Fractal trace of earthworms

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Phys. Rev. E **87**, 052120 — Published 15 May 2013

DOI: 10.1103/PhysRevE.87.052120

Fractal trace of earthworm

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We investigate a process of random walk of a point particle on the two-dimensional square lattice of size $n \times n$ with periodic boundary conditions. A fraction $p \leq 20\%$ of the lattice is occupied by holes (p represents macro-porosity). A site not occupied by a hole is occupied by an obstacle. Upon a random step of the walker, a number of obstacles, M, can be pushed aside. The system approaches equilibrium in $(n \log n)^2$ steps. We determine the distribution of M pushed in a single move at equilibrium. The distribution $F(M)$ is given by M^γ where $\gamma = -1.18$ for $p = 0.1$ decreasing to $\gamma = -1.28$ for $p = 0.01$. Irrespective of the initial distribution of holes on the lattice the final, equilibrium distribution of holes forms a fractal with fractal dimension changing from $a = 1.56$ for $p = 0.20$ to $a = 1.42$ for $p = 0.001$ (for $n = 4,000$). The trace of a random walker forms a distribution with expected fractal dimension 2.

I. INTRODUCTION

Our work is inspired by a natural phenomenon. Earthworms move through soil and the common assumption is that they leave channels that help to aerate and drain the soil ([1]). The distribution of holes left by the earthworm has crucial effects for the soil ecosystem. Uniform distribution of free space is beneficial for the ecosystem while distribution given by large free spaces and equally large compact blocks of soil are considered ill-structured for the ecosystem. One of the parameters used to quantify the soil structure is the fractal dimension of the distribution of pore space, usually measured at two-dimensional cross sections. Fractal dimension decreases with compactification of soil. For aerated soils the dimension is close to 3/2, while for compactified soils drops to 1.35 ([2]). The purpose of this paper is to show the relation between motion of an earthworm (modelled as a random walker) and the fractal dimension of pores created by the motion. Our model has some common elements with typical modelling in ecosystems and bioturbation, where active matter (living organisms) push aside or consume organic matter ([3–5]). Similar mechanism of motion is used in dip pen nanolitography modelling in 2D, where random motion of ink particles is accompanied by serial pushing of molecules deposited from the tip. ([6, 7]).

A recent mathematical paper ([8]) presents a model in which the earthworm is modeled as a sphere moving according to Brownian motion and soil particles (points) are pushed aside by the earthworm (sphere). The paper [8] contains a theorem suggesting that in three dimensions the soil is not compactified in the long run, that is, the spherical earthworm has a beneficial effect on the soil. In the "toy model" considered in [8], the soil particles can occupy the same lattice site, a hardly realistic assumption in view of the incompressible character of the soil. In the following, we simplify the situation by assuming that soil is incompressible and we neglect the fact that small pores may collapse due to activity of worms.

We propose to study a different model that is inspired by the same natural phenomenon. In our model, a large fixed block of the discrete lattice is considered to be the holding volume for soil. Soil grains are represented by point-like particles which can move from one vertex of the lattice to one of its neighbors. The earthworm is also assumed to be a point-like particle and has the trajectory of a simple symmetric random walk. Only the earthworm moves on its own; all other points (soil particles) move only when they are pushed aside. The fundamental difference between this model and that in [8] is that no two soil particles can be present at the same vertex in the new model, by assumption. This is a discrete version of the "incompressibility" condition. The incompressibility condition generates great difficulties on the mathematical side and also our model is hard to analyze in a rigorous way. Hence, we chose computer simulations as the best way to proceed.

In our model, the minority (between 0.1% and 20%) of sites in the lattice are holes (sites unoccupied by soil particles). We are interested in the distribution of "holes" (empty vertices) in the stationary distribution. It is easy to guess and well supported by simulations that the distribution of holes is typically fractal. The fundamental quantitative characteristic of a fractal is its (Hausdorff) dimension. In our model, low fractal dimension of the set of holes would mean little beneficial effect of the earth-

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worm on the soil. On the other hand, high (close to 2) Hausdorff (fractal) dimension of the set of holes would mean that the holes are widely spread, confirming that the earthworm is likely to create widespread canals.

