Topological control of synchronous patterns in systems of networked chaotic oscillators

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Topological control of synchronous patterns in systems of networked chaotic oscillators

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Recent studies of network science have revealed the sensitive dependence of the network collective behaviors on structures, here we employ this feature of topological sensitivity for the purpose of pattern control. By simple models of networked chaotic oscillators, we are able to argue and demonstrate that, by manipulating just a *single* link in the network, the synchronous patterns of the system can be effectively adjusted or controlled. In particular, by changing the weight or the connection of a shortcut link in the network, we find that not only various stable synchronous patterns can be generated from the system, but also the synchronous patterns can be successfully switched among different forms. The stability of the synchronous patterns is analyzed by the method of eigenvalue analysis, and the feasibility of the control is verified by numerical simulations. Our study provides a step forward to the control of sophisticated collective behaviors in more complex networks, as well as giving insights to the evolution and function of some realistic complex systems.

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I. INTRODUCTION

A distinct feature of the complex systems in nature is the sensitive dependence of their dynamics on the initial conditions or system parameters [1]. A well-known example is the chaotic systems, where a small perturbation on the system initial condition will result in a significant change of the system state a moment later, namely the butterfly effect [2]. To tame this dynamical sensitivity, in the past decades there have been extensive studies on the control of chaos [3], where a significant finding is that the chaotic behaviors can be efficiently tamed or controlled by adding small perturbations onto the system states or parameters, e.g., the OGY method [4]. In terms of dynamical sensitivity, a network analogy of chaos could be the dynamics of complex networks, where a slight modification of the network structure could change the collective behaviors of the system as a whole. For instance, the failure of a single transmission line in the power-grid network could lead to a large-scale blackout within a few minutes, due to the mechanism of network cascading [5]. Being aware of this type of dynamics-induced catastrophe, in the past years efforts have been given to the improvement of the network robustness and performance. For instance, it is found that by intentionally removing a few of the network links at the beginning of the cascading, the network damage can be largely reduced [6]. Besides cascading, recently the idea of topological control has been also employed in many other problems in network science, e.g., epidemic propagation[7], global synchronization [8], oscillatory patterns[9], and control optimization [10], etc.

Synchronous and coherent motions are commonly observed in neural and biological systems, and are widely recognized as important to the system operations and functions [11–14]. A typical example is the human brain, where the networked neurons are found to be firing synchronously in a group fashion during the process of information processing (e.g., perception recovery) [15], or when are subjected to some external stimuli (e.g., neural binding) [16]. That is, under certain circumstances the neurons are able to self-organize into different forms of synchronous patterns. These synchronous patterns, as revealed from the experimental data, are very unstable and can be largely changed by small perturbations. For instance, a slight change of the network structure, as caused by the aberrant axonal reorganization of the excitatory dentate granule cell axons onto the neighboring granule neurons, could lead to the emergence of a large-scale synchrony that involves many neuronal assemblies – a network mechanism for epileptic seizures [17]. In network science, an intriguing and challenging question is: *can we stabilize the synchronous patterns in complex systems, or switching the synchronous pattern among different forms, by only a slight modification of the network structure?*

Comparing to other types of network dynamics, e.g., global network synchronization or Turing-like patterns [12–14, 18], the analysis of synchronous patterns in the complex networks is much more difficult and challenging [19]. This is partially due to the fact that the synchronous patterns, if existing, are highly fragmented and scattered, making them difficult to be figured out from the complex network [18, 20]; and also because that the patterns are highly dynamic and unstable, making them difficult to be manipulated [21, 22]. Regarding these difficulties, to investigate the control of synchronous patterns in complex systems, a plausible and meaningful approach would be adopting the simplified network structures that capture some essential features of the general complex networks, e.g., regular networks with a few of random shortcut links. In the present work, employing simple networks of coupled chaotic oscillators, we will investigate how synchronous patterns can be modified or manipulated by a small adjustment of the network structure. Interestingly, we find that in these network models, by adjusting the properties of just a *single* link, not only stable synchronous patterns can be generated, but also the patterns can be switched among different forms.

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The rest of the paper is organized as follows. In Sec. II we will give our model of networked chaotic oscillators, and propose the method of eigenvalue analysis used to characterize the stability of the synchronous patterns. In Sec. III, by different network models, we will demonstrate numerically how the synchronous patterns can be generated and manipulated by a slight change of the network structure. Finally, in Sec. IV we will give our discussions conclusion.

II. THE STABILITY OF SYNCHRONOUS PATTERNS

Consider a complex network of N identical nonlinear oscillators. Let $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ be the node dynamics in the isolated form, and $\mathbf{H}(\mathbf{x})$ be the coupling function among the nodes, then the evolution of the network could be described by the set of equations

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{j=1}^{N} a_{ij}[\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)], \tag{1}$$

with $i, j = 1, \ldots, N$ the node indices, and ε the uniform coupling strength. The network structure is captured by the adjacency matrix $\mathbf{A}$, with $a_{ij} = -1$ if nodes i and j are directly connected, and $a_{ij} = 0$ otherwise. For networks of linearly coupled identical oscillators, the previous studies have shown that by a suitable coupling strength, the trajectories of the oscillators can be converged to the same one after a transient time, i.e., reaching the state of global synchronization. The range of the coupling strength, as well as its dependence to the network structure and the node dynamics, can be analyzed by the method of master stability function (MSF) [23].

