Dispersive shock wave interactions and asymptotics

Mark J. Ablowitz and Douglas E. Baldwin

Phys. Rev. E **87**, 022906 — Published 13 February 2013

DOI: 10.1103/PhysRevE.87.022906

Dispersive shock wave interactions and asymptotics

Mark J. Ablowitz and Douglas E. Baldwin*

Department of Applied Mathematics, University of Colorado, Boulder, Colorado, 80309-0526, USA

Dispersive shock waves (DSWs) are physically important phenomena that occur in systems dominated by weak dispersion and weak nonlinearity. The Korteweg–de Vries (KdV) equation is the universal model for systems with weak dispersion and weak, quadratic nonlinearity. Here we show that the long-time-asymptotic solution of the KdV equation for general, step-like data is a single-phase DSW; this DSW is the 'largest' possible DSW based on the boundary data. We find this asymptotic solution using the inverse scattering transform and matched-asymptotic expansions. So while multi-step data evolve to have multiphase dynamics at intermediate times, these interacting DSWs eventually merge to form a single-phase DSW at large time.

I. INTRODUCTION

Dispersive shock waves (DSWs) have been seen in plasmas [1], fluids (e.g., undular bores) [2, 3], superfluids [4–7], and optics [8–11]. DSWs occur when weak nonlinearity and weak dispersion dominate the physics and there is step-like data. For many weakly dispersive, weakly nonlinear systems, the Korteweg–de Vries (KdV) equation is the leading-order asymptotic equation [12]. Here we find the long-time-asymptotic behavior of the KdV equation with general, step-like data using the inverse scattering transform (IST) and matched-asymptotic expansions. We show that general, step-like data go to a single-phase DSW for the KdV equation in the long-time, fixed-dispersion limit. Our results show that while multi-step data evolve to have multiphase dynamics at intermediate times, these interacting DSWs eventually merge to form a single-phase DSW: each sub-step in well-separated, multi-step data forms its own DSW (Fig. 1a); these DSWs then interact and develop multiphase dynamics at intermediate times (Figs. 1b and 1c); and, in the long-time limit, these DSWs merge to form a single-phase DSW (Fig. 1d). The boundary data determine this single-phase DSW's form; the initial data determine its position. This is similar to interacting viscous shock waves (VSW), where only the single, largest possible VSW remains after a long time. Grava and Tian [13] and Ablowitz et al. [14] suggested this merging of multiphase to single-phase by their two-phase to one-phase results — they used Whitham theory, which applies to slowly varying periodic wavetrains. We anticipate that the IST and matched-asymptotic procedure presented here will be applied to other important, nonlinear integrable systems for general, step-like data, such as the modified Korteweg–de Vries (mKdV) and the nonlinear Schrödinger (NLS) equations.

A shock wave is an abrupt change in the medium that propagates; it often moves faster than the local wave speed. If dissipation and dispersion are ignored, then breaking occurs in finite time; since this is not usually physical, most models include weak dissipation or weak dispersion. When dissipation dominates dispersion, a VSW forms that is smooth but changes rapidly from one value to another; VSWs form in compressible gases and other classical fluids. When dispersion dominates dissipation, a DSW forms that is smooth but has an additional modulated wavetrain that allows transitions from one value to another; DSWs form in cold plasmas, superfluids (like Bose–Einstein condensates), and nonlinear electromagnetic waves in suitable optical materials.

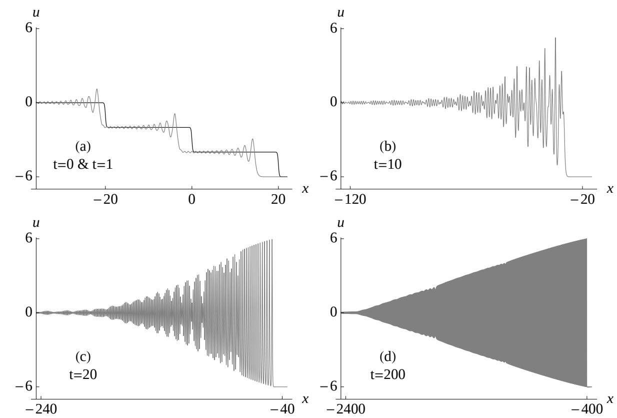

FIG. 1. Numerical simulations (using the scheme in [14]) of three well-separated steps (at $t = 0$, with $\varepsilon^2 = 0.1$ and $c = 1$). Here we see: (a) three single-phase DSWs at $t = 1$; (b) and (c) strong interaction and multiphase dynamics; and (d) eventual merging to form a single-phase DSW.

The KdV and NLS equations are universal models: they are the leading-order asymptotic equations for a wide class of physical phenomena (see [12]). The KdV equation is the leading-order asymptotic equation for systems with weak dispersion and weak, quadratic nonlinearity; it has important applications in shallow water waves, plasmas, lattice dynamics, and elasticity among others. The NLS equation is the leading-order asymptotic equation for quasi-monochromatic, weakly nonlinear systems; it has important applications in nonlinear optics, deep water waves, Bose–Einstein condensates, and magnetic-spin waves among others.

Here we consider the DSWs that the KdV equation describe; the KdV equation, written in dimensionless form, is

$$u_t + uu_x + \varepsilon^2 u_{xxx} = 0, \tag{1}$$

where subscripts denote partial derivatives. We will consider the boundary conditions

$$\lim_{x\to-\infty} u = 0 \quad \text{and} \quad \lim_{x\to+\infty} u = -6c^2. \tag{2}$$

* dsw@douglasbaldwin.com; http://www.douglasbaldwin.com

Here, ε and c are real, positive constants, and ε corresponds to the size of the regularizing dispersive effects. We require that u goes to these limits sufficiently rapidly; so we assume that

$$\int_{-\infty}^{\infty} \left| u(x,0) + 6c^2 H(x) \right| (1 + |x|^n)\, \mathrm{d}x < \infty, \tag{3}$$

for $n = 1, 2, \ldots$ and where $H(x > 0) = 1$ and $H(x \leq 0) = 0$ is the Heaviside function. Since the KdV equation is Galilean invariant, we can transform any constant boundary conditions where $\lim_{x\to-\infty} u > \lim_{x\to+\infty} u$ to (2). We use the IST method (see [15–18]) and matched-asymptotic expansions (see [19, 20]) to find a long-time-asymptotic solution.

IST method

The IST method is the nonlinear analog of the Fourier transform method: we transform the initial data into scattering data; we evolve this scattering data in time; and we then recover the solution from the evolved scattering data. First we associate the nonlinear partial differential equation (PDE) with a (linear) Lax pair. Then we use the scattering equation of the Lax pair to transform the initial data into scattering data. Then we use the other linear equation of the Lax pair to evolve the scattering data. Finally, we use a linear integral equation, the Gel'fand–Levitan–Marchenko (GLM) integral equation, to recover the solution at any time.

Elegant and powerful asymptotic methods based on Riemann–Hilbert problems can also be used to recover the solution at any time. They have been used to find the asymptotic solution for large time with vanishing boundary conditions (see [21, 22]); see [23] for a NLS shock example. For our purposes, the GLM integral equation and our matched-asymptotic method is sufficient.

Hruslov [24] and then Cohen [25] and Cohen and Kappeler [26] studied the IST theory for step-like initial data; we state the IST results that we need to find our asymptotic solution in section II. Hruslov [24], based on [27], presented the GLM integral equations and investigated the soliton train at the DSW's right. Cohen [25] and Cohen and Kappeler [26], using the methods of [16, 27], rigorously studied some scattering-data properties, rederived the GLM integral equations, and analyzed existence for piecewise-constant initial conditions. We derive the GLM integral equations in a different way in appendix B.

