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Phys. Rev. E **87**, 052127 — Published 22 May 2013

DOI: 10.1103/PhysRevE.87.052127

# Weakly Explosive Percolation in Directed Networks

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Percolation, the formation of a macroscopic connected component, is a key feature in the description of complex networks. The dynamical properties of a variety of systems can be understood in terms of percolation, including the robustness of power grids and information networks, the spreading of epidemics and forest fires, and the stability of gene regulatory networks. Recent studies have shown that if network edges are added "competitively" in undirected networks, the onset of percolation is abrupt or "explosive." The unusual qualitative features of this phase transition have been the subject of much recent attention. Here we generalize this previously studied network growth process from undirected networks to directed networks and use finite-size scaling theory to find several scaling exponents. We find that this process is also characterized by a very rapid growth in the giant component, but that this growth is not as sudden as in undirected networks.

PACS numbers: 64.60.ah, 64.60.aq, 05.40.-a, 89.75.Da
Keywords: Explosive percolation, directed networks, complex networks, phase transitions.

## I. INTRODUCTION

A complex network is a collection of nodes, along with a set of edges which join pairs of nodes. In an undirected network, in which each edge may be traversed in both directions, the network can be divided into distinct connected components. As edges are successively added to a large undirected network, it may transition from a non-percolating phase, in which every connected component is microscopic, to a percolating phase, in which there is a single "giant" component which contains a macroscopic fraction of the nodes in the network [1]. The fraction of nodes in the giant component is the order parameter for the percolation phase transition.

The percolation phase transition on undirected networks was independently discovered by Solomonoff and Rapoport [2] and Erdős and Rényi [3] and later generalized by other authors [1, 4]. The network growth process studied by Erdős and Rényi, now the prototypical example of network percolation, may be characterized as follows. The network initially consists of $N \gg 1$ nodes and no edges. Then, on each successive step of the growth process, a pair of nodes is selected randomly and an undirected edge is added between them. The size of the largest connected component is recorded and the process is repeated. The percolation phase transition for networks grown in this manner is second-order (continuous) in the number of edges in the network. However, recent work by Achlioptas et al. demonstrated that simple modifications to this growth algorithm can induce surprisingly different behavior in the growth of the giant component [5]. In particular, they found that introducing "edge competition" during network growth results in "explosive percolation," a delayed, seemingly first-order (discontinuous) transition.

The network growth process proposed by Achlioptas et al. is designed to inhibit the formation of large connected components. At each step, two random candidate edges are considered, with the intention of selecting only one of them for addition to the network. If one of the edges connects two nodes in the same component, it is selected automatically because its addition would not cause any component to grow. If the addition of either edge would connect two distinct components, the product of the sizes of these two components is compared, and only the edge with the smaller product is added to the network [6].

Networks grown in this fashion percolate much later than Erdős-Rényi networks; however, when a giant component eventually forms, it grows extremely rapidly. Based on numerical simulations, Achlioptas et al. conjectured that the phase transition is first-order, but it has now been shown that the Achlioptas process actually produces a second-order transition [7–10]. The abrupt growth observed in numerical experiments is due to a small but positive critical exponent for the growth of the order parameter, along with strong finite-size effects which diminish only very slowly as $N \to \infty$. In spite of this, the Achlioptas process continues to attract considerable interest because, at network sizes that are typical in applications, these finite-size effects give the percolation phase transition an "effectively" first-order appearance that is qualitatively different from that of traditional percolation problems (see Fig. 1). It has also spurred interest in other models which exhibit abrupt phase transitions, including Kuramoto [11] and Ising [12] models, as well as other modified percolation processes [13–21], many of which are believed to exhibit genuine first-order transitions.

