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Energy equipartition in two-dimensional granular systems with spherical intruders

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We study the effects of a line of spherical interstitial particles (or intruders) placed between two adjacent uncompressed chains of larger particles in a square packing of spheres, using experiments and numerical simulations. We excite one of the chains of particles adjacent to the intruders with an impact and show how energy is transmitted across the system until equipartition is reached from the excited (or impacted) chain to the absorbing (or adjacent) chain. The coupling of the two chains, although a purely two-dimensional effect, is modeled by a simplified one-and-a-half-dimensional (1.5-D) system in which transverse motions of the particles are neglected.

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Introduction. Granular crystals are unique nonlinear systems that exhibit interesting properties stemming from the nonlinear contact interactions (Hertzian [1]) between two individual particles. Uniform one-dimensional chains of spheres have been shown to support the propagation of a new type of solitary wave. The width of these waves is independent of its velocity [2–5]. The degree of nonlinearity can be tuned from highly nonlinear to linear by the addition of a precompression force [3, 4, 6]. Additionally, it was shown that homogeneous granular media can support families of strongly nonlinear traveling and standing waves [7], whereas heterogeneous media can exhibit resonance [8] and anti-resonance phenomena [9]. Another interesting property of these materials is the reflection of the solitary waves at an interface between two granular crystals [10–12], which could be used to develop new impulse trapping granular materials [13–15]. Several groups studied the interaction of a solitary wave with defect particles [16–19]. Two-dimensional (2-D) granular crystals have been relatively unexplored, and prior works mainly consisted of numerical studies or experiments visualizing dynamic stress in photo-elastic disks [20–23]. Solitary waves have been observed in 2-D square packings of spherical particles [24]. Granular crystals have been proposed as new structured materials for the control and redirection of stress waves (see for example [10, 13, 14, 25, 28, 33]). The experiments reported in this paper provide the first observation of energy equipartition between two adjacent and nonlinearly coupled chains of particles. In particular, we show that when one chain is excited by an impulse while the other is as rest, the energy is redistributed between the two chains within a short spatial distance. A similar equipartition phenomenon was studied numerically in an earlier work [26]. This phenomenon is of interest to create new acoustic wave guides, delay lines and stress mitigating materials. Energy transfer and equipartition phenomena in weakly coupled one-dimensional granular chains were studied [26, 27], and in [28] through a macroscopic realization of the Landau-Zener tunneling quantum effect. The energy equipartition principle is well known for elastic waves. Seismic waves for example have well known regimes where the P and S wave energy density equilibrates in a unique way that is independent of the details of the scattering. Interaction of solitons in coupled nonlinear lattices (scalar models) have been considered for various classical configurations such as coupled Toda lattices [29], coupled nonlinear Schrodinger equations [30, 31] or coupled Ablowitz-Ladik chains [32] for example. In the case of coupled Toda lattices, it was shown numerically by Kevrikidis et al. [29] that solitonic excitations supplied to each one of the coupled chains may result in the two distinct dynamical regimes (attractors). A non-uniform initial excitation (solitons with different amplitudes and/or phase mismatches) may lead to the formation of the identical solitons on each one of the chains, propagating with the same speed and zero phase mismatch (first attractor) as well as the formation of two unequal solitons (i.e. with different amplitudes and phases also propagating with the same speed, second attractor). Here, we report an extension of energy equipartition phenomena in 2-D granular media perturbed by lines of intruders.

Experimental setup. We designed a 2-D setup to study the response of a 2-D square packing of elastic spheres with the presence of a line of interstitial defets (or intruders). It consisted of a flat polycarbonate base and four delrin walls to support the particles. We assembled a 20 by 20 granular crystal composed of grade 316 stainless steel spheres ($R = 9.525$ mm radius) organized in a square packing configuration, and no precompression force was added to the system. A line of tungsten carbide intruders was placed in between the excited and absorbing chains, as shown in Fig. 1. The intruders are custom made spherical particles with radius $R_d = 3.943mm \pm 0.001mm$ [34]. Custom made teflon stands were lodged in the interstitial sites below the intruders to keep the centers of mass of all the particles in a same horizontal plane. We excited one of the chains of particles adjacent to the intruders with a striker impact. The striker was a stainless steel sphere identical to the beads composing the square packing and its velocity was measured by an optical velocimeter. In this study, $v_{impact} = 0.147m/s$, which corresponds to an impact force of $90N$. The stainless steel and tungsten carbide (TC) beads have densities 8000 and 15800 kg/m^3 and elastic moduli 193 and 400 GPa, respectively [35]. We built custom sensor particles using mini tri-axial accelerometers (PCB 356A01, with sensitivity 0.51 $mV/(m/s^2)$) embedded in spherical particles located in every other position in the excited and absorbing chains. We then compared the output acceleration to numerical simulations.