Long-time asymptotic solution

Our long-time-asymptotic-analysis results are new. We find the long-time-asymptotic solution for non-vanishing boundary conditions (where $c \neq 0$) by using and suitably modifying the methods in [19, 20]. Ablowitz and Segur [19, 20] developed these IST and matched-asymptotic methods to find the long-time-asymptotic solution for vanishing boundary conditions (where $c = 0$). We show, for large time, that $u(x,t)$ goes to a single-phase DSW that has three basic regions (Fig. 2):

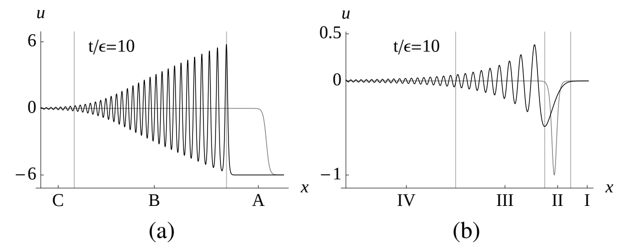

FIG. 2. Numerically computed solutions of the KdV equation for the initial conditions shown in gray. (a) Non-vanishing boundary conditions, $c \neq 0$. This solution has three basic regions: rapid decay in region A, right of the DSW; strong nonlinearity of width $O(t)$ in region B; and an oscillating tail in region C, left of the DSW. (b) A vanishing boundary conditions, $c = 0$. Here, the solution that has four basic regions (see [19]). Region III has strong nonlinearity with height $O[(\log t)^{1/2} t^{-2/3}]$ and width $O[t^{1/3}(\log t)^{2/3}]$.

- an exponentially small solution for $x \geq O(t)$ (region A in Fig. 2a);
- a slowly varying cnoidal-wave solution for $|x| \leq O(t)$ (region B in Fig. 2a), which has a soliton train on its right and an oscillatory wave on its left; and
- a slowly varying oscillatory solution for $(-x) \geq O(t)$ (region C in Fig. 2a).

Comparison with vanishing boundary conditions

The long-time-asymptotic solution of the KdV equation when $c \neq 0$ is quite different from when $c = 0$ (see Fig. 2): the strong nonlinearity when $c = 0$ is only over $|x| \leq O[t^{1/3}(\log t)^{2/3}]$, but when $c \neq 0$ it is over $|x| \leq O(t)$. From [19], the long-time-asymptotic solution when $c = 0$ has four basic regions:

- an exponentially small solution for $x \geq O(t)$ (region I in Fig. 2b);
- a growing similarity solution for $|x| \leq O(t^{1/3})$ (region II in Fig. 2b), which is related to Painlevé II's solution;
- a collisionless-shock solution for $(-x) = O[t^{1/3}(\log t)^{2/3}]$ (region III in Fig. 2b), which is a slowly varying cnoidal wave analogous to a DSW; and
- an oscillatory similarity solution for $(-x) \geq O(t)$ (region IV in Fig. 2b), which has the same form as region C in Fig. 2a.

The amplitude for all these regions when $c = 0$ decays in time at least as $O(t^{-1/2})$; the amplitude when $c \neq 0$ is $O(1)$.

Comparison with the linear problem

The long-time-asymptotic solution of the KdV equation is also quite different from the linear problem ($\tilde{u}_t + \varepsilon^2 \tilde{u}_{xxx} = 0$). Both problems have three basic regions; but the middle regions have different widths: the linear KdV equation has a middle region with strong nonlinearity over $|x| \leq O(t^{1/3})$, while the nonlinear KdV equation has a middle region (region B in Fig. 2a) over $|x| \leq O(t)$. The linear problem's solution in the middle region is

$$\tilde{u}(x,t) \sim U_0(0) \int_{-\infty}^{\eta} \mathrm{Ai}(\eta')\,\mathrm{d}\eta', \qquad \eta = \frac{x}{(3\varepsilon^2 t)^{1/3}},$$

where $\mathrm{Ai}(x)$ is the Airy function and U_0 is the Fourier transform of $\tilde{u}_x(x,0)$.

Comparison with viscous shock waves

In the long-time limit, both DSWs and VSWs merge to form a single shock wave. For shock waves where dissipation dominates dispersion, Burgers' equation is the leading-order asymptotic equation. Burgers' equation, in normalized form, is

$$w_t + ww_x - \nu w_{xx} = 0,$$

where $\nu > 0$ is a measure of dissipation and is typically small. If we take initial data that go rapidly to the boundary conditions $\lim_{x\to-\infty} w(x,t) = 0$ and $\lim_{x\to+\infty} w(x,t) = -h^2$, then the long-time-asymptotic solution is

$$w(x,t) \sim -\frac{h^2}{2}\left\{1 + \tanh\left[\frac{h^2}{4\nu}\left(x - x_0 + \frac{h^2}{2}t\right)\right]\right\},$$

where x_0 is a real constant that depends on the initial data — see appendix A for details. Thus, for both Burgers' and the KdV equation, well-separated step data go to a single shock wave in the long-time limit: the boundary conditions determine its form, and the initial data determine its location. But unlike with Burgers' equation, the solution of the KdV equation can also have a finite number of solitons, which move to the DSW's right in the long-time limit.

Relation to previous work

Single-step data, such as a Heaviside function, have been studied extensively (see [28–31]) using wave-averaging techniques, which are often called Whitham theory [32, 33]. Whitham theory averages over suitable, slowly varying periodic waves to get reduced equations; these reduced equations are a quasi-linear, first-order, hyperbolic system that describes how the periodic wave's parameters slowly evolve.

The evolution of multiphase DSWs to a single-phase DSW was investigated in the two-phase case by Grava and Tian [13] using Whitham theory in the zero-dispersion limit ($\varepsilon \to 0$) for finite time and by Ablowitz et al. [14] using numerical and asymptotic methods in the fixed-dispersion, long-time limit. Both zero-dispersion and long-time are important, but different, limits. Here we study the long-time limit with fixed dispersion. By using the IST method, we find the asymptotic solution directly: we can investigate general, step-like initial data and DSW interactions without having to find the solution at intermediate times. On the other hand, Whitham theory in the zero-dispersion limit requires that the solution is found at intermediate times through a nonlinear hyperbolic system.

The IST method also gives the behavior to the left and right of the DSW; it's nontrivial to get such behavior from the Whitham theory results. For example, it's useful to compare [34] with [19, 20] to see how each matches the solution (for vanishing boundary conditions) in region III (Fig. 2b) to that in region IV. Also compare [35] with [19] to see how each matches the solution in region II to that in region III.

The key result for the fixed-dispersion, long-time limit is that the DSWs from well-separated, multi-step data merge to form a single-phase DSW.

To find the KdV equation's long-time-asymptotic solution: We give the IST results that we need in section II. Then we asymptotically solve the linear GLM integral equation to find the exponentially small solution right of the DSW (sec. III A); use matched asymptotics to get the DSW (sec. III B), which is a slowly varying cnoidal wave that has a soliton train on its right and an oscillatory tail on its left; and find the small, decaying, oscillatory solution left of the DSW (sec. III C) that matches into the DSW. Finally, we draw some conclusions (sec. IV), find Burgers' equation's long-time-asymptotic solution (app. A), derive the GLM integral equation (app. B), and outline (app. C) how we find the solution in section III C.

II. IST SOLUTION

The IST method first associates a Lax pair with the nonlinear PDE. Using the Lax pair's scattering equation, we transform the initial data into the scattering data. We then evolve the scattering data in time using the associated linear equation. The GLM integral equation, a linear integral equation, provides the inversion at any time, and so we can recover the solution at any time.

The Lax pair associated with (1) is

$$v_{xx} + \left(u/6 + \lambda^2\right) v/\varepsilon^2 = 0, \tag{4a}$$

$$v_t = (u_x/6 + \gamma)\,v + \left(4\lambda^2 - u/3\right) v_x, \tag{4b}$$

where λ is the spectral parameter and γ is a constant. This linear pair is compatible ($v_{xxt} = v_{txx}$) when $u = u(x,t)$ satisfies (1) and λ is isospectral ($\partial\lambda/\partial t = 0$).

We use (2) to define the eigenfunctions that satisfy (4a) (for notational simplicity, we often suppress the time dependence):

$$\phi(x;\lambda) \sim \exp(-i\lambda x/\varepsilon), \qquad \bar{\phi}(x;\lambda) \sim \exp(i\lambda x/\varepsilon), \tag{5a}$$

as $x \to -\infty$ and

$$\psi(x;\lambda_r) \sim \exp(i\lambda_r x/\varepsilon), \quad \bar{\psi}(x;\lambda_r) \sim \exp(-i\lambda_r x/\varepsilon), \tag{5b}$$

as $x \to +\infty$, where $\lambda_r \equiv \sqrt{\lambda^2 - c^2}$. We take the branch cut of λ_r to be $\lambda \in [-c, c]$, and the branch cut of λ to be $\lambda_r \in [-ic, ic]$; then $\mathrm{Im}(\lambda_r) \gtrless 0$ when $\mathrm{Im}(\lambda) \gtrless 0$. This branch cut is one of the main differences between vanishing and nonvanishing boundary conditions: vanishing boundary conditions give eigenfunctions that do not have a branch cut.