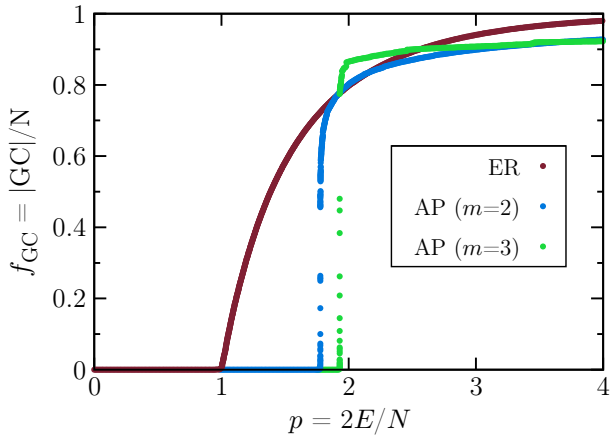


FIG. 1 (color online). The growth of $f_{GC}$, the fraction of nodes in the giant component of an undirected network, for three individual networks with $N = 2^{23}$. The growth process is repeated using the Erdős-Rényi growth process (red or dark gray), the Achlioptas process (blue or medium gray), and a modified Achlioptas process in which three candidate edges, rather than two, are used at each network growth step (green or light gray).

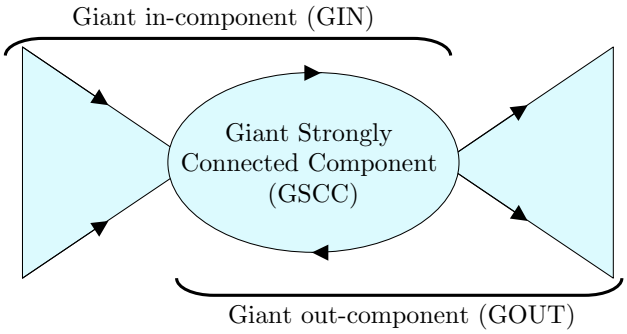


FIG. 2 (color online). An illustration of the "bow-tie" structure of the giant component in a directed network above the percolation threshold (see text).

In this paper, we extend the Achlioptas process to *directed* networks. In a directed network, each edge can only be traversed in one direction. Directed networks are widely used to model gene regulation, food webs, neural networks, citation networks, the world-wide web, and other systems. However, the existing literature on explosive percolation is exclusively focused on undirected networks. Here, we explore a generalization of the Achlioptas process to directed networks and study the scaling properties of this process. We find that competitive edge percolation on directed networks shares some of the qualitative features of the Achlioptas process on undirected networks, but these features are far less pronounced.

Because many modified percolation models exhibit unusual phase transitions, models with significantly different properties have all been labeled explosive in the literature. For the purposes of this paper, the adjective "explosive" will refer to the unusual features which distinguish the critical behavior of the Achlioptas process from both ordinary percolation as well as truly discontinuous models. These features are discussed further in the Results. We have termed the behavior of our directed-network model "weakly explosive" because it shares some of these qualities, but only to a limited extent.

## II. METHODS

In order to define an Achlioptas-like process on directed networks, we first need to define connectedness on a directed network. Although there is a single unambiguous definition of a "connected component" for undirected networks, there are multiple related definitions for directed networks [1]. In the algorithms discussed below, we will study four different types of structures to which a node may belong. In the giant component, these structures are commonly illustrated with the well-known "bow-tie diagram" (Fig. 2) [22]. First, the in-component of a node $i$, IN($i$), is the set of all nodes which have paths to $i$. Likewise, the out-component of $i$, OUT($i$), is the set of all nodes which can be reached on paths from $i$. Next, the strongly connected component of $i$, SCC($i$), is the intersection of IN($i$) and OUT($i$). Finally, we define the full bow-tie, BT($i$), to be the union of IN($i$) and OUT($i$) [23]. Each of these structures is in some sense analogous to the connected component in undirected networks. This comparison extends to the percolation transition in the directed Erdős-Rényi process, in which directed edges are successively added between randomly selected, unconnected pairs of nodes. At the critical point, a giant strongly connected component (GSCC), giant in-component (GIN), and giant out-component (GOUT) form simultaneously [1], comprising the giant bow-tie (GBT). For convenience below, we will use $G$ to denote any one of the parts of the giant component of a directed network (GSCC, GIN, GOUT, or GBT), or for the giant component (GC) of an undirected network. See Table I for a list of acronyms.