Besides the special state of global synchronization, a network may also support other forms of synchronous behaviors. For instance, for the network example plotted in Fig. 1(a), if in numerical simulations we artificially set the initial conditions of the paired nodes, $(2, 5)$ and $(3, 4)$, to be identical, then during the process of the system evolution, the trajectories of the paired nodes will be always identical (since they have the same node dynamics, the same initial conditions, and receiving the same coupling signals during the system evolution). That is, the system will be staying on the state of partial synchronization characterized by the symbol sequence (a, b, c, c, b) [24, 27]. Here, the symbols, a, b and c, represent the trajectories of the oscillators, and are ordered by the node indices in the network. Nodes of the same symbol are regarded as synchronized, and they form an individual synchronous cluster. The number of clusters in the system thus is counted as the number of different symbols in the pattern sequence. It is straightforward to find that the synchronous patterns that can be supported by a network is closely dependent on the network symmetry. Specifically, to have a specific form of synchronous pattern, the network structure must own the corresponding symmetry. For instance, for the network plotted in Fig. 1(a), the synchronous pattern (a, b, c, c, b) is supported by the reflection symmetry $\mathbf{S}$.

A network may possess different topological symmetries, but not all the corresponding patterns are stable. The stability of a synchronous pattern can be analyzed by the method of eigenvalue analysis, with the details the following. (This method is originated from the group-theory analysis proposed in Ref. [25], and is also a generalization of the method proposed in Refs. [26, 27].) Let $\mathbf{x}_s$ be the synchronous manifold of the system (the manifold for global synchronization), and $\delta\mathbf{x}_i = \mathbf{x}_i - \mathbf{x}_s$ be infinitesimal perturbations added to the oscillator trajectories, then the evolutions of the perturbations are mainly governed by the equations

$$\delta\dot{\mathbf{x}}_i = \mathbf{DF}(\mathbf{x}_s) - \varepsilon \sum_{j=1}^{N} a_{ij}\mathbf{DH}(\mathbf{x}_s)(\delta\mathbf{x}_j - \delta\mathbf{x}_i), \tag{2}$$

where $\mathbf{DF}$ and $\mathbf{DH}$ are the Jacobian matrices of the corresponding vector functions evaluated on $\mathbf{x}_s$. Projecting $\{\delta x_i\}$ into the eigenspace spanned by the eigenvectors of the network coupling matrix $\mathbf{C} = \mathbf{A} + \mathbf{K}$ ($\mathbf{K}$ is the diagonal matrix whose elements are the degree of the corresponding node, i.e. $k_{ii} = \sum_j a_{ij}$), then the set of equations described by Eq. (2) can be transformed into N decoupled equations

$$\delta\dot{\mathbf{y}}_i = [\mathbf{DF}(\mathbf{x}_s) - \varepsilon\lambda_i\mathbf{DH}(\mathbf{x}_s)]\delta\mathbf{y}_i, \tag{3}$$

where $0 = \lambda_1 < \lambda_2 < \ldots < \lambda_N$ are the eigenvalues of $\mathbf{C}$, and $\delta\mathbf{y}_i$ denotes the ith mode of the perturbations. Let Λ_i be the largest Lyapunov exponent calculated from Eq. (3) for the ith mode, then the stability of this mode is determined by the sign of Λ_i: it is stable if $\Lambda_i \leq 0$, and is unstable if $\Lambda_i > 0$. The mode of λ_1 represents the motion parallel to the synchronous manifold, which is always unstable due to the chaotic nature of the node dynamics.

Network symmetry sets in when dividing the eigenvalues (modes) into groups. For any given symmetry, $\mathbf{S}$, of the network structure, we can construct the corresponding permutation matrix $\mathbf{P}_{N\times N}$: $p_{ij} = p_{ji} = 1$ if the exchange of nodes i and j according to $\mathbf{S}$ does not change the network structure, and $p_{ij} = 0$ otherwise. It is straightforward to find that $\mathbf{PP}^{-1} = \mathbf{P}^2 = \mathbf{I}$, with $\mathbf{I}_{N\times N}$ the identity matrix. Let $\mathbf{M}$ be the transformation matrix of $\mathbf{P}$, i.e., $\mathbf{M}^{-1}\mathbf{PM} = \mathbf{P}'$ (with $\mathbf{P}'$ the diagonal matrix), then the network coupling matrix can be transformed into the following blocked form

$$\mathbf{G} = \mathbf{M}^{-1}\mathbf{CM} = \begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{D} \end{pmatrix}, \tag{4}$$

where $\mathbf{B}$ and $\mathbf{D}$ are, respectively, n_1 and n_2 dimensional matrices, with $n_1 + n_2 = N$. Because $\mathbf{G}$ and $\mathbf{C}$ are similar matrices, they have the same set of eigenvalues. However, in the blocked matrix $\mathbf{G}$, the eigenvalues are divided into two groups: n_1 eigenvalues in $\mathbf{B}$ and n_2 eigenvalues in $\mathbf{D}$.