The Wronskian, $W(f,g) \equiv fg_x - f_x g$, is constant (in x) for (4a) by Abel's identity; so, from (5), $W(\phi,\bar{\phi}) = 2i\lambda/\varepsilon$ and $W(\psi,\bar{\psi}) = -2i\lambda_r/\varepsilon$. The scattering eigenfunctions and scattering data a and b associated with (4a) satisfy

$$\phi(x;\lambda) = a(\lambda,\lambda_r)\bar{\psi}(x;\lambda_r) + b(\lambda,\lambda_r)\psi(x;\lambda_r) \tag{6}$$

for $\lambda_r \neq 0$, $\lambda_r \in \mathbb{R}$ (or, equivalently, $|\lambda| > c$, $\lambda \in \mathbb{R}$). The scattering data can be written as

$$a = \frac{\varepsilon}{2i\lambda_r} W(\phi,\psi) \quad \text{and} \quad b = \frac{\varepsilon}{2i\lambda_r} W(\bar{\psi},\phi). \tag{7}$$

We can use this to extend a to $|\lambda| < c$, $\lambda \in \mathbb{R}$ (where λ_r is pure imaginary); when $|\lambda| < c$, $\lambda \in \mathbb{R}$, ψ is real and exponentially decaying. This also gives that $a = -b$ for $|\lambda| \leq c$, $\lambda \in \mathbb{R}$ and $|a|^2 - |b|^2 = \lambda/\lambda_r$ for $|\lambda| > c$, $\lambda \in \mathbb{R}$.

It's convenient to define the transmission coefficient $T \equiv 1/a$ and the reflection coefficient $R \equiv b/a$. Then (6) can be written as

$$T(\lambda,\lambda_r)\phi(x;\lambda) = \bar{\psi}(x;\lambda_r) + R(\lambda,\lambda_r)\psi(x;\lambda_r). \tag{8}$$

We use (2), (4b), and (8) to find how T and R evolve in time:

$$T(\lambda,\lambda_r;t) = T(\lambda,\lambda_r;0)\exp[i(4\lambda^2\lambda_r - 4\lambda^3 + 2c^2\lambda_r)t/\varepsilon]$$

and

$$R(\lambda,\lambda_r;t) = R(\lambda,\lambda_r;0)\exp[i(8\lambda^2\lambda_r + 4c^2\lambda_r)t/\varepsilon].$$

For vanishing boundary conditions ($c = 0$), the transmission coefficient T does not depend on time. But here, where $c \neq 0$, the transmission coefficient does depends on time; this dependance when $|\lambda| < c$, $\lambda \in \mathbb{R}$ is not pure phase.

The associated GLM integral equation — see appendix B for a derivation — is

$$G(x,y;t) + \Omega(x+y;t) + \int_x^\infty \Omega(y+z;t)G(x,z;t)\,\mathrm{d}z = 0, \tag{9}$$

where

$$\Omega(\xi;t) = \frac{1}{2\varepsilon\pi}\int_{-\infty}^{\infty} Re^{i\lambda_r\xi/\varepsilon}\,\mathrm{d}\lambda_r + \sum_j c_j e^{-\tilde{\kappa}_j\xi/\varepsilon} + \frac{1}{2\varepsilon\pi}\int_0^c |\lambda T/\lambda_r|^2 e^{-\sqrt{c^2-\lambda^2}\xi/\varepsilon}\,\mathrm{d}\lambda,$$

the constants $\{i\kappa_j\}_{j=1}^N$ are the (simple) poles of $T(i\kappa_j,\lambda_r(i\kappa_j);t)$, $\tilde{\kappa}_j = \sqrt{\kappa_j^2 + c^2}$, $c_j = -i\mu_j/[\varepsilon\partial_{\lambda_r}a(i\kappa_j)]$, $\phi(x;i\kappa_j,t) \equiv \mu_j(t)\psi(x;i\kappa_j,t)$, and $0 < \kappa_1 < \cdots < \kappa_N$ are real. The kernel Ω has contributions from the reflection coefficient, from the poles, and from the branch cut (the $|\lambda T/\lambda_r|^2$ term) — there is no branch-cut contribution in the $c = 0$ case. We will omit any contributions from poles in our asymptotics. These poles relate to the solitons, which move to the right of the DSW, and so do not affect the DSW in the long-time limit.

From G, we recover $u(x,t)$ from

$$u(x,t) = -6c^2 + 12\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}x}G(x,x;t). \tag{10}$$

III. LONG-TIME ASYMPTOTICS

For large time, we use (9) to asymptotically compute the behavior right of the DSW (sec. III A). When this asymptotic solution breaks down, we use the matched-asymptotic method introduced in [19] to find the DSW's slowly varying elliptic-function solution (sec. III B). This naturally leads to Whitham's equations, which Whitham [32] originally found by an averaging method; Luke [36] later developed a perturbative method (and [37] used such a method on the KdV equation). Then we use the method in [20] to determine the small-amplitude, slowly varying, oscillatory solution to the left of the DSW (sec. III C); this matches the slowly varying elliptic-function solution in the middle region.

A. Shock front

Right of the DSW: we asymptotically compute $\Omega(\xi;t)$ for large time, use Ω to compute G using a Neumann series, and use G in (10) to find u. When our asymptotic expansion for Ω breaks down, the Neumann series for G becomes disordered: this gives us the boundary condition for the DSW's right edge.

Far right of the DSW, where $x \gg -2c^2t$, the reflection coefficient's contribution to Ω dominates. Using the steepest-descent method (see [38, 39]) gives

$$\Omega(\xi;t) = -\frac{R_+(\lambda_*)e^{-2t[\xi/(6t)+2c^2]^{3/2}/\varepsilon}}{8\sqrt{\varepsilon\pi}[\xi/(6t)+2c^2]^{1/4}\sqrt{t}}\left[1 + O(t^{-1/2})\right] + \text{cc},$$

where $\lambda_* = \sqrt{c^2/2 - \xi/(24t)}$ and cc is the complex conjugate. We can then find G using the Neumann series from the iterates $G^{(0)}(x,y;t) = -\Omega(x+y;t)$ and

$$G^{(n)}(x,y;t) = -\Omega(x+y;t) - \int_x^\infty \Omega(y+z;t)G^{(n-1)}(x,z;t)\,\mathrm{d}z.$$

Using (10) then gives the exponentially small solution

$$u(x,t) = -6c^2 + \frac{\operatorname{Re}\{R_+(\lambda_*)\}\,e^{-2t[x/(3t)+2c^2]^{3/2}/\varepsilon}}{4\sqrt{\varepsilon\pi}[x/(3t)+2c^2]^{1/4}\sqrt{t}}\left[1 + O(t^{-1/2})\right],$$

for $x \gg -2c^2t$.

Near the DSW's right, the transmission coefficient's contribution to Ω dominates. The contribution from $\lambda = 0$ dominates and is

$$\Omega(\xi;t) = \frac{-e^{-ct(\xi/t+4c^2)/\varepsilon}\sqrt{\varepsilon}}{16\sqrt{\pi}[6c - \xi/(2ct)]^{3/2}}\left[H_2(0)t^{-3/2} + O(t^{-5/2})\right],$$

where

$$H_j(\lambda_*) \equiv \left[\frac{\partial^j}{\partial\lambda^j}|T(\lambda,\lambda_r(\lambda);0)|^2\right]_{\lambda=\lambda_*}.$$

The terms in the Neumann series become disordered when

$$[x + 2c^2t + 3\varepsilon/(4c)\log(6c^2t - x)] = O(1).$$

This is the DSW's right edge. (See the asymptotic principles discussed in [40].) When we sum the Neumann series, we find that

$$u(x,t) \sim -6c^2 + 12c^2 \operatorname{sech}^2\left[\frac{c}{\varepsilon}(\zeta - \zeta_0)\right], \tag{11}$$

where

$$\zeta_0 = \frac{\varepsilon}{2c}\log\left\{\frac{32\sqrt{\pi}}{H_2(0)c^{1/2}\varepsilon^{3/2}}\right\},$$

$$\zeta = -x - 2c^2 t - \frac{3\varepsilon}{4c}\log(6c^2 t - x) + A_1(x/t)t^{-1} + \cdots, \tag{12}$$

and

$$A_1(x/t + 6c^2) = \frac{3\varepsilon^2}{8c^2 x/t} + \frac{135\varepsilon^2}{16(x/t)^2} + \frac{3c^2\varepsilon^2 H_4(0)}{8(x/t)^2 H_2(0)}.$$

This provides the boundary condition on the DSW's right edge.