Now, we describe a new network growth processes on directed networks. We will refer to this process as the directed competition process (DCP) to distinguish it from the Achlioptas process (AP), the Erdős-Rényi process (ER), and the directed Erdős-Rényi process (DER). It also consists of repeatedly choosing two random directed candidate edges $i_1 \to j_1$ and $i_2 \to j_2$ from the set of all distinct unoccupied edges, then selecting one for addition to the network. As in the Achlioptas process, we automatically select one of the edges if that edge is redundant to the connectedness of the network, i.e., if there is already a path from $i$ to $j$. Otherwise, we select the edge for which $|\text{IN}(i)| \cdot |\text{OUT}(j)|$ is minimized. Here, the vertical bars denote cardinality, so $|\text{IN}(i)|$ refers to the number of nodes in IN($i$). As in [24, 25], we also consider generalizations of both AP and DCP in which $m$ edges (rather than two edges) are chosen for consideration at each step in the growth process, and we will discuss results for both

| Acronym | Definition |
|---|---|
| ER | Erdős-Rényi process |
| AP | Achlioptas process |
| DER | Directed Erdős-Rényi process |
| DCP | Directed competition process |
| GC | Giant component |
| GSCC | Giant strongly connected component |
| GIN | Giant in-component |
| GOUT | Giant out-component |
| GBT | Giant bow-tie |

TABLE I. Acronyms commonly used in the text.

$m = 2$ and $m = 3$. Note that the $m = 1$ case of AP corresponds to ER, and the $m = 1$ case of DCP corresponds to DER. Finally, in order to emphasize that DCP is a generalization of AP, we also note that the two processes are identical when applied to an undirected network.

The DCP edge selection rule may also be motivated by noting that it minimizes the "throughput" which is created by the addition of each edge in a way which is analogous to the Achlioptas product rule. More formally, let $P_{ij}$ indicate whether or not there is a path from $i$ to $j$, i.e., $P_{ij} = 1$ if there is such a path and $P_{ij} = 0$ if there is not. The throughput of the network can be defined as $T = \langle P \rangle$, where the average is taken over all node pairs $i$ and $j$ ($i \neq j$). Well below the percolation threshold, when there are few paths from nodes in IN($i$) to nodes in OUT($j$), adding an edge from $i$ to $j$ on average increases $T$ by approximately $|\mathrm{IN}(i)| \cdot |\mathrm{OUT}(j)|/N^2$. Similarly, in the Achlioptas process for an undirected network, the change in $T$ from the addition of a single edge to a network well below the percolation threshold is approximately $2|\mathrm{C}(i)| \cdot |\mathrm{C}(j)|/N^2$, where C($i$) and C($j$) are the components to which $i$ and $j$ belong. Thus, both rules may be construed as minimizing $T$ early in the network growth process. This, in turn, leads to an explosive phase transition by creating what has been termed a "powder keg" [26] of mesoscopic components which "ignites" at the critical point, when edge competition can no longer prevent them from merging.

For the order parameter of each phase transition, we will use the normalized size $f_G$ of a giant component,

$$f_G = \frac{|G|}{N}. \tag{1}$$

We define the GSCC to be the largest strongly connected component in the network, the GIN and GOUT to be its in- and out-components, and the GBT to be the union of the two [29]. For the tuning parameter, we will use the average degree of the network, $p$. For undirected networks, $p = 2E/N$, whereas for directed networks, $p = E/N$ [27]. Note that, for undirected networks, our use of $p$ as the tuning parameter differs slightly from the usual convention of using $E/N$ as a tuning parameter. Our use of the average degree is motivated by the observation that both undirected and directed Erdős-Rényi networks percolate at the same average degree ($p_c = 1$), so $p$ is a natural scale for comparison between the directed and undirected cases.

