at $q = 0$ for different n, which is the limit of a single network i.e., the classical percolation of a single network under targeted attack [35]. Figure 11 also shows that when $q = 0$ the central network becomes more vulnerable when the higher-k nodes have a higher failure probability.

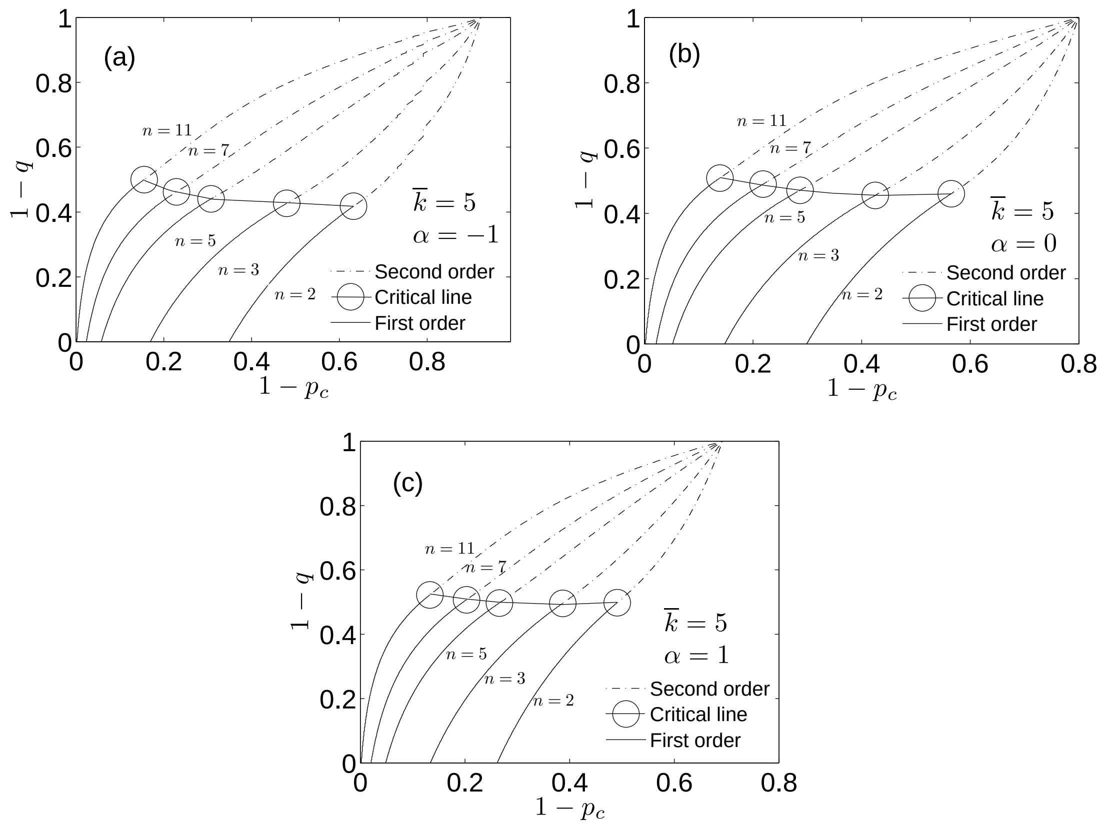

FIG. 11: The relation between $1 - q$ and $1 - p_c$ for different n for (a) $\alpha = -1$, (b) $\alpha = 0$ and (c) $\alpha = 1$. The curves connecting the circles show the critical lines of q_c, below which the system shows a first order phase transition and above a second order phase transition.

In summary, by introducing a new targeted attack probability function which avoids singularities for degree $k_i = 0$ and $\alpha < 0$, we address the question of robustness of two types of NON under targeted attack: (i) a treelike network comprised of n fully interdependent Erdős-Rényi or Scale-free networks, and (ii) a starlike network comprised of n partially interdependent Erdős-Rényi networks. For the treelike NON of n fully interdependent ER networks under targeted attack, we find an exact solution for the giant component P_∞ when $\alpha = 1$, as a function of $1 - p$ initial fraction of removed nodes from each of the n networks. Analytical solutions are found for the critical fraction p_c that leads to the fragmentation of n interdependent networks, and the minimum average $\bar{k}_{\rm min}$ at which the NON becomes so vulnerable that the failure of a single node will bring down the entire system. For different values of α, we present the numerical solution for the giant component and the critical fraction p_c as a function of $\bar{k}$ and n for ER and SF networks. When a treelike network of n fully interdependent ER networks is under targeted attack, increasing the average degree of each of the networks, increasing the probability of targeted attacks on lower degree nodes, and decreasing the number of networks will make the tree more robust. For a starlike network of n partially interdependent ER networks we can use these same strategies, but we can also decrease the coupling strength to make the starlike NON more robust. We also find the critical threshold q^c for different α values below which the system exhibits a second order transition and above which the system shows a first order transition. When the coupling strength $q = 0$, the NON system become n single and independent n networks. When α is increasing or decreasing, the network becomes more vulnerable or robust, which coincides with the classic percolation of a single network to targeted network.