This procedure gives the DSW's phase, ζ_0. This phase only depends on $H_2(0)$ (since $H_0(0) = H_1(0) = 0$). In the vanishing case, [19, Eq. (2.25c)] found a similar phase term: $r''(0) - [r'(0)]^2/r(0)$, where r is the corresponding reflection coefficient. Burgers' equation's long-time-asymptotic solution also has a phase term that depends on the initial data in a similar way (see app. A).

B. DSW

For the DSW, we find the slowly varying, cnoidal-wave solution using matched asymptotics. First we make a variable change in (1) based on (11). Then we use the multiple-scales method (see [12]) to determine how its solution slowly varies: the secularity and compatibility conditions lead to three conservation laws, which we can transform into Whitham's equations [32]. Matching to (11) and assuming a similarity solution determines the DSW's long-time-asymptotic solution.

Analogous to [19], we look for a solution of the form

$$u(x,t) = -6c^2 + g(\zeta, t),$$

based on (11), where ζ is defined in (12). We substitute this into (1). Then we introduce the slow-variables $Z \equiv \delta\zeta$ and $T \equiv \delta t$, where $\delta = O(t^{-1})$ is a small parameter. Grouping terms in like powers of δ gives

$$\begin{aligned}\varepsilon^2 g_{\zeta\zeta\zeta} + g g_\zeta - 4c^2 g_\zeta - g_t \\ = \delta\left\{\frac{3\varepsilon(3\varepsilon^2 g_{\zeta\zeta\zeta} + g g_\zeta - 12c^2 g_\zeta)}{4c(8c^2 T + Z)}\right\} + \cdots.\end{aligned} \tag{13}$$

To leading order, (13) has the special solution

$$\begin{aligned}g(\zeta,t) \sim 4c^2 - V + 4\varepsilon^2\kappa^2(1-2k^2) \\ + 12k^2\varepsilon^2\kappa^2 \operatorname{cn}^2\left[\kappa(\zeta - \zeta_0 - Vt), k\right],\end{aligned} \tag{14}$$

where $\operatorname{cn}(z,k)$ is the Jacobian elliptic 'cosine' (see [41]); it can be found using the methods in [42]. If we neglect the right-hand side of (13), κ, k, and V are arbitrary constants but vary slowly in general. In the special case $k = 1$, $\kappa = c/\varepsilon$, and $V = 0$, $g(\zeta,t) = 12c^2\operatorname{sech}^2[c(\zeta-\zeta_0)/\varepsilon]$, which exactly matches (11).

As in [36], we use the multiple-scales method — with a fast variable θ — to determine how κ, k, and V vary with the slow-variables Z and T. This leads to three conservation laws from a compatibility condition and two secularity conditions; we can transform these conservation laws into a convenient diagonal system of quasilinear, first-order equations, which were first found by Whitham [32].

To get the compatibility condition, we introduce the rapid-variable $\theta(\zeta, t)$ with

$$\theta_\zeta \equiv \kappa(Z,T) \quad \text{and} \quad \theta_t \equiv -\omega(Z,T) \equiv -\kappa V. \tag{15}$$

This leads to the compatibility condition $(\theta_\zeta)_t = (\theta_t)_\zeta$ or

$$\kappa_T + \omega_Z = 0, \tag{16}$$

which is a conservation law.

To get the secularity conditions: we rewrite (13) in terms of θ, and then require that the leading-order solution is periodic in θ. We use $\partial_t = -\omega\partial_\theta + \delta\partial_T$ and $\partial_\zeta = \kappa\partial_\theta + \delta\partial_Z$ to transform (13) into

$$\begin{aligned}&\varepsilon^2\kappa^3 g_{\theta\theta\theta} + \kappa g g_\theta + (\omega - 4c^2\kappa)g_\theta \\ &\quad = \delta\Bigg[\frac{3\varepsilon\kappa}{4c(8c^2T+Z)}\Big(3\varepsilon^2\kappa^2 g_{\theta\theta\theta} + g g_\theta - 12c^2 g_\theta\Big) \\ &\qquad + g_T - \Big(3\varepsilon^2\kappa(\kappa g_{\theta\theta})_Z + g g_Z - 4c^2 g_Z\Big)\Bigg] + \cdots.\end{aligned} \tag{17}$$

Then we expand $g(\theta,Z,T) = g_0(\theta,Z,T) + \delta g_1(\theta,Z,T) + \delta^2 g_2(\theta,Z,T) + \cdots$ and group the terms in like powers of δ. The $O(1)$ equation is

$$\varepsilon^2\kappa^3 g_{0,\theta\theta\theta} + \kappa g_0 g_{0,\theta} + (\omega - 4c^2\kappa)g_{0,\theta} = 0; \tag{18}$$

the $O(\delta)$ equations is

$$\begin{aligned}&\varepsilon^2\kappa^3 g_{1,\theta\theta\theta} + \kappa(g_0 g_1)_\theta + (\omega - 4c^2\kappa)g_{1,\theta} \\ &\quad = \frac{3\varepsilon\kappa}{4c(8c^2T+Z)}\left(3\varepsilon^2\kappa^2 g_{0,\theta\theta\theta} + g_0 g_{0,\theta} - 12c^2 g_{0,\theta}\right) \\ &\qquad + g_{0,T} - 3\varepsilon^2\kappa(\kappa g_{0,\theta\theta})_Z - g_0 g_{0,Z} + 4c^2 g_{0,Z} \equiv F.\end{aligned} \tag{19}$$

To eliminate secular terms (that is, terms that grow arbitrarily large), we enforce the periodicity of $g_0(\theta,Z,T)$ in θ:

$$\int_0^1 F\,\mathrm{d}\theta = 0 \quad \text{and} \quad \int_0^1 g_0 F\,\mathrm{d}\theta = 0.$$

Using

$$\int_0^1 \frac{\partial^i g_0}{\partial\theta^i}\,\mathrm{d}\theta = 0, \qquad \int_0^1 g_0\frac{\partial^j g_0}{\partial\theta^j}\,\mathrm{d}\theta = 0,$$

for $i = 1, 2, 3, \ldots$ and $j = 1, 3, 5, \ldots$, and

$$\int_0^1 g_0 g_{0,\theta\theta}\,\mathrm{d}\theta = -\int_0^1 g_{0,\theta}^2\,\mathrm{d}\theta,$$

we get from $\int_0^1 F\,\mathrm{d}\theta = 0$ that

$$\frac{\partial}{\partial T}\int_0^1 g_0\,\mathrm{d}\theta + \frac{\partial}{\partial Z}\left(4c^2\int_0^1 g_0\,\mathrm{d}\theta - \frac{1}{2}\int_0^1 g_0^2\,\mathrm{d}\theta\right) = 0 \quad (20)$$

and from $\int_0^1 g_0 F\,\mathrm{d}\theta = 0$ that

$$\frac{\partial}{\partial T}\int_0^1 g_0^2\,\mathrm{d}\theta + \frac{\partial}{\partial Z}\left(4c^2\int_0^1 g_0^2\,\mathrm{d}\theta - \frac{2}{3}\int_0^1 g_0^3\,\mathrm{d}\theta + 3\varepsilon^2\kappa^2\int_0^1 g_{0,\theta}^2\,\mathrm{d}\theta\right) = 0. \quad (21)$$

The solution of (18) is

$$g_0(\theta, Z, T) = a(Z,T) + b(Z,T)\,\mathrm{cn}^2[2(\theta-\theta_0)K, k(Z,T)], \quad (22)$$

where $K \equiv K(k(Z,T))$ is the complete elliptic integral of the first kind,

$$\kappa^2 = \frac{b}{48\varepsilon^2 k^2 K^2}, \quad \text{and} \quad a = 4c^2 - V - \frac{2}{3}b + \frac{b}{3k^2}. \quad (23)$$

We can use these to rewrite the conservation law (16) as

$$\frac{\partial}{\partial T}\left(\frac{1}{4\sqrt{3}\varepsilon K}\sqrt{\frac{b}{k^2}}\right) + \frac{\partial}{\partial Z}\left(\frac{V}{4\sqrt{3}\varepsilon K}\sqrt{\frac{b}{k^2}}\right) = 0.$$