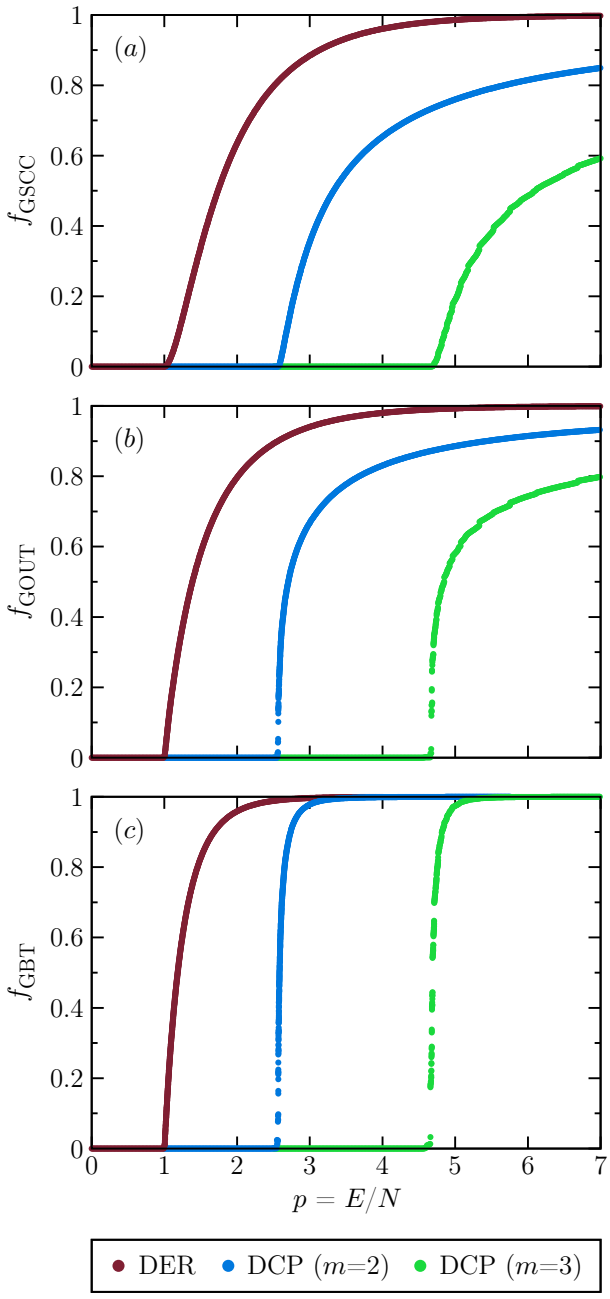


FIG. 3 (color online). The formation of ($a$) the giant strongly connected component, ($b$) the giant out-component, and ($c$) the giant bowtie in a directed network with $N = 2^{23}$. In each panel, the results for the directed Erdős-Rényi process (red or dark gray) are compared to those for the directed competition process using either $m = 2$ (blue or medium gray) or $m = 3$ (green or light gray). Results for GIN are omitted due to symmetry with GOUT.

| Network Growth Rule | | $p_c$ | $\theta$ | Component | $\beta$ | $\eta$ | $\lambda$ |
|---|---|---|---|---|---|---|---|
| $m = 1$ | ER | 1 | $1/3$ | GC | 1 | 0.328(7) | 0.30(9) |
| | DER | 1 | $1/3$ | GOUT | 1 | 0.329(3) | 0.31(9) |
| | | | | GSCC | 2 | 0.64(7) | 0.50(5) |
| $m = 2$ | AP | 1.7769(8) | 0.5(0) | GC | 0.0861(5) | 0.0645(5) | —— |
| | DCP | 2.565(9) | 0.44(1) | GOUT | 0.34(5) | 0.14(1) | 0.12(9) |
| | | | | GSCC | 1.2(9) | 0.55(8) | 0.53(3) |
| $m = 3$ | AP | 1.92(9) | 0.50(1) | GC | 0.03(0) | 0.020(7) | —— |
| | DCP | 4.86(1) | 0.42(7) | GOUT | 0.30(0) | 0.10(5) | 0.09(3) |
| | | | | GSCC | 1.(4) | 0.40(7) | 0.4(9) |

TABLE II. Critical exponents for each process (see text). For ER and DER, $p_c$, $\theta$, and $\beta$ are well-known exact results (see, for example, [3] and [27]). For AP with $m = 2$, we reproduce $p_c$, $\theta$, and $\beta$ from [8] and $\eta$ from [28]; refer to [8] for additional comments about the interpretation of $\theta$. All other exponents listed above are derived from our numerical simulations, as described below. Due to symmetry, results for GIN are identical to those for GOUT, and results for GBT are not listed because, in most cases, they are similar to those for GOUT.