Let $\mathbf{D}$ be the matrix that contains the eigenvalue $\lambda_1 = 0$, then we know from the function of the transformation matrix that the synchronous manifold of the pattern is embedded in the n_2 dimensional subspace spanned by the eigenvectors of $\mathbf{D}$. We order the eigenvalues of $\mathbf{D}$ as $0 = \lambda_1^{syn} < \lambda_2^{syn} \leq \ldots \leq \lambda_{n2}^{syn}$, and call the spanned subspace the *synchronous subspace*. In a similar way, the eigenvalues of $\mathbf{B}$ are ordered as $\lambda_1^{tr} \leq \lambda_2^{tr} \leq \ldots \leq \lambda_{n1}^{tr}$. Since the subspace spanned by the eigenvectors of $\mathbf{B}$ characterizes the perturbations transverse to

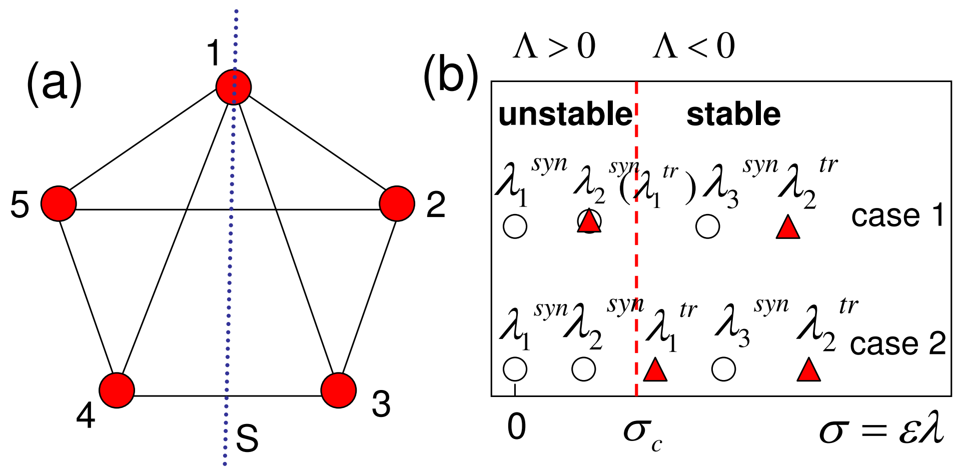

FIG. 1: (Color online) (a) The model of 5-node network used in our analysis. The network has the reflection symmetry $\mathbf{S}$, which may support the synchronous pattern (a, b, c, c, b). (b) A schematic plot used to analyze the stability of the synchronous pattern. The eigenvalues are divided into two groups: ($\triangle$) for the transverse modes and ($\bigcirc$) for the synchronous modes. σ_c is a critical parameter characterizing the boundary of the stable regime: a mode is stable if its eigenvalue satisfies $\varepsilon\lambda = \sigma > \sigma_c$. Case 1: For the pattern (a, b, c, c, b), the distribution of the eigenvalues calculated from the network in (a). Since $\lambda_1^{tr} = \lambda_2^{syn}$, which not satisfies the eigenvalue condition, the pattern thus is unstable. Case 2: A possible distribution of the eigenvalues that may generate a stable pattern (satisfying the eigenvalue condition $\lambda_1^{tr} > \lambda_2^{syn}$), which is expected to be realized by a slight modification of the network structure or properties.

the synchronous manifold, we thus give it the name *transverse subspace*. To have a stable pattern, it is necessary that all the transverse modes in the transverse subspace should be damping with time. More specifically, we should have $\Lambda(\lambda_l^{tr}) < 0$ for $l = 1, \ldots, n_1$. Meanwhile, to avoid the trivial pattern of global network synchronization, it is also necessary that as least one of the non-trivial modes in the synchronous subspace still be unstable, i.e., $\Lambda(\lambda^{syn}) > 0$ for some mode (modes) of $\mathbf{D}$. These are the two necessary conditions for generating stable synchronous patterns in a complex network.

By the above method, we now give an analysis to the stability of the pattern (a, b, c, c, b) in the network shown in Fig. 1(a). Firstly, from the reflection symmetry, $\mathbf{S}$, we can construct the permutation matrix, which reads

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Then, by calculating the eigenvectors of $\mathbf{P}$, we can construct the transform matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{pmatrix}. \tag{6}$$

Finally, by $\mathbf{M}$, we can transform the coupling matrix, $\mathbf{C}$, into the blocked matrix, $\mathbf{G}$, in which

$$\mathbf{B} = \begin{pmatrix} -4/3 & -1/3 \\ -1/3 & -4/3 \end{pmatrix}, \tag{7}$$

and

$$\mathbf{D} = \begin{pmatrix} -1 & \sqrt{2}/4 & \sqrt{2}/4 \\ \sqrt{2}/3 & -2/3 & 1/3 \\ \sqrt{2}/3 & 1/3 & -2/3 \end{pmatrix}. \tag{8}$$

For $\mathbf{B}$, we have $(\lambda_1^{tr}, \lambda_2^{tr}) = (1, 1.67)$; while for $\mathbf{D}$, we have $(\lambda_1^{syn}, \lambda_2^{syn}, \lambda_3^{syn}) = (0, 1, 1.33)$. Since the null eigenvalue belongs to $\mathbf{D}$, the synchronous and transverse subspaces thus are spanned by the eigenvectors of $\mathbf{D}$ and $\mathbf{B}$, respectively.