We can also use (22) to rewrite the conservation laws (20) and (21) in terms of b/k^2, V, and k. Using (22) and elliptic-function properties (see Byrd and Friedman [43, formulas 312 and special values 122]), we can write $\int_0^1 g_0\,\mathrm{d}\theta, \ldots, \int_0^1 g_0^3\,\mathrm{d}\theta$, and $\omega^2\int_0^1 g_{0,\theta}^2\,\mathrm{d}\theta$ in terms of b/k^2, V, k, and the complete first (K) and second (E) elliptic integrals. After simplification, we get the conservation laws

$$\frac{\partial}{\partial T}\left[(4c^2 - V) + \frac{1}{3}\left(3\frac{E}{K} + v - 2\right)\frac{b}{k^2}\right] + \frac{\partial}{\partial Z}\left[\frac{1}{2}(4c^2 - V)(4c^2 + V) + \frac{V}{3}\left(3\frac{E}{K} + k^2 - 2\right)\frac{b}{k^2} - \frac{1}{18}(1 - k^2 + k^4)\left(\frac{b}{k^2}\right)^2\right] = 0$$

and

$$\frac{\partial}{\partial T}\left[(4c^2 - V)^2 + \frac{2(4c^2 - V)}{3}\left(3\frac{E}{K} + k^2 - 2\right)\frac{b}{k^2} + \frac{1}{9}\left(1 - k^2 + k^4\right)\left(\frac{b}{k^2}\right)^2\right] + \frac{\partial}{\partial Z}\left[\frac{2}{3}(4c^2 - V)^2(2c^2 + V) + \frac{2V}{3}(4c^2 - V)\left(3\frac{E}{K} + k^2 - 2\right)\frac{b}{k^2} - \frac{2}{9}(2c^2 - V)(1 - k^2 + k^4)\left(\frac{b}{k^2}\right)^2 + \frac{1}{81}(2 - k^2)(1 + k^2)(2k^2 - 1)\left(\frac{b}{k^2}\right)^3\right] = 0.$$

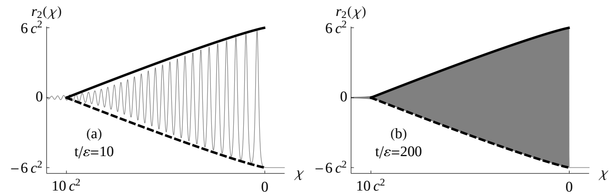

FIG. 3. The value of $r_2(\chi)$ found numerically for $0 < \chi < 10c^2$, where $\chi \equiv \zeta/t$. For comparison, we include $-r_2(\chi)$ as a dashed line and a numerical simulation of $u(x,t)$ in gray (inside the envelope of r_2 and $-r_2$) for a single-step at (a) $t/\varepsilon = 10$ and (b) $t/\varepsilon = 200$. Note that $\chi = 0$ corresponds to $x \sim -2c^2 t$ and $\chi = 10c^2$ to $x \sim -12c^2 t$.

These three conservation laws determine b, k, and V.

We can transform these conservation laws into Whitham's equations. Make the variable changes

$$\frac{b}{k^2} = 2(r_3 - r_1), \quad k^2 = \frac{r_2 - r_1}{r_3 - r_1}, \quad V = 4c^2 - \frac{r_1 + r_2 + r_3}{3}.$$

Simplifying then gives the convenient diagonal system

$$\frac{\partial r_i}{\partial T} + v_i(r_1, r_2, r_3)\frac{\partial r_i}{\partial Z} = 0, \qquad i = 1, 2, 3, \quad (24)$$

where $v_1 = V + bK/[3(K - E)]$, $v_2 = V + b(1 - k^2)K/[3(E - (1 - k^2)K)]$, $v_3 = V - b(1 - k^2)K/(3k^2 E)$, and

$$g_0(\theta, Z, T) = r_1 - r_2 + r_3 + 2(r_2 - r_1)\,\mathrm{cn}^2\left[2(\theta - \theta_0)K, k\right].$$

Whitham first found (24) in [32] (see also [14, 28]). Here, θ is found through integrating with (15).

For large time, the solution tends to a self-similar solution. We assume that $r_i = r_i(\chi)$ with $\chi \equiv Z/T = \zeta/t$. Taking $r_1 = 0$ and $r_3 = 6c^2$ satisfies the boundary conditions; so (24) reduces to $(v_2 - \chi)r_2'(\chi) = 0$ or

$$v_2 = 2c^2 - r_2 + \frac{2}{3}\frac{r_2 E\left(\sqrt{\frac{r_2}{6c^2}}\right)}{E\left(\sqrt{\frac{r_2}{6c^2}}\right) - \left(1 - \frac{r_2}{6c^2}\right)K\left(\sqrt{\frac{r_2}{6c^2}}\right)} = \chi.$$

We can numerically solve this implicit equation for r_2 (Fig. 3). We can also directly compute the DSW's left- and right-edge speed: At the right edge, we take the limit $r_2 \to r_3$, and get that $v_2 \to 0$ or $x \sim -2c^2 t$ — the leading soliton's speed. At the left edge, we take the limit $r_2 \to r_1$, and get that $v_2 \to 10c^2$ or $x \sim -12c^2 t$. Moreover, at the left edge where $0 < (10c^2 - \chi) \ll 1$, we have that $r_2 = 2(10c^2 - \chi)/3 + O[(10c^2 - \chi)^2]$; using this and taking $x \to -12c^2 t$ gives $u = (2/3)(10c^2 - \chi)\cos[16c^3 t/\varepsilon + O(\log t)]$.

C. Trailing edge

The solution left of the DSW has the same form for both vanishing ($c = 0$) and non-vanishing ($c \neq 0$) boundary conditions. In both cases, the GLM integral equation formulated from $-\infty$ to x has the same form. The scaling symmetry of

(1) — $(u,x,t) \to (\gamma^2 u, \gamma^{-1}x, \gamma^{-3}t)$ — leads to a similarity solution; in this region, we find that the slowly varying, asymptotic similarity solution is

$$u(x,t) = 2A\frac{X^{1/4}}{\sqrt{\tau}}\cos(\theta) - \frac{A^2(1-\cos 2\theta)}{3\tau\sqrt{X}} + O(\tau^{-3/2}), \quad (25)$$

where $X = -x/(3t)$, $\tau = 3t$, and

$$\theta = \frac{\tau}{\varepsilon}\left[\frac{2}{3}X^{3/2} - \frac{A^2}{18}\frac{\log(\tau X^{3/2})}{\tau} + \frac{\theta_0}{\tau} + O(\tau^{-2})\right].$$

We can use several methods to find A and θ_0 in terms of the scattering data.

One method for finding A and θ_0 is to use the GLM integral equation formulated from $-\infty$ to x. Following the same procedure as section III A requires that we sum the whole Neumann series: that is, unlike section III A, we cannot get A and θ_0 from the Neumann series's first few terms. But the first few terms are sufficient to show our main result: the long-time limit of general, step-like data is a single-phase DSW.

While we don't need expressions for A and θ_0 to show our main result, we can use the method as [20] to find A and θ_0. We find that

$$A^2(X) \sim -\frac{9\varepsilon}{\pi}\log\left(1 - \left|R\left(\sqrt{X}/2\right)\right|^2\right), \quad (26)$$

where $R(\sqrt{X}/2) \equiv R(\lambda = \sqrt{X}/2, \lambda_r(\lambda), t = 0)$, and

$$\begin{aligned}\frac{\theta_0}{\varepsilon} \sim \frac{\pi}{4} - \arg\{\tilde{r}(\lambda)\} - \arg\left\{\Gamma\left(1 - \frac{iA^2(4\lambda^2)}{18\varepsilon}\right)\right\} \\ - \frac{c^2A^2(4c^2)}{9\varepsilon\lambda^2}\log\left(\frac{c-\lambda}{c+\lambda}\right) - \frac{A^2}{6\varepsilon}\log 2 \\ - \frac{1}{9\lambda^2\varepsilon}\int_c^\lambda \left(\xi^2A^2(4\xi^2)\right)_\xi \log\left(\frac{\xi-\lambda}{\xi+\lambda}\right)\mathrm{d}\xi, \quad (27)\end{aligned}$$

where $\lambda = \sqrt{X}/2$, $\tilde{r} \equiv \tilde{b}/\tilde{a}$, and $\phi \to \tilde{a}(\lambda)e^{-i\lambda x/\varepsilon} + \tilde{b}(\lambda)e^{i\lambda(x+8\lambda^2 t)/\varepsilon}$ as $x \to -12c^2t$. This $\tilde{r}$ can be related to a and b through the GLM integral equation formulated from $-\infty$ to x. See appendix C for details.