Computationally, percolation simulations are more time-intensive for directed networks than undirected networks. While only $\mathcal{O}(N)$ operations are needed to simulate an entire network growth process in an undirected network [30], a naïve algorithm for competitive edge percolation in a directed network would require at least $\mathcal{O}(N^2)$ operations, because there are $\mathcal{O}(N)$ edge additions, between each of which several processes with up to $\mathcal{O}(N)$ steps must occur. These processes include checking for a path from $i$ to $j$ for each prospective edge $i \to j$, finding IN($i$) and OUT($j$), and decomposing the network into strongly connected components [31]. In order to improve computational performance, we track each part of the giant component during the network growth process and use knowledge of the giant component to speed up or eliminate the first two processes. For example, if $i$ is in GIN and $j$ is in GOUT, checking for a path from $i$ to $j$ is unnecessary because one must exist. Additionally, we report results only for the giant component, not the distribution of other component sizes, to avoid the third process. This results in an algorithm which scales approximately as $\mathcal{O}(N^{1.5})$, where most of the time is spent in the critical region where more than one macroscopic or near-macroscopic component exists. This improvement enables the simulation of networks with significantly larger $N$ than would otherwise be feasible.

## III. RESULTS

Plots of the order parameters versus $p$ are shown in Fig. 3 for large-$N$ single-network realizations of the DER and DCP growth processes. When edge competition is present, the emergence of all four parts of the giant component are delayed, and the GOUT and GBT display sudden growth at the critical point which is qualitatively similar to (though less marked than) that of the Achlioptas process (Fig. 1). By symmetry, results for GIN are the same as those for GOUT. In order to make quantitative comparisons between DCP and AP, we measure several scaling exponents which can be used to characterize the features of explosive percolation [8–10, 28, 32]. In fact, the Achlioptas process is striking precisely because these exponents are small (see Table II), but it is continuous because they are nonzero.

The first such measure is the critical exponent $\beta$, defined by

$$\langle f \rangle \sim (p - p_c)^\beta \tag{2}$$

as $p \to p_c$ from above, for networks in the thermodynamic limit $N \to \infty$. The average $\langle \cdot \rangle$ is taken over the ensemble of grown networks. Clearly, $\beta > 0$ indicates a continuous transition, and it has been observed that $0 < \beta \ll 1$ for AP [8, 9]. Next, we report another exponent $\eta$, defined by

$$\langle \max(\Delta f) \rangle \sim N^{-\eta}, \tag{3}$$

where $\max(\Delta f)$ is the largest jump in $f$ upon the addition of a single edge during a network growth process. In a discontinuous phase transition, the maximum jump would approach a nonzero constant as $N \to \infty$, corresponding to $\eta = 0$, but $\eta$ has also been observed to be small and positive for AP [10, 28].

Finally, we introduce a third scaling exponent $\lambda$, defined by

$$\max_p \left( \mathrm{Var}[f] \right) \sim N^{-\lambda} \tag{4}$$

for sufficiently large $N$. This is motivated by the observation in [8] that, for the Achlioptas process, the maximum variance of $f$ initially increases as $N$ grows, then begins to decrease very slowly when $N$ is extremely large. This is related to other unusual finite-size effects in AP; see [8] for a thorough discussion. In a continuous transition, we expect that $\mathrm{Var}[f] \to 0$ for all $p$ in the thermodynamic limit, so $\lambda > 0$. Moreover, a small value of $\lambda$ indicates that for finite $N$, there may be large changes in $f$ near the critical point.

These observations suggest the use of the following descriptions for the critical behavior of percolation models for large but finite networks. However, we emphasize that these categories are merely useful heuristics for describing qualitative behavior, rather than precise definitions.

- Discontinuous: Cases in which $\beta = 0$, $\eta = 0$, and $\lambda = 0$.
- Explosive: Cases in which $0 < \beta \ll 1$, $0 < \eta \ll 1$, and $0 < \lambda \ll 1$.
- Weakly explosive: Intermediate cases which cannot be clearly designated as either "explosive" or "ordinary."
- Ordinary: Cases in which $\beta$ is on the order of 1 and $\eta$ and $\lambda$ are not small. (In practice, a natural standard for comparison is ER, in which both $\eta$ and $\lambda$ are approximately $1/3$.)

In order to avoid confusion, we note that our terminology is not directly related to the language of [10], which distinguishes between "strongly" and "weakly" discontinuous transitions [33].