Previous studies of MSF have shown that [28], for the typical nonlinear systems, the value of $\Lambda(\varepsilon\lambda)$, as calculated from Eq. (3), is negative only when $\varepsilon\lambda = \sigma > \sigma_c$, with σ_c a parameter jointly determined by the node dynamics and the coupling function. The meaning of σ_c, as well as the distribution of the two groups of eigenvalues, are schematically plotted in Fig. 1(b). From this figure, it is straightforward to find that the pattern (a, b, c, c, b) is unstable, as it not satisfies the eigenvalue condition [case 1 in Fig. 1(b)]. More specifically, when the transverse mode of λ_1^{tr} is inside of the stable regime (which can be achieved by changing the coupling strength), all other non-trivial modes of the system ($\lambda_{2,3}^{syn}$ and λ_2^{tr}) will be also inside of the stable regime, which will lead to the global network synchronization, instead of the synchronous pattern.

Is there any method to stabilize the pattern in the network? The remedy lies in the modification of the network structure. As the network collective behavior is sensitively dependent on its structure, it is possible that, by a slight change of the network structure, the unstable pattern changes to stable. Say, for example, if by introducing a new link into the network the eigenvalues can be redistributed in such a way that $\lambda_1^{tr} > \lambda_2^{syn}$ [as illustrated by case 2 in Fig. 1(b)], then the pattern (a, b, c, c, b) may be stabilized. In the following section, employing a typical chaotic oscillator as the node dynamics, we will show how this idea of structure-based pattern control can be realized in some simple network models.

III. TOPOLOGICAL CONTROL OF SYNCHRONOUS PATTERNS

We first demonstrate how the pattern (a,b,c,c,b) can be stabilized by adjusting the *weight* of a single link in the network. To keep the network symmetry unaffected (so as to support the same pattern), we will change only the weight of the link $L_{2,5}$ in the network, wile keeping the weights of other links fixed. For the sake of simulation convenience, here we adopt the normalized coupling scheme for the weighted network: $c_{ij}=-w_{i,j}/\sum w_{i,j}$ for the non-diagonal elements, and $c_{ii}=1$ for the diagonal elements [29, 30]. Here $w_{i,j}$ represents the weight of the network links, which is to be adjusted for the link $L_{2,5}$, while is fixed to $w_{i,j}=1$ for other links. We first check whether the eigenvalue condition, $\lambda_1^{tr}>\lambda_2^{syn}$, can be satisfied by this modification. In Fig. 2(a), we plot the variations of the 4 nontrivial eigenvalues, $\lambda_{1,2}^{tr}$ and $\lambda_{2,3}^{syn}$, as a function of $w_{2,5}$. It is clearly seen that, as $w_{2,5}$ exceeds the critical value $w_c=1$, λ_1^{tr} is larger than λ_2^{syn}. The crossover of λ_1^{tr} and λ_2^{syn} thus suggests that, in the regime of $w_{2,5}>1$, the eigenvalue condition is satisfied.

In simulations, we adopt the chaotic Lorenz oscillator as the node dynamics, which in the isolated form is described by equations $(dx/dt,dy/dt,dz/dt)^T=(\alpha(y-x),rx-y-xz,xy-bz)^T$. By the parameters $\alpha=10$, $r=35$, and $b=8/3$, the oscillator is chaotic, with the largest Lyapunov exponent being about 0.94. (This oscillator will be employed throughout the paper, but the same results have been also observed in other node dynamics, including the chaotic Rössler oscillators and logistic maps.) By the coupling function $\mathbf{H}([\mathbf{x},\mathbf{y},\mathbf{z}]^{\mathbf{T}})=[x,0,0]^T$ (i.e., coupling through the x component of the oscillator), the critical parameter characterizing the stable regime in MSF analysis is $\sigma_c\approx 10$, which is calculated from Eq. (3) by requiring $\Lambda=0$. Thus, to make the transverse mode λ_1^{tr} stable, it is necessary that the coupling strength should be larger than $\varepsilon_1=\sigma_c/\lambda_1^{tr}$. In the meantime, to prevent the system from reaching the state of global synchronization, we should also keep the coupling strength be smaller than $\varepsilon_2=\sigma_c/\lambda_2^{syn}$. These are the conditions for the choosing the coupling strength. For example, if we use $w_{2,5}=1.5$, the two boundary eigenvalues of the coupling matrix are $\lambda_1^{tr}=1.06$ and $\lambda_2^{syn}=0.92$. According to the above analysis, to make the pattern stable, the coupling strength should be chosen from the range $\varepsilon\in(9.35,10.82)$.

By $\varepsilon=9.6$, we plot in Fig. 2(b) the time evolution of the normalized synchronization error for the oscillators, Δx_i. Here, $\Delta x_i=(x_i-\langle x\rangle)/\Delta x^{ave}$, with $\langle x\rangle$ the averaged state of the network and $\Delta x^{ave}=\langle x_i-\langle x\rangle\rangle$ a scaling factor. If during the system evolution two nodes have the same value of Δx_i, then they are identified as synchronized. (The use of Δx is just for the purpose of a clear presentation, which can be replaced by other quantities, e.g., the state variables, which does not affect the form of the synchronous patterns.) The