II. MODEL

We will consider particle systems on two-dimensional square lattice with periodic boundary conditions. Boundary effects may be included in the future work on the same model. We work with the two-dimensional state space rather than the more natural three-dimensional state space for several reasons. First, simulated two-dimensional random walk converges to equilibrium much faster than three-dimensional random walk hence providing us with more reliable estimates. In particular, this allows us to obtain more reliable fractal dimension estimates than we would be able to generate in the three-dimensional case. Second, the results of two-dimensional simulations are easier to illustrate and, therefore, to interpret at the heuristic level, than results of three-dimensional simulations. Two-dimensional random walk is recurrent (unlike the three-dimensional random walk) so two-dimensional simulations can serve as a toy model for systems with high correlations. Finally, there are cases when the earthworms move in cracks or channels which can be approximated as 2D systems.

The lattice in our simulations has size $n \times n$, with n ranging from 800 to $6,000$. Typically, we restricted simulations to $n \leq 4,000$ so that the process would attain the equilibrium regime in a realistic time.

We placed two types of particles on the lattice. There was a single representative X of the first type of the particle. The process X represents the earthworm and is a simple (nearest neighbor) symmetric random walk. The other particles $Y^1, Y^2, \dots, Y^N$ were "inert" in the sense that their positions are totally determined by the motion of X. There was no extra randomness in the motion of particles Y^k.

The dynamics of the process was the following. No two particles could ever occupy the same site of the lattice. Hence, we always had $N + 1 \leq n \times n$. In fact, typically, the number of unoccupied sites ("holes") was always substantial. On the other hand, the fraction of the holes among the total number of sites was always small, not larger than 20%, to make this aspect of the process a realistic representation of the motivating natural phenomenon. This was implemented by placing initially all particles on distinct sites of the lattice. A step in the simulations, to be described below, preserved this property of the state of the process (distinct locations for distinct particles).

When particle made a step from a location to a new location unoccupied by any of the particles Y^k then none of the particles Y^k moved. If X moved to a site occupied by a particle Y^k then all adjacent particles Y^k in the same direction moved by one step. In other words, the row of adjacent particles Y^k in the direction of X's move, between the new location of X and the closest hole moved by one step in the direction of the hole. If there was no hole, that is, if all the sites in the direction of the step of X were occupied (wrapped around lattice) then all particles on this line moved by one step. See Figure 1 for examples illustrating the dynamics of the process.

III. SIMULATION RESULTS

A. Convergence to equilibrium

Since the state space of the process is finite and the process is Markovian, it has a stationary distribution (equilibrium). It is easy to see that the stationary distribution is unique. Note, however, that the process is not time reversible.