Now we turn to the numerical estimation of the critical exponents. Both $\eta$ and $\lambda$ may be determined by a straightforward fit to a power law using a weighted sum of squares; see Figs. 4 and 5. The critical exponent $\beta$, as well as the critical point $p_c$, are more difficult to estimate. To do this we analyze the finite-size scaling properties of the system. Sufficiently close to the critical point of a continuous phase transition, the order parameter $f$ is hypothesized to obey the finite-size scaling relation

$$\langle f \rangle = (p - p_c)^{\beta} g\left(N^{\theta}(p - p_c)\right), \tag{5}$$

where $\theta$ determines the scaling of the width of the critical region and $g$ is a universal scaling function [8, 34]. This may be written in the equivalent form

$$\langle f \rangle = N^{-\beta\theta} h\left(N^{\theta}(p - p_c)\right), \tag{6}$$

where $h(z) = z^{\beta} g(z)$ is another universal scaling function.

Unlike $g(z)$, $h(z)$ is not singular at $z = 0$ [32, 34]. Therefore, Eq. (6) may be interpreted by saying that plots of $\langle f \rangle$ versus $z = N^{\theta}(p - p_c)$ for various values of $N$ will all collapse, when appropriately scaled, onto $h(z)$, when $z$ is near 0 (i.e., $p \approx p_c$). We choose $\beta$, $\theta$, and $p_c$ to optimize this data collapse; see Fig. 6. Specifically,

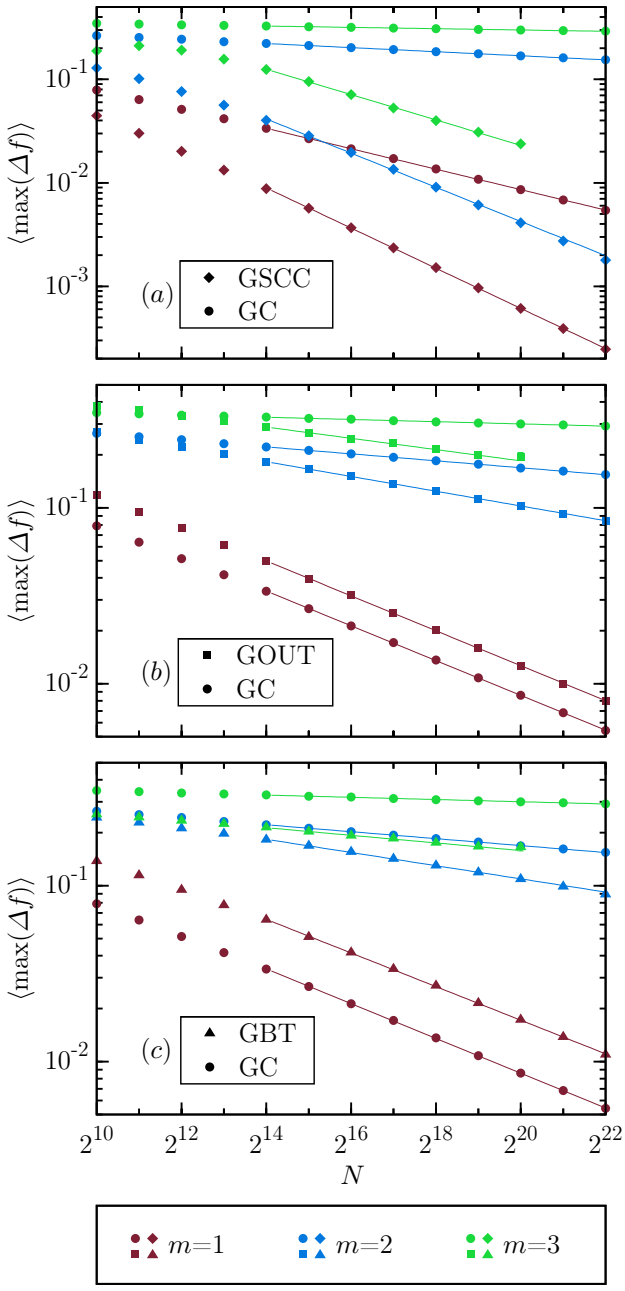


FIG. 4 (color online). Scaling of the maximum jump in the each part of the giant component of directed and undirected networks as a function of $N$. The results for ($a$) the GSCC (diamonds), ($b$) the GOUT (squares), and ($c$) the GBT (triangles) are compared in each panel to the results for undirected networks (circles). In addition, ER and DER (red or dark gray) are compared to AP and DCP with $m = 2$ (blue or medium gray) and $m = 3$ (green or light gray). Lines are power-law fits, whose slopes are given as $\eta$ in Table II. Each point is averaged over many network growth trials (50 to 10,000, depending on $m$ and $N$). Error bars (one standard deviation of the mean) are smaller than the point size for all points.