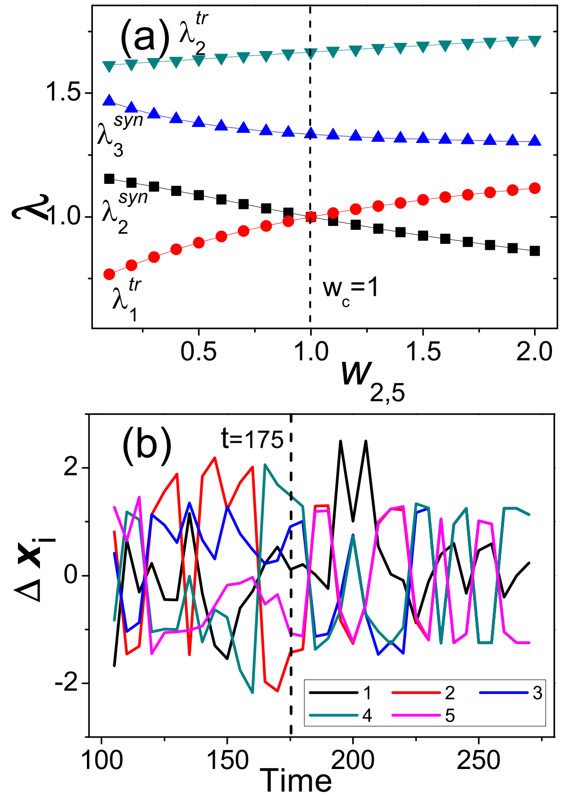

FIG. 2: (Color online) For the network plotted in Fig. 1(a), the stabilization of the pattern (a,b,c,c,b) by changing the weight of the shortcut link $L_{2,5}$. (a) The variations of the non-trivial eigenvalues, $\lambda_{1,2}^{tr}$ and $\lambda_{2,3}^{syn}$, as a function of the link weight, $w_{2,5}$. When $w_{2,5}>w_c=1$, we have $\lambda_1^{tr}>\lambda_2^{syn}$, indicating the possible existence of a stable pattern in this regime. (b) By $\varepsilon=9.6$, the time evolution of the normalized synchronization errors, Δx_i. The control is activated at $t=175$, where $w_{2,5}$ is changed from 1 to 1.5. It is seen that, with the control, the system is gradually transferred from non-synchronization to synchronous pattern (a,b,c,c,b).

control is activated at time $t=175$, where $w_{2,5}$ is changed from 1 to 1.5. In Fig. 2(b), it is shown that, before the control, the synchronization errors are well separated from each other, indicating the absence of synchronization among any pair of the nodes; after the control, the 5 synchronization errors are gradually merged into 3 individual ones. Specifically, from the time $t\approx 250$ on, we have simultaneously $\Delta x_2=\Delta x_5$ and $\Delta x_3=\Delta x_4$, and this synchronization relation keeps unchanged as the time increases, i.e., the system is stabilized onto the synchronous pattern (a,b,c,c,b).

We next demonstrate how the form of a synchronization pattern can be adjusted by *removing or rewiring* a link in the network. Employing still the unweighted 5-node network [Fig. 3(a)] and the normalized coupling scheme, but this time we start from the state of global network synchronization, and the targeting states are chosen as different synchronous patterns. We first make the network be globally synchronized, which is accomplished by a larger coupling strength, $\varepsilon=10.8$. The evolution of the network dynamics is plotted in Fig.

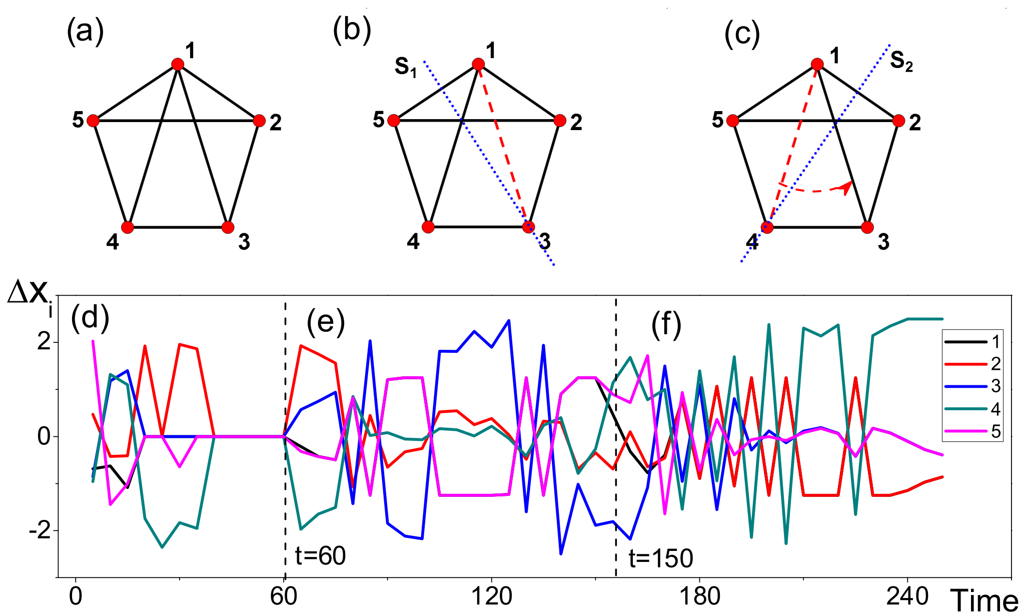

FIG. 3: (Color online) The control of the network synchronization by removing or rewiring a shortcut link. (a) The structure of the original network, which is globally synchronized under $\varepsilon = 10.8$. (b) The modified network where the link $L_{1,3}$ in (a) is removed. The new network owns the reflection symmetry $\mathbf{S}_1$, and supports the pattern (a, b, c, b, a). (c) The new network constructed from (b) by rewiring the link $L_{1,4}$. The new network owns the reflection symmetry $\mathbf{S}_2$, and supports the pattern (a, a, b, c, b). (d-f) The time evolution of the synchronization errors, Δx_i, for the networks in (a-c). The link $L_{1,3}$ in (a) is removed at $t = 60$, which leads to the pattern (a, b, c, b, a). The link $L_{1,4}$ in (b) is rewired to $L_{1,3}$ at $t = 150$, which results in the new pattern (a, a, b, c, b).