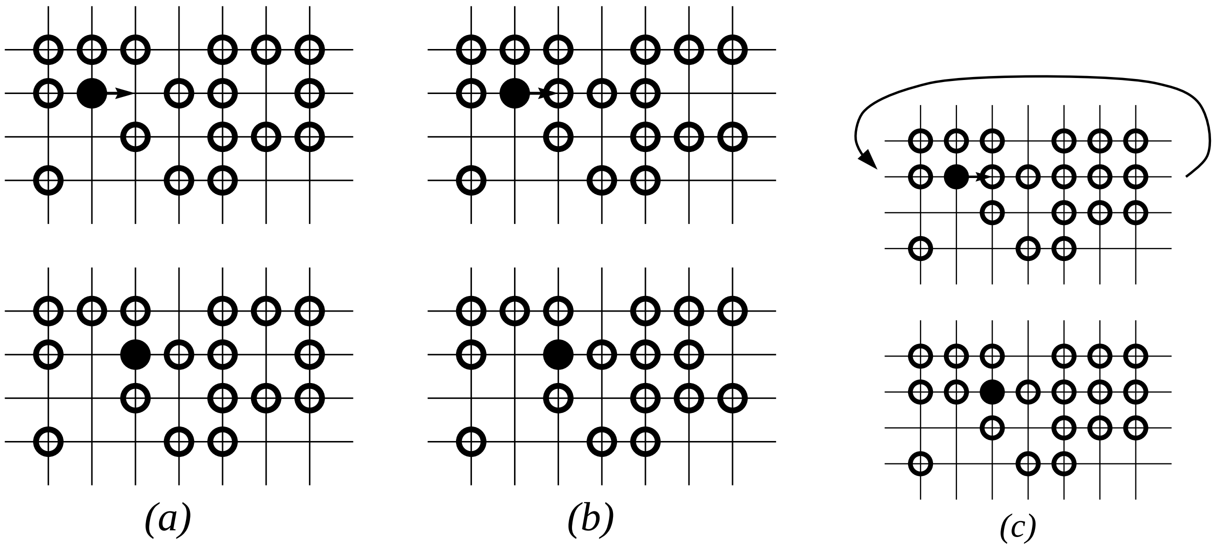
The process of all particles cannot reach equilibrium before X itself reaches equilibrium. The time needed for X to reach equilibrium is of order n^2. The time for the whole process to reach equilibrium is also bounded below by the time which X needs to visit all points on the lattice. The time needed by X to reach every site on the lattice is of order $(n \log n)^2$ (see [9]). We determined the characteristic time for the process to reach equilibrium by simulations; the results are in good agreement with the theoretical estimates mentioned above.

First, we started with the uniform distribution of holes in the lattice. Figures 2 (a)-(c) illustrate the transition from the uniform distribution of holes to the totally fractal (non-uniform) distribution of holes in all parts of the state space. In these simulations, the lattice edge had length $n = 2,000$, and the porosity parameter was $p = 0.08$.

Next, we started with the maximally compact distribution of holes; the holes were located in a solid square. Figures 3 (a)-(c) illustrate the process of smearing out of the initial distribution of holes. Once again, $n = 2,000$, and the porosity parameter was $p = 0.08$.

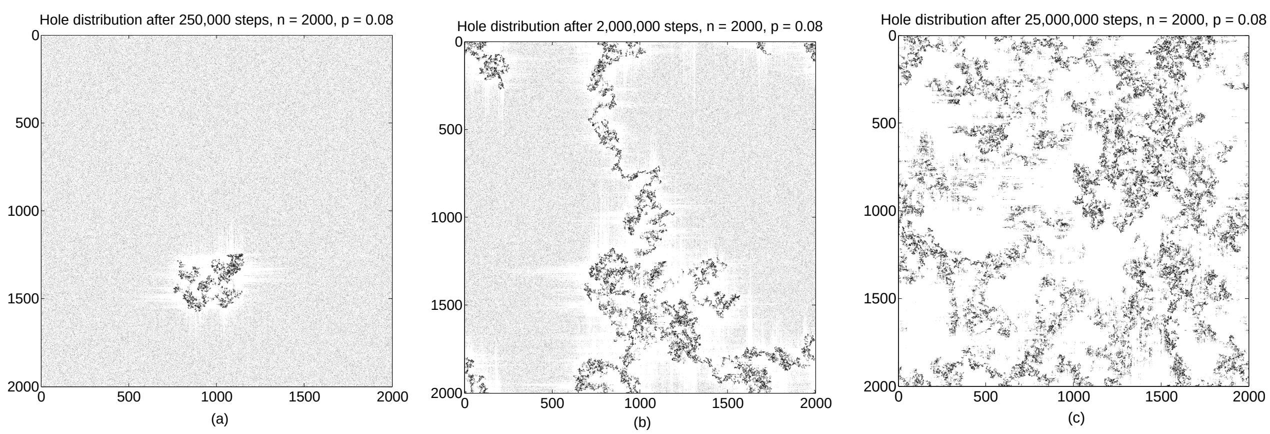
B. Random walk trace and distribution of holes