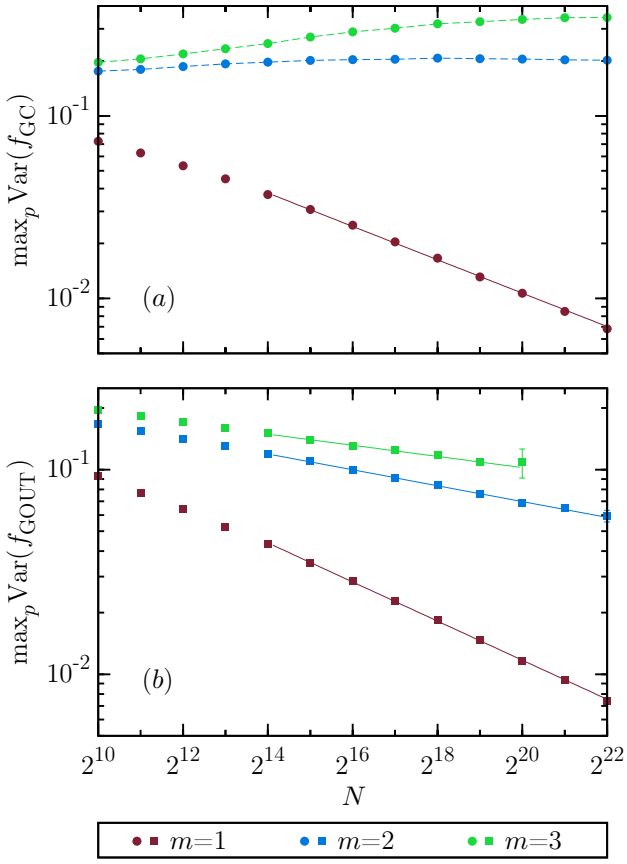


FIG. 5 (color online). Scaling of the largest variance in (*a*) $f_{\mathrm{GC}}$ (circles) and (*b*) $f_{\mathrm{GOUT}}$ (squares) as a function of $N$, for $m = 1$ (red or dark gray), $m = 2$ (blue or medium gray), and $m = 3$ (green or light gray), using data from the same simulations as in Fig. 4. Solid lines are power-law fits whose slopes are given as $\lambda$ in Table II; dashed lines merely connect the data points to guide the eye of the reader. One unusual feature of AP is that the maximum variance of $f$ increases with $N$, for $N$ not too large (see text), but then eventually decreases; DCP does not share this feature.

we choose $\beta$, $\theta$, and $p_c$ to minimize the function

$$V(\beta, \theta, p_c) = \frac{1}{\Delta z} \int_{-\Delta z}^{\Delta z} \mathrm{Var}_N \left[ N^{\beta\theta} \langle f(z, N) \rangle \right] dz. \quad (7)$$

For further details, see [34]. Since there is no straightforward way to estimate the range of validity of Eq. (6), which also depends on $N$, we regard $\Delta z$ as an external parameter in Eq. (7) and we choose the value of $\Delta z$ which gives us the smallest minimum in $V$. For all values of $m$ and all components in Table II, $\Delta z$ was between 1 and 3.

The results in Table II summarize the important features of DCP and how they relate to both DER (the analogous non-explosive case) and AP (the analogous undirected case). For the GSCC, $\beta$ and $\eta$ are lower in DCP than in DER, but are not small enough to lead to interesting behavior; therefore, we will focus on GOUT from here