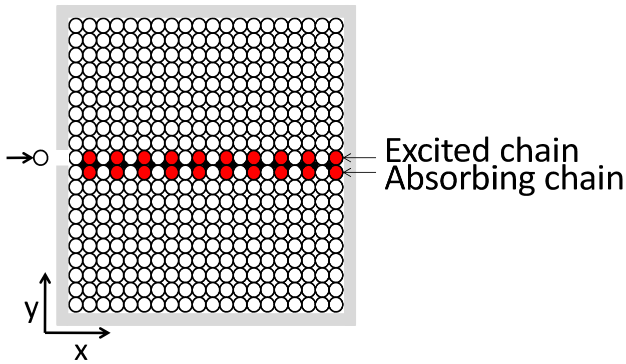

FIG. 1: (Color online) Schematic diagram of the experimental setup. The small (black) particles correspond to the spherical intruders. The red (gray) particles show the positions chosen for the placement of the sensor particles.

Numerical setup. We numerically model a square packing of 400 particles, plus the intruders and one striker particle as point masses connected by nonlinear springs, accordingly to the (conservative) Hertz interaction. The walls are modelled as still spherical particles of infinite radii. The equation of motion for particle i is:

$$m_i \ddot{\vec{X}}_i = -\sum_j A_{i,j}[(\vec{X}_i - \vec{X}_j).\vec{e_{ij}}]_+^{\frac{3}{2}} \vec{e_{ij}}, \tag{1}$$

where $\vec{X}_i$ is the vectorial displacement of the i^{th} particle from its equilibrium uncompressed position, j sums over the number of neighbors directly interacting (i.e. in contact) with particle i, $A_{i,j} = \frac{4}{3}\sqrt{\frac{R_i R_j}{R_i+R_j}}\left(\frac{1-\nu_i^2}{E_i} + \frac{1-\nu_j^2}{E_j}\right)^{-1}$ and $\vec{e_{ij}}$ unit vector connecting the centers of beads i and j, oriented form i to j. $[Y]_+$ denotes the positive part of Y as no interaction exists between particles when they lose contact. We numerically integrated Eq. 1 using a fourth order Runge-Kutta scheme, and chose a timestep of $5 \times 10^{-8} s$. This ensures conservation of the total energy with an accuracy better than $3 \times 10^{-7}\%$.

Results and discussion. We study the temporal evolution of the transmitted acceleration along the x-direction in the chains of particles adjacent to the intruders (Fig. 2). We compare results obtained from experiments (Fig. 2(a) and Fig. 2(b)) with the corresponding numerical simulations (Fig. 2(c) and Fig. 2(d)). Each curve corresponds to a different sensor location placed in all even particles in the impacted chain and in the chain adjacent to the intruders (Fig. 1). Without the presence of the line of intruders, a solitary wave forms and propagates along the excited chain (pseudo 1-D case, see [24]). The presence of intruders introduces a nonlinear coupling between the two chains, leading to reciprocal energy and momuntum transfer between them. We observe in experiments and numerical simulations that the input energy of the leading propagating pulse (minus what is radiated in the transverse chains) is equally split between the excited and absorbing chains after approximately 8 particles, as two distinct pulses with similar velocity and amplitude form and propagate down the two chains. We use the term “equipartition” to describe the phenomenon of equal division of energy and momentum between two adjacent granular chains, after one of the two is excited by an impulse.

The intruders are also responsible for the scattering of energy in the y-direction, and we observe the formation of solitary waves travelling perpendicular to the excited and absorbing chains. For each particle in these two chains, the amplitude of the velocity in the y-direction is small ($\approx 20\%$) in comparison to that in the x direction. This corresponds to $\approx 4\%$ of the kinetic energy, and we neglect the leakage of energy in the transverse direction.

The amplitude of the wave traveling in the excited chain decreases drastically within the first few particles as energy is transfered to the absorbing chain through each light intruder. It was shown in [36] that 82.3% of the energy of a solitary wave reaching a TC intruder is transmitted along the chain after interacting with the intruder. The rest of the energy is scattered in adjacent and perpendicular chains, or reflected. Reversely, the amplitude of the wave propagating in the absorbing chain increases from 0 to the same level as the signal in the excited chain. It is important to note that these systems are tunable: the material selected for the intruders affects the speed of the energy transfer since weaker interactions between the line of intruders and the excited and absorbing chains will result in a slower transfer of energy. Numerical simulations and experiments were performed by the authors for a line of Teflon particles (Young modulus $1.26GPa$) in order to evidence this effect (not shown here).