IV. CONCLUSION

DSWs appear when weak dispersion and weak nonlinearity dominate the physics; they arise in many physical systems, including fluid dynamics, plasmas, superfluids, and nonlinear optics. For systems with weak dispersion and weak, quadratic nonlinearity, the KdV equation is the leading-order asymptotic equation. Here we showed that the long-time-asymptotic solution of the KdV equation for general, step-like initial data tend to a single-phase DSW; we found this long-time-asymptotic solution using the IST method and matched-asymptotic expansions. Therefore, a single-phase DSW eventually forms from well-separated, multi-step initial data, despite having more complex multiphase dynamics at intermediate times. We anticipate that our IST and matched-asymptotic procedure for general, step-like data will be applied to other important nonlinear integrable systems.

The long-time-asymptotic solution of the KdV equation for general, step-like initial data has three basic regions: an exponentially small region right of the DSW; the main DSW region, which is a slowly varying cnoidal wave with a soliton-train on its right and oscillatory behavior on its left; and a small, decaying, oscillatory region left of the DSW. The DSW region is over $|x| \le O(t)$ and has height $O(1)$. Compare this with the linear KdV equation with step-like data and the nonlinear KdV equation with vanishing data: the linear KdV equation with step-like data has a middle region with strong nonlinearity over $|x| \le O(t^{1/3})$ and has height $O(1)$; the nonlinear KdV equation with vanishing data has a collisionless-shock region over $(-x) = O[t^{1/3}(\log t)^{2/3}]$ and has height $O[(\log t)^{1/2}t^{-2/3}]$. The merging of shocks from multistep data is similar for both the KdV and Burgers' equations: in both, the boundary conditions determine its form and the initial data determine its position — but the KdV equation can also have a finite number of solitons.

ACKNOWLEDGMENTS

We wish to acknowledge the support of the National Science Foundation under grant DMS-0905779 and the Air Force Office of Scientific Research under grant FA9550-12-1-0207.

Appendix A: Long-time asymptotic solution of Burgers' equation

We can transform Burgers' equation,

$$w_t + ww_x - \nu w_{xx} = 0, \quad (A1)$$

into the heat equation ($\phi_t = \nu\phi_{xx}$) using the Hopf–Cole transformation,

$$w = -2\nu\frac{\phi_x}{\phi}. \quad (A2)$$

For simplicity, we take

$$w(x, t=0) = w_0(x) = \begin{cases} 0, & x \le x_\ell \\ f(x), & x_\ell \le x \le x_r \\ -h^2, & x \ge x_r \end{cases},$$

where h is real and f is bounded. So, from (A2),

$$\begin{aligned}\phi(x, t=0) = \phi_0(x) &= \exp\left(\frac{-1}{2\nu}\int_{-\infty}^x w_0(x')\,\mathrm{d}x'\right) \\ &= \begin{cases} 1, & x \le x_\ell \\ \exp\left[-1/(2\nu)\int_{x_\ell}^x f(x')\,\mathrm{d}x'\right], & x_\ell \le x \le x_r \\ \exp\left[h^2(x - \tilde{x}_0)/(2\nu)\right], & x \ge x_r \end{cases},\end{aligned}$$

where $\tilde{x}_0 \equiv \int_{x_\ell}^{x_r} f(x')/h^2 \,\mathrm{d}x'$. Solving the heat equation gives

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\left(\int_{-\infty}^{x_\ell} + \int_{x_\ell}^{x_r} + \int_{x_r}^{\infty}\right)\phi_0(x')e^{-(x-x')^2/(4\nu t)}\,dx' \equiv I_1 + I_2 + I_3.$$

And, in the long-time limit,

$$I_1 \sim \frac{1}{2}, \quad I_2 \to 0, \quad \text{and} \quad I_3 \sim \exp\left[\frac{h^2}{2\nu}\left(x - \tilde{x}_0 + \frac{h^2}{2}t\right)\right].$$

Therefore, from (A2),

$$w(x,t) \sim -\frac{h^2}{2}\left\{1 + \tanh\left[\frac{h^2}{4\nu}\left(x - x_0 + \frac{h^2}{2}t\right)\right]\right\},$$

where $x_0 \equiv \tilde{x}_0 + (2\nu/h^2)\log 2$.

Appendix B: Derivation of GLM integral equation

To find (9), the GLM integral equation, we need to know the eigenfunctions's and the scattering data's analyticity; see [26, 44] for more details. Using Green's functions, we can write $e^{i\lambda x/\varepsilon}\phi$, $e^{i\lambda x/\varepsilon}\bar{\phi}$, $e^{i\lambda_r x/\varepsilon}\psi$, and $e^{i\lambda_r x/\varepsilon}\bar{\psi}$ as Volterra integral equations, which can be solved using Neumann series. From these Neumann series, we find that

- $e^{i\lambda x/\varepsilon}\phi$ is analytic for $\mathrm{Im}(\lambda) > 0$,
- $e^{-i\lambda x/\varepsilon}\bar{\phi}$ is analytic for $\mathrm{Im}(\lambda) < 0$,
- $e^{-i\lambda_r x/\varepsilon}\psi$ is analytic for $\mathrm{Im}(\lambda_r) > 0$, and
- $e^{i\lambda_r x/\varepsilon}\bar{\psi}$ is analytic for $\mathrm{Im}(\lambda_r) < 0$.

From (7), we have that a is analytic for $\mathrm{Im}(\lambda) > 0$. If $n = 1, 2, \ldots, N$ in (3), then: $e^{i\lambda x/\varepsilon}\phi$ and $e^{-i\lambda x/\varepsilon}\bar{\phi}$ are N-fold differentiable (with respect to λ) on $\mathrm{Im}(\lambda) = 0$, $\lambda \neq 0$ and $(N-1)$-differentiable at $\lambda = 0$; $e^{-i\lambda_r x/\varepsilon}\psi$ and $e^{i\lambda_r x/\varepsilon}\bar{\psi}$ are N-fold differentiable (with respect to λ_r) on $\mathrm{Im}(\lambda_r) = 0$, $\lambda_r \neq 0$ and $(N-1)$-differentiable at $\lambda_r = 0$. Likewise, if $u(x,t)$ satisfies

$$\int_{-\infty}^{\infty} |u(x,t) + 6c^2 H(x)| e^{d|x|}\,\mathrm{d}x < \infty, \qquad 0 < d \in \mathbb{R},$$

then $e^{i\lambda x/\varepsilon}\phi$ and $e^{-i\lambda x/\varepsilon}\bar{\phi}$ are analytic in $-d < \mathrm{Im}(\lambda) < d$, $e^{-i\lambda_r x/\varepsilon}\psi$ and $e^{i\lambda_r x/\varepsilon}\bar{\psi}$ are analytic in $-d < \mathrm{Im}(\lambda_r) < d$, and b is analytic, from (7), in $-d < \mathrm{Im}(\lambda)$ and $\mathrm{Im}(\lambda_r) < d$.