3(d), where it is seen that after a transient period the system is globally synchronized. Having reached the state of global synchronization, we then at the moment $t = 60$ remove the link $L_{1,3}$, so that the network structure is modified to the structure plotted in Fig. 3(b). In the meantime, small perturbations are added onto the oscillators, so as to diverge the trajectories from the global-synchronization manifold. The modified network [Fig. 3(b)] has the reflection symmetry, $\mathbf{S}_1$, which can support the pattern (a, b, c, b, a), given the two conditions are satisfied. From the network coupling matrix, we find that $\lambda_1^{tr} = 1.0$ and $\lambda_2^{syn} = 0.86$. The eigenvalue condition thus is satisfied. Meanwhile, since we have set $\varepsilon = 10.8$, which is just between the two critical strengths: $\varepsilon_1 = \sigma_c/\lambda_1^{tr} \approx 10$, $\varepsilon_2 = \sigma_c/\lambda_2^{syn} \approx 11.63$. The condition for the coupling strength thus is also satisfied. The numerical simulation verifies this analysis. As shown in Fig. 3(e), after removing the link $L_{1,3}$, the network is gradually changed from global synchronization to the synchronous pattern (a, b, c, b, a). In Fig. 3(c), we further modify the network structure by rewiring the link $L_{1,4}$ in Fig. 3(b). Since the networks in Figs. 3(b) and (c) are essentially the same (with a clockwise rotation of 0.4π), the network of Fig. 3(c) therefore supports the synchronous pattern (a, a, b, c, b), as verified by the numerical simulations [Fig. 3(f)]. We would like to note that, although the two patterns, (a, b, c, b, a) and (a, a, b, c, b), characterize essentially the same network dynamics, the change of the pattern from (a, b, c, b, a) to (a, a, b, c, b) is still nontrivial, as the synchronization relations of the nodes have been modified.

Finally we demonstrate how the network dynamics can be switched between *different forms of synchronous patterns*, by adding or removing a single link in the network. To illustrate this type of control, we adopt the network structure plotted in Fig. 4(a), which contains 6 nodes and one shortcut link. As depicted in Fig. 4(a), this network owns two reflection symmetries: $\mathbf{S}_1$ and $\mathbf{S}_2$. A checking of their eigenvalues shows that only $\mathbf{S}_1$ satisfies the eigenvalue condition $\lambda_1^{tr} > \lambda_2^{syn}$, which corresponds to the pattern (a, b, c, c, b, a). Since $\lambda_1^{tr} = 0.833$ and $\lambda_2^{syn} = 0.5$, to make the pattern stable, the coupling strength should be chosen within the range $\varepsilon \in (12, 20)$. By $\varepsilon = 12.8$, we plot in Fig. 4(b) the time evolution of the synchronization errors, where the formation of the pattern (a, b, c, c, b, a) is shown. To switch the pattern to another form, we add a new link, $L_{3,6}$, onto the network of Fig. 4(a), with the new network structure is shown in Fig. 4(b). The new network also possesses two reflection symmetries, $\mathbf{S}_3$ and $\mathbf{S}_4$, which may support different synchronous patterns. By analyzing the distributions of their eigenvalues, we find that only the former satisfies the condition $\lambda_1^{tr} > \lambda_2^{syn}$. As such, the stable pattern for Fig. 4(b) is only (a, b, c, d, c, b). From λ_1^{tr} and λ_2^{syn}, we can also obtain the range of the coupling strength, $\varepsilon \in (10, 15)$. Since $\varepsilon = 12.8$ is within this range, the switching from pattern (a, b, c, c, b, a) [Fig. 4(a)] to pattern (a, b, c, d, c, b) [Fig. 4(b)] thus is expected to be workable. This is confirmed by numerical simulations, as shown in