It follows from the dynamics of the process that if there is a hole at a lattice site then the random walk X must have (recently) visited the site. It is also clear that many sites that were recently visited by X hold particles Y^k, that is, they are not empty. Figure 4 illustrates the difference between the size (fractal dimension) of the trace of the random walk and that of the set of holes. Not surprisingly, the trace of random walk appears to be more space-filling than the set of holes. In fact, the fractal dimension of the random walk trace is equal to 2, the same as for the state space. The figure shows the last $100,000$ sites visited by X. The process was in equilibrium. The lattice size was given by $n = 800$. The porosity parameter was $p = 0.1$.



(a) (b) (c)

FIG. 1. Particle X is depicted in solid black. Particles Y^k are depicted as circles. The initial position of all particles is shown in the figures at the top. (a) After particle X moves one step to the right, the positions of other particles do not change, as shown in the bottom figure. (b) There is a row of three adjacent particles Y^k blocking X's move to the right. When particle X moves one step to the right, all three blocking particles also move one step to the right, as shown in the bottom figure. (c) There are no holes present on the line of the direction of X's move. All particles move one step, including wrapping around the lattice.



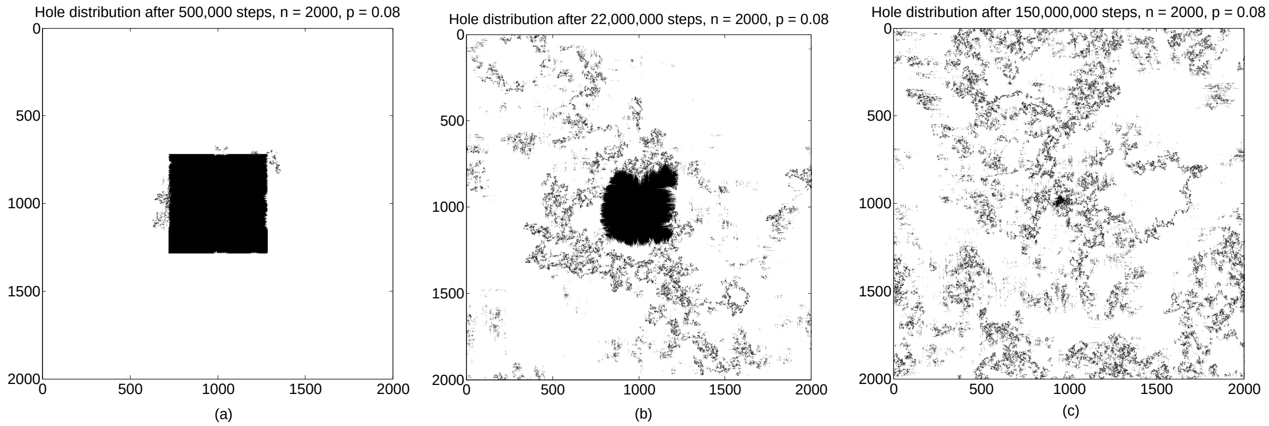
(a) (b) (c)

FIG. 2. The lattice size is given by $n = 2,000$. The porosity parameter is $p = 0.08$. The initial distribution of particles is uniform. The figure shows the hole distribution after (a) 2.5×10^5 steps, (b) 2×10^6 steps, (c) 2.5×10^7 steps.

C. Fractal dimension of the distribution of holes

The fractal dimension of the set of holes in equilibrium is the quantity of primary interest to us because it indicates the structure of this set. It is also conceivable that the fractal dimension can be estimated rigorously, so the results of the simulation may serve as a guide for the corresponding rigorous mathematical project.