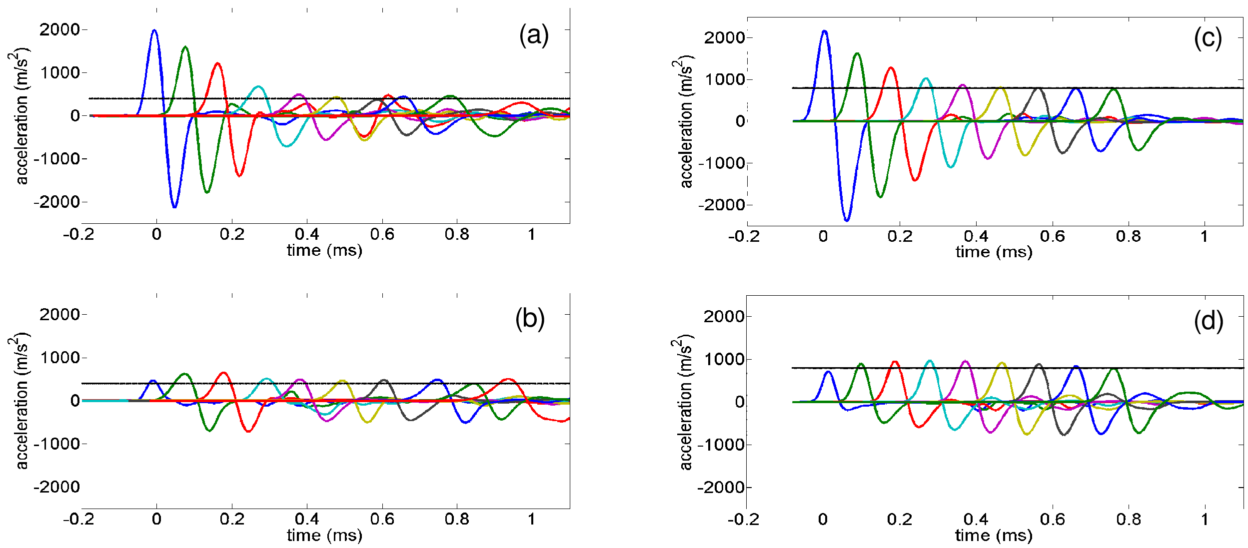

FIG. 2: (Color online) Acceleration-time diagrams showing the evolution of the transmitted signal in the two chains of particles adjacent to the line of intruders, along the x-direction. (a) Experimental results in the excited chain (at all even particles in the chain). (b) Experimental results in the absorbing chain. (c) Numerical data corresponding to (a). (d) Numerical data corresponding to (b). In all panels, the system is impacted by a stainless steel sphere of radius 9.525 mm with initial velocity 0.147 m/s. The horizontal black lines show the stabilized amplitude of the leading acceleration pulse ($\approx$ 800 m/s^2 in the numerical simulations and 400 m/s^2 in the experiments).

We find excellent agreement between our numerical and experimental results (compare Fig. 2(a) with Fig. 2(c) for the excited chain; Fig. 2(b) with Fig. 2(d) for the absorbing chain). The effect of dissipation is visible in the experimental data, and the stabilized amplitude reached in the two chains in experiments is smaller than in our numerical calculations. After equipartition, we observe in the numerical data a slow decrease of the amplitude of the two waves, due to radiations in the y-direction, as the central light intruders push the heavy steel particles away from the centerline. Although we expect the same effect to be present in the experimental data, it is difficult to determine how much of the loss is accounted for by the radiations versus dissipation.

In the next section, we present a new analytical model which explains how this fully 2-D phenomenon can be described by a simplified 1.5-D system, and compare this simplified theoretical model to our experimental and numerical results for the 2-D system (similar in form to the one studied in [26–28]).

Simplified 1.5-D modelization and comparison with full 2-D system. Upon arrival of the incoming pulse, the presence of each intruder induces a displacement of the two larger spheres located after it in both the x and y directions. However, the amplitude of the waves traveling in the y-direction is small. In our model, we neglect the energy loss in the y-direction, constraining the particles of the excited and absorbing chains, and the intruders, to move only horizontally. For simplicity, we limit our analysis to modeling the dynamics of the intruders and the particles of the excited and absorbing chains. Although all particles are contrained to move in the x-direction, energy transfer between the excited and absorbing chains in the y-direction is still possible due to the strongly nonlinear coupling induced by the line of intruder (1.5-D system).