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[1] D. J. Watts and S. H. Strogatz, Nature **393**, 440 (1998).
[2] A. -L. Barabási and R. Albert, Science **286**, 509(1999).
[3] R. Albert and A. -L. Barabasi, Rev. Mod. Phys. **74**, 47(2002).
[4] R. Cohen, K. Erez, D. Ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000); Phys. Rev. Lett. 86, 3682 (2001).
[5] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).
[6] S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW(Physics)* (Oxford Univ. Press, New York, 2003).
[7] R. P. Satorras and A.Vespignani, *Evolution and Structure of the Internet: A Statistical Physics Approach* (Cam-bridge Univ. Press, England, 2006).
[8] A. Bashan et al, Nature Comm. **3**, 702 (2012).
[9] C. Song, S. Havlin, and H. A. Makse, Nature **433**, 392 (2005); Nature Physics, 2, 275 (2006).
[10] R. Cohen and S. Havlin, *Complex Networks: Structure, Robustness and Function* (Cambridge Univ. Press, England, 2010).
[11] G. Caldarelli and A. Vespignani, Large scale Structure and Dynamics of Complex Webs (World Scientific, Singapore, 2007).
[12] M. E. J. Newman, *Networks: An Introduction* (Oxford Univ. Press, New York, 2010).
[13] R. Cohen et al, Phys. Rev. Lett. **91**, 247901 (2003).
[14] S. Son, G. Bizhani, C. Christensen, P. Grassberger and M. Paczuski, Europhys. Lett. **84**, 16006 (2012).
[15] Z. Wang, A. Szolnoki, and M. Perc, Europhys. Lett. **97**, 48001 (2012).
[16] G. J. Baxter et al., Phys. Rev. Lett. **109**, 248701 (2012).
[17] Z. Wang, A. Szolnoki, and M. Perc, Scientific Reports **3**, 1183 (2013).
[18] S. Amina, G. A. Schwartzb, and S. S. Sastryb, Automatica **49**, 1 (2013).
[19] J. F. Castet, and J. H. Saleh, PLOS one **8**, e60402 (2013).
[20] C. D. Brummitta, R. M. DSouzab, and E. A. Leichtf, Proc. Natl. Acad. Sci. **109**, E680 (2012).
[21] C. M. Schneider,A.A.Moreira, J. S.Andrade, Jr., S. Havlin, and H. J. Herrmann, Proc. Natl. Acad. Sci. **108**, 3838 (2011).
[22] J. C. Laprie, K. Kanoun and M. Kaniche, SAFECOMP **4680**, 54 (2007).
[23] J. Goldenberg et al, Nature Physics 184, 1 (2005).
[24] P. Pederson, D. Dudenhoeffer, S. Hartley and M. Permann, Technical Report INL/EXT-06-11464.
[25] V. Rosato, Int. J. Crit. Infrastruct. **4**, 63 (2008).
[26] S. Rinaldi et al., Int. J. Crit. Infrastruct. **21**, 11 (2001).
[27] S. V. Buldyrev et al., Nature **464**, 1025 (2010).
[28] R. Parshani, S. V. Buldyrev, and S. Havlin, Phys. Rev. Lett. **105**, 048701 (2010).
[29] J. Gao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, Phys. Rev. Lett. **107**, 195701 (2011).
[30] K. Zhao, G. Bianconi, Percolation on interacting antagonistic networks, arXiv:1210.7498 (2012).
[31] X. Q. Huang et al., Phys. Rev. E **83**, 065101(R) (2011).
[32] G. Dong, J. Gao, L. Tian, R. Duo, and Y. He, Phys. Rev. E **85**, 016112 (2012).
[33] I. Simonsen, L. Buzna, K. Peters, S. Bornholdt, and D. Helbing, Phys. Rev. Lett. **100**, 218701 (2008).
[34] A. E. Motter and Ying-Cheng Lai, Phys. Rev. E **66**, 065102 (2002).
[35] L. K. Gallos, R. Cohen, P. Argyrakis, A. Bunde, and S. Havlin Phys. Rev. Lett. **94**, 188701 (2005).
[36] J. Gao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, Phys. Rev. E **85**, 066134 (2012).
[37] S. Havlin et al., arXiv:1012.0206v1.
[38] J. Shao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, Phys. Rev. E **83**, 036116 (2011).
[39] Y. Hu, B. Ksherim, R. Cohen, and S. Havlin, Phys. Rev. E **84**, 066116 (2011).
[40] J. Gao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, Nature physics **8**, 40 (2012).
[41] Q. Li, L. A. Braunstein, S. Havlin, and H. E. Stanley, Phys. Rev. E **84**, 06601 (2011).
[42] M.E.J. Newman, Phys. Rev. E **66**, 016128 (2002).
[43] Shao, J., Buldyrev, S. V., Braunstein, L. A., Havlin, S. and H. E. Stanley, Phys. Rev. E **80**, 036105 (2009).
[44] R. Parshani, S.V. Buldyrev, S. Havlin, PNAS **108**, 1007 (2011).