Consider a hole at a lattice site. Let d denote the lattice distance between the hole and other sites. Recall that p denotes the fraction of holes in the lattice. Let t denote the number of steps of the simulation (one unit of time represents one step of X). Let $k = f(d, p, t)$ be the average number of holes at the distance at most d from a given hole, after t steps, assuming that the fraction



(a) (b) (c)

FIG. 3. The lattice size is given by $n = 2,000$. The porosity parameter is $p = 0.08$. The initial distribution of holes is a solid square. The figure shows the hole distribution after (a) 5×10^5 steps, (b) 2.2×10^7 steps, (c) 1.5×10^8 steps.

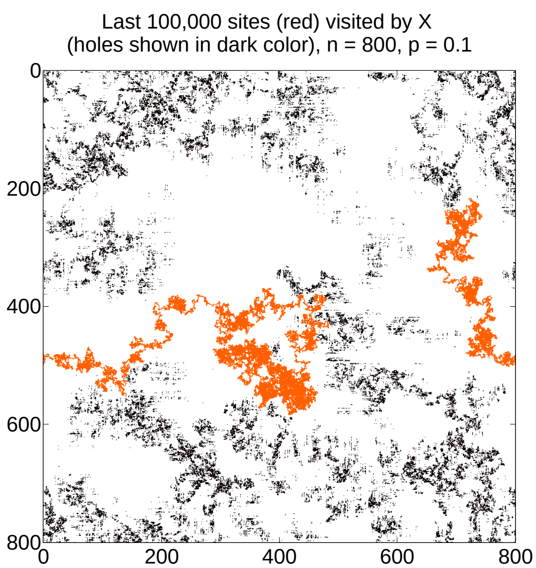


FIG. 4. (Color online) The figure shows the last 100,000 sites visited by X in orange (gray). The holes are shown in black. The fractal dimension of the random walk trace is equal to 2, the same as for the state space. The fractal dimension of the distribution of holes is about 1.51. The process is in equilibrium. The lattice size is given by $n = 800$. The porosity parameter is $p = 0.1$.

of holes in the whole lattice is p. One expects the set of holes to have a fractal structure, that is, one expects $k \approx d^a$ for some fractal dimension $a > 0$.

We determined the number of simulation steps needed to reach equilibrium, from the point of view of the fractal dimension. Roughly speaking the number of steps needed to achieve this goal was between 10^7 and 10^8. More precise results are illustrated as follows. Figure 5 shows that the occupation density for neighboring sites with holes stabilizes before $t = 5 \times 10^7$, for $n = 3,000$, for five pairs of parameters p and d, ranging from $p = 0.02$ to $p = 0.15$ and from $d = 4$ to $d = 19$. The initial occupation density for neighboring sites with holes was very low because it was equal, more or less, to the average density of holes in the whole square, until the motion of the earthworm created fractal-like patterns. The effect appeared in both cases, when the initial distribution of holes was uniform or square-like.

To determine the fractal dimension a in the formula $k = cd^a$, we postulated the following relationship $\log(k) = a\log(d) + b$ and then we used linear regression to find a and b. The graph of the regression line $\log(k) = a\log(d)+b$ for $p = 0.039$ is presented in Figure 6. The number of steps in the simulation was $t = 2 \times 10^8$.

The results of fractal dimension simulations are given in Figure 7. The lattice sizes were given by $n = 2,000, 3,000$ and $4,000$. The estimates of the dimension a do not seem to depend on n in this range beyond the small random fluctuations inherent in the model.

In our simulations, the fractal dimension of the set of holes depends on the occupation density of the lattice and varies from 1.56 to 1.42 when p varies from 0.20 to 0.001 (for $n = 4,000$). Our results are generally consistent with the empirical results presented in [2, Fig. 2], in the sense that in both cases the range of dimension values is centered at 1.5. Also, in both cases, a is close to 1.5 for $p = 0.1$. Our range of values of the fractal dimension is narrower than that in [2, Fig. 2], for p ranging from 0.05 to 0.2.

The fractal dimension varies in an interval that contains 3/2 and is far from 0 and 2. We propose it as a challenge, realistic in our opinion, to prove both assertions in the rigorous mathematical sense, when the size of the system n goes to infinity.