We denote by m, ν, E and ρ the mass, Poisson's ratio, Young's modulus and density of the heavy steel particles. The same quantities with subscript d correspond to the properties of the light TC defect particles. We then use the normalized displacement $x = \frac{X}{R}$ and time $\tau = t\sqrt{\frac{E}{\pi R^2 \rho}}$ to rewrite the set of equations (1) into the non-dimensional equations describing the motions of the intruders (2), the particles composing the excited chain (3) and particles composing the absorbing chain (4):

$$\begin{aligned}\epsilon x_i^{d''} &= \beta\left[f(x_i^e - x_i^d) - f(x_{i+1}^a - x_i^d)\right] \\ &\quad + \beta\left[f(x_i^e - x_i^d) - f(x_{i+1}^e - x_i^d)\right], \end{aligned} \tag{2}$$

$$x_i^{e\prime\prime} = \beta\left[f(x_i^e - x_{i-1}^d) - f(x_i^e - x_i^d)\right]$$
$$+ \alpha\left[g(x_{i-1}^e - x_i^e) - g(x_i^e - x_{i+1}^e)\right], \tag{3}$$

$$x_i^{a\prime\prime} = \beta\left[f(x_{i-1}^d - x_i^a) - f(x_i^a - x_i^d)\right]$$
$$+ \alpha\left[g(x_{i-1}^a - x_i^a) - g(x_i^a - x_{i+1}^a)\right], \tag{4}$$

where $\alpha = \frac{1}{2\sqrt{2}(1-\nu^2)}$, $\beta = \sqrt{\frac{2-\sqrt{2}}{2}}\frac{E_d}{E(1-\nu_d^2)+E_d(1-\nu^2)}$ and $\epsilon = \frac{m_d}{m}$ is our small parameter. We used $f(x) = \left(\sqrt{2} - \sqrt{x^2+1}\right)_+^{3/2}$ and $g(x) = (x+2)_+^{3/2}$. x is the scalar non-dimensional displacement of the particles (constrained in the x direction) and x'' represents the second derivative with respect to the normalized time τ. The superscript d corresponds to the defect particles, e corresponds to the particles in the excited chain, and a to the particles in the absorbing chain. The subscript i indicates the position of every particle in their chains, oriented along the positive x axis.

For the materials chosen in our study, $\epsilon = 0.134$, $\beta = 1.022$ and we assume the ratio $\frac{\epsilon}{\beta}$ to be small enough to neglect the inertia of the intruders. Equating the right term of Eq. (2) to 0 leads to:

$$f(x_i^e - x_i^d) - f(x_i^d - x_{i+1}^a) = -[f(x_i^a - x_i^d) - f(x_i^d - x_{i+1}^e)]. \tag{5}$$

We numerically showed (not presented in this paper) that the two sides of Eq. (5) are independently small and we can consequently assume:

$$f(x_i^e - x_i^d) \approx f(x_i^d - x_{i+1}^a),$$
$$f(x_i^a - x_i^d) \approx f(x_i^d - x_{i+1}^e). \tag{6}$$

This means that, to a first order approximation, the diagonal forces applied on a defect particle are independent from each other. Eq. (6) yields:

$$x_i^d \approx \frac{x_i^e + x_{i+1}^a}{2} \approx \frac{x_i^a + x_{i+1}^e}{2}. \tag{7}$$

Further analytical treatment assumes that the right-hand side and left-hand side of Eqs. (6) and (7) are equal. We can replace x_i^d and x_{i-1}^d in Eqs. (3) and (4) to obtain a system of equations for the excited and absorbing chains only. To incorporate the radiation effect, we finally add a linear damping term in order to account for the decay caused by the energy leakage in the transverse direction:

$$x_i^{e\prime\prime} = \beta\left[f\left(\frac{x_{i-1}^a - x_i^e}{2}\right) - f\left(\frac{x_i^e - x_{i+1}^a}{2}\right)\right]$$
$$+ \alpha\left[g(x_{i-1}^e - x_i^e) - g(x_i^e - x_{i+1}^e)\right] - \lambda x_i^{e\prime}, \tag{8}$$

$$x_i^{a\prime\prime} = \beta\left[f(x_{i-1}^e - x_i^a) - f(x_i^a - x_{i+1}^e)\right]$$
$$+ \alpha\left[g(x_{i-1}^a - x_i^a) - g(x_i^a - x_{i+1}^a)\right] - \lambda x_i^{a\prime}. \tag{9}$$

We used a vectorized fourth order Runge-Kutta scheme to integrate this simplified set of equations. The numerical setup consists of two adjacent chains of 20 particles each (absorbing and excited chains) and 19 interstitial intruders. Similarly as before, the walls are modeled as fixed spherical particles of infinite radius. The excited chain is impacted by a striker particle with initial velocity obtained from experiments. We show the numerical solution in Fig. 3. We clearly observe the energy equipartition previously evidenced in our experiments and in the fully 2-D numerical simulations. Similarly to what observed earlier, also in the 1.5-D model the velocity stabilizes in the two chains after approximately 8 particles.