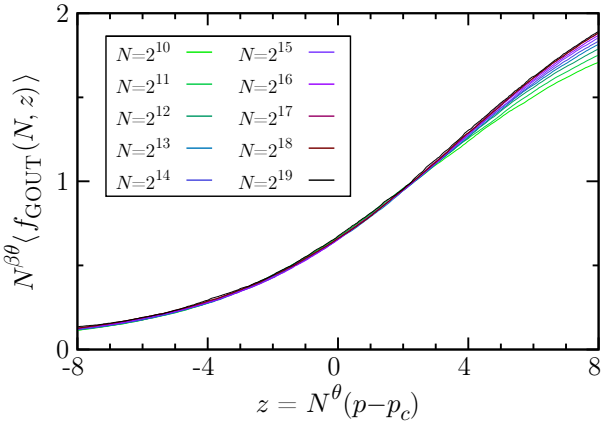


FIG. 6 (color online). Collapse of $\langle f_{\mathrm{GOUT}} \rangle$ for DCP ($m = 2$) using various values of $N$ onto the universal scaling function $h(z)$, according to Eq. (6). From the bottom curve to the top curve, $N$ increases from $2^{10}$ (light green or light gray) to $N = 2^{19}$ (black). The first eight curves are averaged over 10,000 network growth processes, and the last two are averaged over 5,000 and 2,500 respectively. For these values of $N$, the collapse is excellent up to $z \approx 4$. Similar collapses are used to fit the values of $p_c$, $\theta$, and $\beta$ reported in Table II.

forward. We see that $\beta$ and $\eta$ are significantly smaller in DCP than in DER, but not nearly as small as in AP. This provides quantitative support for our characterization of DCP as weakly explosive, in contrast to both the explosive behavior of AP and the ordinary behavior of DER. It is clear that DCP belongs somewhere between these two previously-studied regimes.

Several other features of Table II are worth noting. For example, in the Achlioptas process, $\beta$ and $\eta$ change quite significantly when $m$ is changed from 2 to 3, but the corresponding changes for DCP are comparatively small. This suggests again that the amount of edge competition has a more pronounced effect on the critical behavior of undirected networks than directed networks. However, the opposite is true of the critical point $p_c$, which, for successive values of $m$, increases by a much greater factor for directed networks than for undirected networks. If one views the purpose of edge competition as *delaying* the formation of a giant component rather than producing an explosive transition, then this goal is better achieved by DCP than by AP.

Finally, in Fig. 5, we see that DCP lacks some of the unusual scaling behavior observed for AP in [8]. Although the values of $\lambda$ for the giant out-component in DCP are smaller than those for DER, again indicating weakly explosive behavior, it is nonetheless clear that they are positive. On the other hand, in AP, a much more detailed analysis is required to show that $\mathrm{Var}[f]$ eventually approaches 0 for all $p$ as $N \to \infty$ (see [8]). Therefore, we do not report $\lambda$ for AP, but merely note the qualitative differences between AP and DCP.

## IV. DISCUSSION

We have shown that an extension of the Achlioptas process to directed networks exhibits critical behavior which is, in many respects, partway between classical percolation and explosive percolation, which we have termed weakly explosive percolation. This has several interesting ramifications for future research on controlling or modifying percolation phase transitions. One fundamental open question is how general the phenomenon of explosive percolation is, and whether the explosiveness of a percolation process can be predicted in a relatively straightforward way. From the perspective of classical percolation, the primary distinguishing features of the Achlioptas network growth process are that it is irreversible [9] and uses nonlocal information [8]; however, there are clearly such processes which are not explosive (see, for example, [35]). The strong explosiveness of the Achlioptas process may be contingent on several factors, and the present work suggests that the use of undirected networks is one of these factors.

Another avenue for further research is the possibility of tailoring percolation transitions with particular features. For example, different growth rules may create different complex network structures. In Fig. 3, nearly all network nodes have joined the giant bowtie soon after the critical point, but this is not true of the giant in- or out-components until $p$ is quite large [36]. While it is beyond the scope of this paper to investigate this feature, it suggests that there is additional interesting structure in networks grown through the directed competition process which cannot exist in undirected networks. More importantly, it may be possible to control the critical point and the critical behavior of the giant component by using a mix of directed and undirected edges in the network growth process. Because the Achlioptas process produces a more explosive transition, but the directed competition process delays the onset of criticality for longer, this may produce some degree of control for both features. Along with the above results, this suggests that further study of competitive percolation processes on directed networks will widen the known repertoire of percolation behavior in fascinating ways.

## V. ACKNOWLEDGMENTS

This work was supported by ARO grant W911NF-12-1-0101 and is also based upon work supported by the National Science Foundation under Grant Number PHY-1156454. Any opinions, conclusions, or recommendations expressed in this article are those of the authors, and do not necessarily reflect the views of the NSF.

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