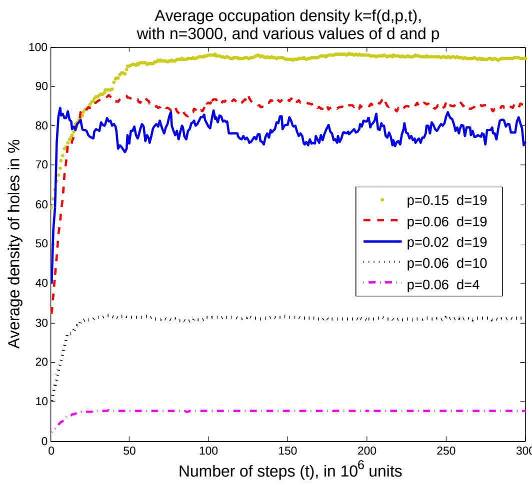


FIG. 5. (Color online) The occupation density for neighboring sites with holes stabilizes before $t = 5 \times 10^7$. The lattice size is given by $n = 3,000$. Number of steps t is on the horizontal axis (in 10^6 units). The average density of holes $k = f(d, p, t)$ is on the vertical axis. Dark yellow (top) line represents the porosity parameter $p = 0.15$ and distance $d = 19$. Red (dashed, second from the top) line represents the porosity parameter $p = 0.06$ and distance $d = 19$. Blue (middle) line represents the porosity parameter $p = 0.02$ and distance $d = 19$. Black dotted line represents the porosity parameter $p = 0.06$ and distance $d = 10$. Violet (bottom) line represents the porosity parameter $p = 0.06$ and distance $d = 4$.

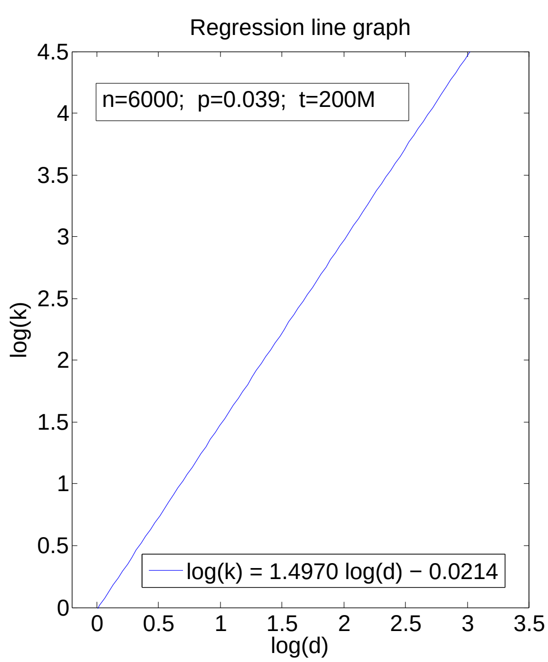


FIG. 6. Regression line $\log(k) = a\log(d) + b$ for $p = 0.039$. $k = f(d, p, t)$ is the average number of holes at the distance at most d from a given hole. The number of steps was $t = 2\times10^8$. The lattice size was given by $n = 6,000$.

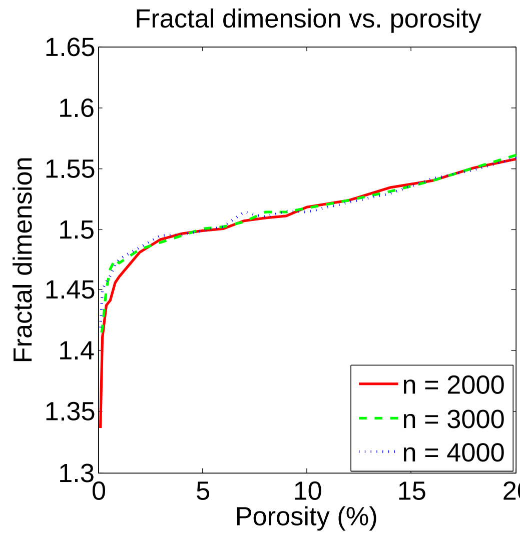


FIG. 7. (Color online) The fractal dimension a as a function of the porosity parameter p. Red (solid) line represents $n = 2,000$. Green (dashed) line represents $n = 3,000$. Blue (dotted) line represents $n = 4,000$. The process is in equilibrium.