The results obtained from the simplified 1.5-D model are compared with the fully 2-D system in Fig. 4. The only fitting parameter used in the 1.5-D model is the linear damping coefficient λ (dimensionless), which is taken to be 0.006. The results show that the 1.5-D approximation captures very well the main features of the 2-D system: the

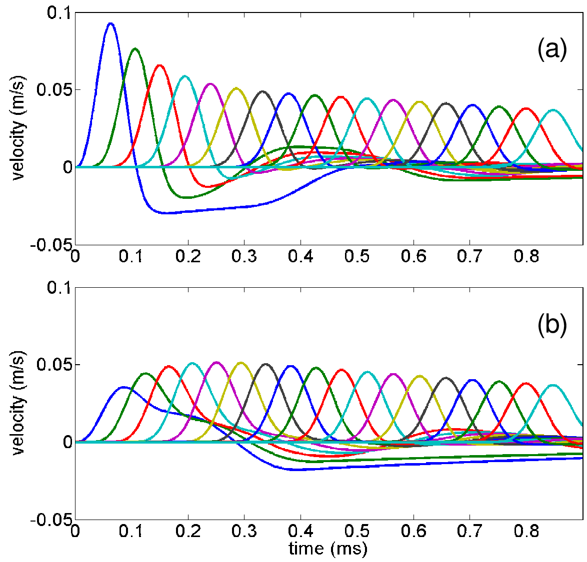

FIG. 3: (Color online) x-velocity *vs* time signals obtained for the 1.5-D model with linear damping coefficient λ equal to 0.006. Each curve represents the velocity of every particle in the excited chain (a) and the absorbing chain (b). The system is impacted by a stainless steel sphere of radius 9.525 mm with initial velocity 0.147 m/s.

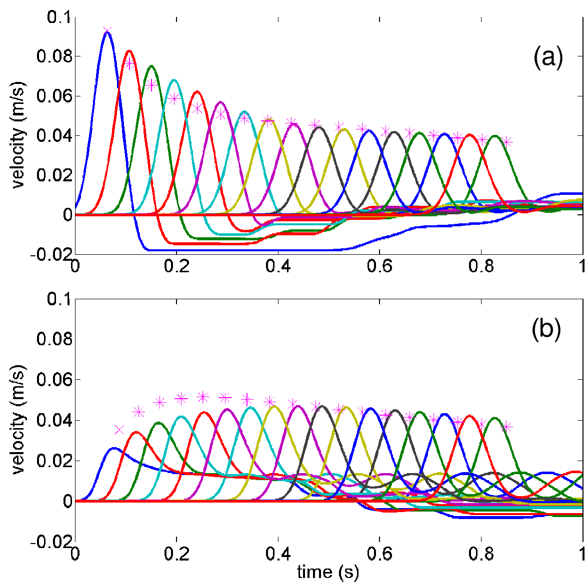

FIG. 4: (Color online) Comparison of the numerical x-velocity *vs* time signals obtained from the 2-D and simplified 1.5-D models (from Fig. 3). The results for the excited chain are shown in (a), and (b) shows the results for the absorbing chain. In both plots, the curves are obtained from the 2-D model, whereas the (purple) stars indicate the peak amplitudes of the corresponding signals obtained from the 1.5-D model.

energy equipartition is reached after the same number of particles, and the amplitudes of the leading pulses are in very good agreement.

Conclusion. In this paper we showed with experiments and numerical simulations that the mechanism of solitary waves equipartition previously observed in linearly coupled, nonlinear lattices persists in full 2-D granular setups with a more complex type of coupling (strongly nonlinear, non-smooth, diagonal coupling via light interstitial intruders). This energy equipartition phenomenon could be used for the creation of novel acoustic delay-lines, wave guides and protective materials. The regime of the primary pulse transmission in the strongly-nonlinear, heterogeneous, 2-D granular crystals is well captured by our simplified, reduced order model (granular scalar model). The results of the present work will pave way for further analytical, numerical and experimental studies of the mechanisms of energy transfer and wave redirection in higher dimensional granular crystals. Moreover, the mechanism of equipartition of solitary pulses in granular crystals realized through the placement of interstitial intruders is of great practical importance in the design of granular shock absorbers being able to efficiently distribute the initially localized shock over the entire granular medium.

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