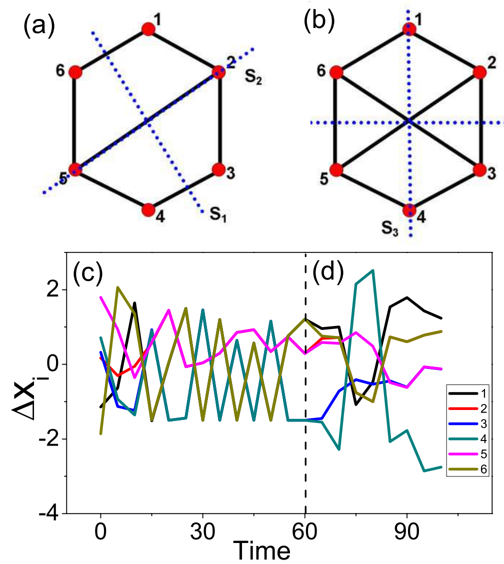

FIG. 4: (Color online) The switching of the synchronous patterns between different forms. The networks are unweighted, and the normalized coupling scheme is employed. $S_{1,2,3,4}$ are the network symmetries. (a) The original network. (b) The modified network by adding the new link $L_{3,6}$ onto the network of (a). (c) and (d), by $\varepsilon = 12.8$, the time evolutions of the synchronization errors, Δx_i, for the network structures in (a) and (b), respectively. In (c), the system dynamics is stabilized onto the pattern (a, b, c, c, b, a), which is supported by $\mathbf{S}_1$ in (a). In (d), the system dynamics is switched to the pattern (a, b, c, d, c, b), which is supported by $\mathbf{S}_3$ in (b). The topological control, i.e. the connection between nodes 3 and 6 in the network of (a), is activated at $t = 60$.

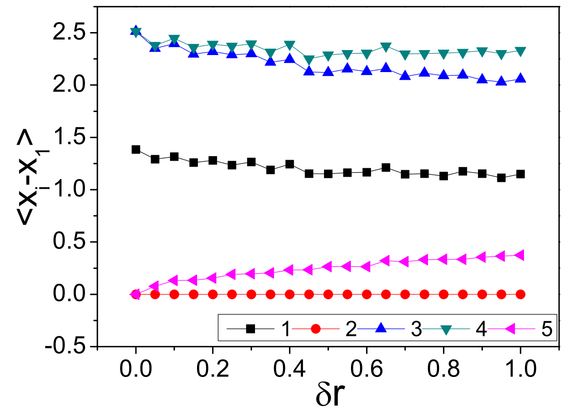

FIG. 5: (Color online) For the same network model used in Fig. 2(b) [which shows stable synchronous pattern of the form (a, b, c, c, b)], the effect of the parameter mismatch on the pattern. The mismatch is introduced to the parameter r in the Lorenz oscillator, which is implemented by randomly chosen r from the range $[35 - \delta r, 35 + \delta r]$. The synchronization errors are evaluated by $\langle x_i - x_1 \rangle$, with $\langle \ldots \rangle$ the time average over a period of $t = 1 \times 10^3$ and over 100 system realizations. It is seen that that, for a smaller parameter mismatch ($\delta r < 1$), the system dynamics is still strongly governed by the synchronous pattern (a, b, c, c, b).

Figs. 4(c) and (d).

IV. DISCUSSIONS AND CONCLUSION

The topological control we have investigated is distinct from the existing studies of network control in literature [31–33]. Firstly, in topological control the targeting states are chosen as the synchronous patterns (selected according to the network symmetries), which are spatially nonuniform; while in previous studies of network control the targeting states are normally uniform in space. For instance, in the pinning synchronization of complex networks [31], all the network nodes are controlled to the same trajectory defined by the external controller, i.e., the network is globally synchronized. For this difference, the analysis of network controllability in topological control is very different from the ones used in previous studies, e.g., it requires a separation of the phase space into two orthogonal subspaces. Secondly, dislike most of the existing studies where the controlling signals are added onto the node state [32, 33], here in topological control the perturbations are made on the network structure. While state perturbation is popular in engineering systems, topological perturbations may have more applications in biological and neural systems, e.g., in understanding the evolution and functions of the human brain [34]. Finally, in previous studies of network control once the system is controlled, the instant states of every nodes can be precisely predicted (from the trajectory of the controller), which is impossible in topological control, as the manifold of the synchronous pattern is self-organized by the network nodes.

Although established on the simplified models of clear network symmetries, the control method proposed in the present work could be potentially applied to the large-size and complex networks. In terms of the network size, in simulations we have successfully applied this method to the control of synchronous pattern for symmetric networks of size up to $N = 100$. In terms of complex networks, this method may also be helpful and constructive, due to the ubiquitous existence of topological symmetry in complex networks, either globally or locally. Firstly, for some special types of networks, e.g., the commander and control system, the network has a strict hierarchical structure, resulting in perfect network symmetries [35]. Secondly, for the general complex networks of practical interest, e.g., the small-world and scale-free complex networks, although in general it is difficult to find a perfect symmetry for the whole network, their local network structures do present some regular and symmetric features, due to either the high clustering coefficient (for small-world networks) or the abundant motif and community structures (for scale-free networks) [36]. Finally, even for the completely random networks, e.g., the Erdös-Rènyi network, there still could be some kinds of weak symmetries in the network structure (i.e., a permutation of a few of the network nodes does not affect the network structure) [37]. All these symmetries, according to our analysis, could provide a plenty of room for the control of synchronous patterns in complex systems.

However, in terms of controlling the realistic complex systems, the current study is still in its infancy, and many important issues need to be investigated. Among others, the non-identical node dynamics, the directed and weighted links, and the identification of the topological symmetries in large-size complex networks are three of the most fundamental questions to be addressed. For non-identical node dynamics, our preliminary simulations show that [Fig. 5], given the parameter mismatch among the oscillators is not significant, the system dynamics will be still governed by synchronous patterns. For symmetry identification in complex networks, we hope the rapid progress of network research would provide solutions in the near future. For instance, the newly developed algorithms for network partition have already shed some new lights on the identification of topological symmetries in large-size complex networks [38]. It is worthy of mentioning that, in controlling a realistic complex network, both the analysis method and the control strategies used in the present work should be largely improved, say, for example, it will be necessary to adjust a number of the network links simultaneously in order of an effective control of synchronous patterns in a large-scale complex network [22, 39] – a promising issue deserves further studies.