D. Correlations in number of blocks pushed by a single step

We measured, in a sense, the long-range dependence in the model by investigating the distribution of the number M of Y^k's pushed by X in a single step. Figure 8 contains a comparison of the distributions of $\log M$ for various occupation densities p, from $p = 0.01$ to $p = 0.15$. The process is in equilibrium. The lattice size is given by $n = 2,000$. The initial parts of the graphs are remarkably similar for a large band of values of p. The distribution $F(M)$ of M has the form $F(M) = cM^\gamma$ for $M = 1, \ldots, 110$ with $\gamma = -1.23$ for $p = 0.15$, $\gamma = -1.18$ for $p = 0.1$, $\gamma = -1.22$ for $p = 0.05$, and $\gamma = -1.28$ for $p = 0.01$. In the case $p = 0.01$, the right tail of the frequency graph increases because of the effect of wrapping of soil particle motion around the torus (see Figure 1 (c)). With porosity index as low as $p = 0.01$, it becomes more likely for very long columns or rows of soil particles to be moved simultaneously, due to their high density.

IV. FUTURE RESEARCH

We are grateful to anonymous referees for many suggestions for improvement, including the following suggestions for future research. (i) Soil is compressible, as opposite to solids such as rocks. Future simulations may incorporate the effects of the force the earthworm is capable of applying to the soil particles, the soil compressibility and the ratio between the work size and the mean particle size. (ii) One could study the dynamically changing fractal structure of the holes as the effect of the earthworm

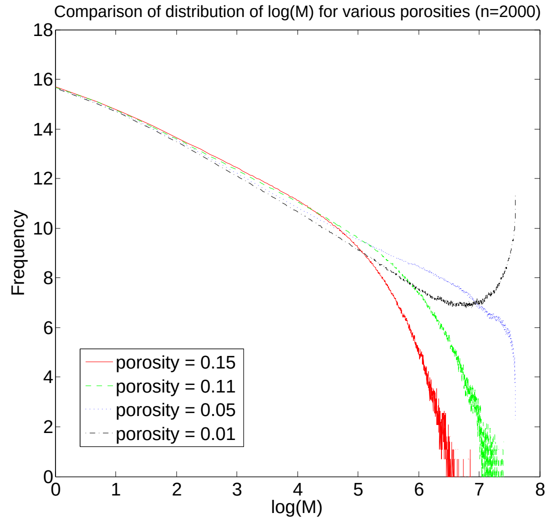


FIG. 8. (Color online) Comparison of the distributions of $\log M$ for various occupation densities p. Red line (lowest right tail) represents $p = 0.15$. Green line (second lowest right tail) represents $p = 0.11$. Blue line (second highest right tail) represents $p = 0.05$. Black line (highest right tail) represents $p = 0.01$. The process is in equilibrium. The lattice size is given by $n = 2,000$. Quantities on both axes are on the logarithmic scale.

motion. (iii) One could consider asymmetric motions of the earthworm, for example, those with different probabilities of motion in horizontal and vertical directions, and their effect on the fractal dimension. (iv) One could consider a model in which the earthworm does not push randomly but in the direction of minimum number of solid particles.

V. CONCLUSIONS

In our discrete model of tunnels created by an earthworm, the earthworm trace has the fractal character with the fractal dimension close to $3/2$ for various values of the simulation parameters. Although the simulations were performed in the two-dimensional case, the results suggest that the trace is sufficiently large to support the claim that earthworms have the beneficial impact on the soil by creating a large number of tunnels if they perform local random walk.

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