Using the eigenfunctions's and the scattering-data's analyticity, we find the GLM integral equation by: assuming that ψ and $\bar{\psi}$ have triangular forms; substituting these forms into (8); and operating on this equation with $(2\varepsilon\pi)^{-1} ⨍_{-\infty}^{\infty} \mathrm{d}\lambda_r$ to get (9). (We use $⨍$ to denote the principle-value integral omitting $\lambda_r = 0$.) Following [18], we assume that ψ and $\bar{\psi}$ have the triangular forms

$$\begin{aligned} \psi(x;\lambda_r;t) &= e^{i\lambda_r x/\varepsilon} + \int_x^{\infty} G(x,s;t)e^{i\lambda_r s/\varepsilon}\,\mathrm{d}s, \\ \bar{\psi}(x;\lambda_r;t) &= e^{-i\lambda_r x/\varepsilon} + \int_x^{\infty} G(x,s;t)e^{-i\lambda_r s/\varepsilon}\,\mathrm{d}s, \end{aligned} \tag{B1}$$

with $G(x,s;t) \equiv 0$ when $s < x$. Substituting (B1) into (8) gives

$$T\phi = e^{-i\lambda_r x/\varepsilon} + \int_x^{\infty} G(x,s;t)e^{-i\lambda_r s/\varepsilon}\,\mathrm{d}s + R\left\{e^{i\lambda_r x/\varepsilon} + \int_x^{\infty} G(x,s;t)e^{i\lambda_r s/\varepsilon}\,\mathrm{d}s\right\};$$

multiplying by $e^{i\lambda_r y/\varepsilon}$ and rearranging gives

$$\left(T\phi e^{i\lambda_r x/\varepsilon} - 1\right)e^{i\lambda_r(y-x)/\varepsilon} = \int_x^{\infty} G(x,s;t)e^{i\lambda_r(y-s)/\varepsilon}\,\mathrm{d}s + R\left\{e^{i\lambda_r(x+y)/\varepsilon} + \int_x^{\infty} G(x,s;t)e^{i\lambda_r(y+s)/\varepsilon}\,\mathrm{d}s\right\}. \tag{B2}$$

Now we operate on (B2) with $(2\varepsilon\pi)^{-1} ⨍_{-\infty}^{\infty} \mathrm{d}\lambda_r$, interchange integrals, and use that $\delta(x) = (2\varepsilon\pi)^{-1}\int_{-\infty}^{\infty} e^{i\lambda_r x/\varepsilon}\,\mathrm{d}\lambda_r$. So, for example,

$$\begin{aligned} \frac{1}{2\varepsilon\pi} ⨍_{-\infty}^{\infty} \int_x^{\infty} G(x,s;t)e^{i\lambda_r(y-s)/\varepsilon}\,\mathrm{d}s\,\mathrm{d}\lambda_r &= \int_x^{\infty} G(x,s;t)\left(\frac{1}{2\varepsilon\pi}\int_{-\infty}^{\infty} e^{i\lambda_r(y-s)/\varepsilon}\,\mathrm{d}\lambda_r\right)\mathrm{d}s \\ &= \int_x^{\infty} G(x,s;t)\delta(y-s)\,\mathrm{d}s = G(x,y;t) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2\varepsilon\pi} ⨍_{-\infty}^{\infty} R(\lambda,t)\int_x^{\infty} G(x,s;t)e^{i\lambda_r(y+s)/\varepsilon}\,\mathrm{d}s\,\mathrm{d}\lambda_r &= \int_x^{\infty} G(x,s;t)\left(\frac{1}{2\varepsilon\pi} ⨍_{-\infty}^{\infty} R(\lambda,t)e^{i\lambda_r(y+s)/\varepsilon}\,\mathrm{d}\lambda_r\right)\mathrm{d}s \\ &= \int_x^{\infty} G(x,s;t)F(y+s;t)\,\mathrm{d}s, \end{aligned}$$

where

$$F(z;t) \equiv \frac{1}{2\varepsilon\pi} ⨍_{-\infty}^{\infty} R(\lambda,t)e^{i\lambda_r z/\varepsilon}\,\mathrm{d}\lambda_r. \tag{B3}$$

Thus,

$$G(x,y;t) + F(x+y;t) + \int_x^{\infty} F(y+z;t)G(x,z;t)\,\mathrm{d}z = I,$$

where

$$I \equiv \frac{1}{2\varepsilon\pi} ⨍_{-\infty}^{\infty} \left(T\phi e^{i\lambda_r x/\varepsilon} - 1\right)e^{i\lambda_r(y-x)/\varepsilon}\,\mathrm{d}\lambda_r.$$

We find I by closing in the upper-half λ_r-plane because $\phi e^{i\lambda_r x/\varepsilon}$ is analytic in $\mathrm{Im}(\lambda) > 0$. We get that $I \equiv -I_b - I_p$, where I_b is the contribution from the branch cut and I_p is the contribution from the zeros of a.

To find I_b, we recall that the branch cut of λ is $\lambda_r \in [-ic, ic]$, and the branch cut of λ_r is $\lambda \in [-c, c]$. So

$$\begin{aligned} I_b &= \frac{1}{2\varepsilon\pi}\left(\int_{0-0^-}^{ic-0^-} - \int_{0+0^+}^{ic+0^+}\right)\left\{\frac{\phi e^{i\lambda_r x/\varepsilon}}{a} - 1\right\}e^{i\lambda_r(y-x)/\varepsilon}\,\mathrm{d}\lambda_r \\ &= \frac{1}{2\varepsilon\pi}\int_0^{ic}\left\{\left(\frac{\phi}{a}\right)_{\lambda=-|\lambda|} - \left(\frac{\phi}{a}\right)_{\lambda=|\lambda|}\right\}e^{i\lambda_r y/\varepsilon}\,\mathrm{d}\lambda_r. \end{aligned}$$

Now we define α and β — the scattering data from the left — so that

$$\psi \equiv \alpha\bar{\phi} + \beta\phi, \qquad \lambda \neq 0; \tag{B4}$$

then

$$\alpha = \frac{\varepsilon}{2i\lambda} W(\phi, \psi) = \frac{\lambda_r a}{\lambda} \quad \text{and} \quad \beta = \frac{\varepsilon}{2i\lambda} W(\psi, \bar{\phi}).$$

For $\lambda_r \in [0, ic]$, $\phi^* = \bar{\phi}$, $\psi = \psi^* = \bar{\psi}$, and $\alpha^* = \beta$ from (5), where $*$ denotes the complex conjugate. So

$$I_b = \frac{1}{2\varepsilon\pi} \int_0^{ic} \left\{ \left(\frac{\phi}{\lambda\alpha}\right)_{\lambda=-|\lambda|} - \left(\frac{\phi}{\lambda\alpha}\right)_{\lambda=|\lambda|} \right\} e^{i\lambda_r y/\varepsilon} \lambda_r \, \mathrm{d}\lambda_r.$$

Using $(\phi/\alpha)_{\lambda=-|\lambda|} = (\phi^*/\alpha^*)_{\lambda=|\lambda|}$, from (5a), and noting λ's sign change, gives

$$\begin{aligned} I_b &= -\frac{1}{2\varepsilon\pi} \int_0^{ic} \left(\frac{\phi}{\lambda\alpha} + \frac{\phi^*}{\lambda\alpha^*} \right)_{\lambda=|\lambda|} e^{i\lambda_r y/\varepsilon} \lambda_r \, \mathrm{d}\lambda_r \\ &= -\frac{1}{2\varepsilon\pi} \int_0^{ic} \left[\frac{1}{\lambda\alpha^*} \left(\phi^* + \frac{\alpha^*}{\alpha}\phi \right) \right]_{\lambda=|\lambda|} e^{i\lambda_r y/\varepsilon} \lambda_r \, \mathrm{d}\lambda_r. \end{aligned}$$

Using the identities $\psi^* = \bar{\psi}$ and $\alpha^* = \beta$ for $\lambda_r \in [0, ic]$ and then using (B4) gives

$$\begin{aligned} I_b &= -\frac{1}{2\varepsilon\pi} \int_0^{ic} \left[\frac{1}{\lambda\alpha^*} \left(\bar{\phi} + \frac{\beta}{\alpha}\phi \right) \right]_{\lambda=|\lambda|} e^{i\lambda_r y/\varepsilon} \lambda_r \, \mathrm{d}\lambda_r \\ &= -\frac{1}{2\varepsilon\pi} \int_0^{ic} \left[\frac{1}{\lambda|\alpha|^2} \psi \right]_{\lambda=|\lambda|} e^{i\lambda_r y/\varepsilon} \lambda_r \, \mathrm{d}\lambda_r. \end{aligned}$$

Making the change of variable from λ_r to λ and using that $T \equiv 1/a$ gives

$$I_b = \frac{1}{2\varepsilon\pi} \int_0^{c} |\lambda T/\lambda_r|^2 \psi e^{-y\sqrt{c^2-\lambda^2}/\varepsilon} \, \mathrm{d}\lambda.$$

To find I_p, we use the residue theorem to get

$$\begin{aligned} I_p &= -\frac{i}{\varepsilon} \sum_j \operatorname{Res}\left(\frac{\phi e^{iy\lambda_r/\varepsilon}}{a}, \lambda = \lambda_j \right) \\ &= \sum_j c_j \psi(x; i\kappa_j, t) e^{-\tilde{\kappa}_j y/\varepsilon}, \end{aligned}$$

where the constants $\{i\kappa_j\}$ are the simple zeros of $a(\lambda, t)$, $\tilde{\kappa}_j = \sqrt{\kappa_j^2 + c^2}$,

$$c_j = -\frac{i\mu_j}{\varepsilon[\partial_{\lambda_r} a]_{\lambda=i\kappa_j}}, \qquad \phi(x; i\kappa_j, t) \equiv \mu_j(t)\psi(x; i\kappa_j, t),$$

and $0 < \kappa_1 < \cdots < \kappa_N$ are real.