In summary, using the sensitivity feature of the network dynamics on structure, we have proposed the idea of topological control of the synchronous patterns in complex systems, and demonstrated it on some simple network models of coupled chaotic oscillators. Although based on the simplified models, our studies provide a new viewpoint to the control of network dynamics, which, after some improvements, might be applied to the control of large-scale complex networks, as well as giving insights to the operation and functioning of some realistic complex systems.

[1] M.M. Waldrop, *Complexity: The emergening science at the edge of order and chaos* (Simon & Schuster, 1993).

[2] E. Ott, *Chaos in dynamical systems* (Cambridge University Press, UK, 2002).

[3] M.A.F Sanjuán, C. Grebogi (eds.), *Recent Progress in Controlling Chaos* (World Scientific, Singapore, 2010).

[4] E. Ott, C. Grebogi, and J.A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).

[5] A.E. Motter and Y.-C. Lai, Phys. Rev. E **66**, 065102 (2002).

[6] A.E. Motter, Phys. Rev. Lett. **93**, 098701 (2004).

[7] T. Gross, C.J.D. D'Lima, and B. Blasius, Phys. Rev. Lett. **96**, 208701 (2006).

[8] A. Hagberg and D.A. Schult, Chaos **18**, 037105 (2008).

[9] Y. Qian, X.D. Huang, G. Hu, and X.H. Liao, Phys. Rev. E **81**, 036101 (2010).

[10] W.-X. Wang, X. Ni, and Y.-C. Lai, Phys. Rev. E **85**, 026115 (2012).

[11] A.S. Pikovsky, M.G. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Science* (Cambridge University Press, Cambridge, UK, 2001).

[12] A. Arenas, A. Diaz-Guilera, J. Kurths, Y. Moreno, and C.S. Zhou, Phys. Rep. **469**, 93 (2008).

[13] F. Dörfer, M. Chertkov, and F. Bullo, *Proc. Natl Acad. Sci. USA* **110**, 2005 (2013).

[14] A.E. Motter, S.A. Myers, M. Anghel, and T. Nishikawa, Nature Physics advance online publication, 10, Feb. 2013, (DOI 10.1038/nphys2535).

[15] E. Rodriguez, N. George, J.P. Lachaux, J. Martinerie, B. Renault, and F.J. Varela, Nature **397**, 430 (1999).

[16] L.M. Ward, Trends Cogn. Sci. **7**, 553 (2003).

[17] J.M. Parent and D.H. Lowenstein, Curr. Opin. Neurol. **10**, 103 (1997).

[18] H. Nakao and A.S. Mikhailov, Nature Physics **6**, 544 (2010).

[19] J.P. Crutchfield, Nature Physics **8**, 17 (2012).

[20] Y. Qian, X.D. Huang, G. Hu, and X.H. Liao, Phys. Rev. E **81**, 036101 (2010).

[21] X.G. Wang, S. Guan, Y.-C. Lai, B. Li, and C.-H. Lai, Europhys. Lett. **88**, 28001 (2009).

[22] X.G. Wang, Eur. Phys. J. B **75**, 285 (2010).

[23] M. Barahona and L.M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).

[24] G. Hu, Y. Zhang, H.A. Cerdeira, and S. Chen, Phys. Rev. Lett. **85**, 3377 (2000).

[25] L.M. Pecora, Phys. Rev. E **58**, 347 (1998).

[26] Y. Zhang, G. Hu, H.A. Cerdeira, S. Chen, T. Braun, and Y. Yao, Phys. Rev. E **63**, 026211 (2001).

[27] B. Ao and Z.G. Zheng, Europhys. Lett. **74**, 229 (2006).

[28] L. Huang, Q. Chen, Y.-C. Lai, and L.M. Pecora, Phys. Rev. E **80**, 036204 (2009).

[29] A.E. Motter, C.S. Zhou, and J. Kurths, Europhys. Lett. **69**, 334 (2005).

[30] X. G. Wang, Y.-C. Lai, and C.-H. Lai, Phys. Rev. E **75**, 056205 (2007).

[31] X. Li, X. F. Wang, and G. Chen, IEEE Trans. Circuits Syst., **51**, 2074 (2004).

[32] Y.-Y. Liu, J.-J. Slotine, and A.L. Barabási, Nature **473**, 167 (2011).

[33] G. Yan, J. Ren, Y.-C. Lai, C.-H. Lai, and B. Li, Phys. Rev. Lett **108**, 218703 (2012).

[34] E. Basar, *Brain Function and Oscillation* Springer, 1st edition, 1998).

[35] E. Ravasz and A.L. Barabási, Phys. Rev. E **67**, 026112 (2003).

[36] M. E. J. Newman, SIAM Rev. **45**, 167 (2002).

[37] Y.H. Xiao, W.T. Wu, H. Wang, M. Xiong, and W. Wang, Physica A **387**, 2611 (2008).

[38] F. Santo, Phys. Rep. **486**, 174 (2010).

[39] C. Fu, H. Zhang, M. Zhan, and X.G Wang, Phys. Rev. E **85**, 066208 (2012).