Using (B1) again gives (9),

$$G(x, y; t) + \Omega(x + y; t) + \int_x^\infty \Omega(y + z; t) G(x, z; t) \, \mathrm{d}z = 0,$$

where

$$\Omega(\xi; t) = \frac{1}{2\varepsilon\pi} \int_{-\infty}^{\infty} R e^{i\lambda_r \xi/\varepsilon} \, \mathrm{d}\lambda_r + \sum_j c_j e^{-\tilde{\kappa}_j \xi/\varepsilon} + \frac{1}{2\varepsilon\pi} \int_0^c |\lambda T/\lambda_r|^2 e^{-\sqrt{c^2-\lambda^2}\xi/\varepsilon} \, \mathrm{d}\lambda.$$

To get u from G: we differentiate (B1) twice with respect to x; multiply (B1) by $\lambda_r^2/\varepsilon^2$; and substitute these into (4a) (where we've used that $\lambda^2 = \lambda_r^2 + c^2$) to get

$$\begin{aligned} &e^{i\lambda_r x/\varepsilon} \left(\frac{u(x,t)}{6\varepsilon^2} + \frac{c^2}{\varepsilon^2} - 2\frac{\mathrm{d}}{\mathrm{d}x} G(x, x; t) \right) \\ &\quad + \int_x^\infty \left[\frac{\partial^2}{\partial x^2} G(x, s; t) - \frac{\partial^2}{\partial s^2} G(x, s; t) \right. \\ &\quad \left. + \left(\frac{u(x,t)}{6\varepsilon^2} + \frac{c^2}{\varepsilon^2} \right) G(x, s; t) \right] e^{i\lambda_r s/\varepsilon} \, \mathrm{d}s = 0. \end{aligned}$$

Therefore,

$$u(x, t) = -6c^2 + 12\varepsilon^2 \frac{\mathrm{d}}{\mathrm{d}x} G(x, x; t)$$

and

$$\frac{\partial^2}{\partial x^2} G(x, s; t) - \frac{\partial^2}{\partial s^2} G(x, s; t) + \left(\frac{u(x,t)}{6\varepsilon^2} + \frac{c^2}{\varepsilon^2} \right) G(x, s; t) = 0.$$

Appendix C: Determining the amplitude and phase left of the DSW

To determine A and θ_0, we use the method in [20] (see also [17, sec. 1.7.c]): We substitute (25) into (4a) and use the boundary values $v \to \phi$ as $x \to -\infty$ and $v \to \tilde{a}e^{-i\lambda x/\varepsilon} + \tilde{b}e^{i\lambda(x+8\lambda^2 t)/\varepsilon}$ as $x \to -12c^2 t$. Here, $\tilde{a}$ and $\tilde{b}$ can be found through the GLM integral equation from the left or by relating them to a and b through the asymptotic forms of u for $-12c^2 t \ll x \ll \infty$. Then we asymptotically solve for the eigenfunction ϕ; this is a WKB-type problem that leads to a matched-asymptotic problem. From the asymptotic form of ϕ, we get A and θ_0 in terms of $\tilde{r} \equiv \tilde{b}/\tilde{a}$.

To get a WKB-type problem for the eigenfunctions: we substitute (25) and $v = \phi = \phi_1 e^{i\lambda x/\varepsilon} + \phi_2 e^{-i\lambda x/\varepsilon}$ into (4a), break it into two consistent relations, and keep only the leading-order terms. This gives

$$\begin{aligned} \frac{\partial\phi_1}{\partial x} &\sim i\frac{AX^{1/4}}{12\lambda\varepsilon\sqrt{\tau}} \left(e^{i(\theta-\eta)} + e^{-i(\theta+\eta)} \right) \phi_2, \\ \frac{\partial\phi_2}{\partial x} &\sim -i\frac{AX^{1/4}}{12\lambda\varepsilon\sqrt{\tau}} \left(e^{i(\theta+\eta)} + e^{-i(\theta-\eta)} \right) \phi_1, \end{aligned} \tag{C1}$$

where $\eta \equiv 2\lambda x/\varepsilon$. This has *two* rapidly varying phases, $(\theta + \eta)$ and $(\theta - \eta)$. Then we expand ϕ_1 and ϕ_2 as

$$\phi_i = \phi_{i,0}(\theta, \eta, X) + \tau^{-1/2}\phi_{i,1}(\theta, \eta, X) + \tau^{-1}\phi_{i,2}(\theta, \eta, X) + \cdots,$$

substitute this into (C1), and group terms with like powers of τ. At $O(\tau^{-1/2})$, we find a resonance or turning-point region near

$$1 \pm \frac{\eta_x}{\theta_x} = 0 \quad \text{or} \quad X \sim 4\lambda^2,$$

where secular terms appear. This gives three regions to consider: $X \gg 4\lambda^2$, $X \sim 4\lambda^2$, and $4\lambda^2 \gg X \gg 4c^2$.

In the left-most region, where $4\lambda^2 \ll X < \infty$, perturbation theory gives

$$\begin{aligned}
&\phi_{1,0}(X) = 0,\\
&\phi_{2,0}(X) \sim \exp\left\{\tfrac{i}{144\lambda^2\varepsilon^2}\int_X^\infty A^2(z)\sqrt{z}\left(\tfrac{1}{\theta_x(z)+\eta_x} - \tfrac{1}{\theta_x(z)-\eta_x}\right)\mathrm{d}z\right\},
\end{aligned}$$

after matching to $\phi_1 \to 0$ and $\phi_2 \to 1$ as $x \to -\infty$. In the limit as $X \to 4\lambda^2$, we get that

$$\phi_{2,0}(X) \sim (X - 4\lambda^2)^\nu (4\lambda)^{-2\nu} e^{I(\lambda)},$$

where $\nu \equiv iA^2(4\lambda^2)/(18\varepsilon)$ and

$$I(\lambda) \equiv \frac{i}{18\lambda^2\varepsilon}\int_\lambda^\infty \left(\xi^2 A^2(4\xi^2)\right)_\xi \log\left(\frac{\xi-\lambda}{\xi+\lambda}\right)\mathrm{d}\xi.$$

In the middle region, where $X \sim 4\lambda^2$, we can represent the solution in terms of parabolic cylinder functions:

$$\begin{aligned}
w_{1,0}(Y) &= e^{-\tilde{Y}^2/4}\left(c_1 U\left(\tfrac{1}{2}-\nu, \tilde{Y}\right) + c_2 U\left(\tfrac{1}{2}-\nu, -\tilde{Y}\right)\right),\\
w_{2,0}(Y) &= e^{\tilde{Y}^2/4}\left(c_3 U\left(-\tfrac{1}{2}-\nu, \tilde{Y}\right) + c_4 U\left(-\tfrac{1}{2}-\nu, -\tilde{Y}\right)\right),
\end{aligned}$$

where U is the parabolic cylinder function (see [41]),

$$Y \equiv (X - 4\lambda^2)\sqrt{\tau} \equiv -\frac{x - x_0}{\sqrt{3t}}, \qquad \tilde{Y} \equiv \frac{Y e^{i\pi/4}}{\sqrt{4\lambda\varepsilon}},$$

and

$$c_1/c_3 = -c_2/c_4 = \frac{A(4\lambda^2)}{3\sqrt{2\varepsilon}}\exp\left\{i\left(\frac{\pi}{4} - \frac{2\lambda x_0 + \tilde{\theta}_0}{\varepsilon}\right)\right\}.$$

Matching w_i as $Y \to +\infty$ to ϕ_i as $X \to 4\lambda^2$ gives

$$c_3 = e^{-i\pi\nu/4}\left(\frac{\varepsilon}{(4\lambda)^3\tau}\right)^{\nu/2} e^{I(\lambda)} \quad \text{and} \quad c_4 = 0.$$

Taking the limit in the other direction, $Y \to -\infty$, then gives

$$\begin{aligned}
\phi_{1,0}(Y) = {}& \frac{\sqrt{2\pi\nu}}{\Gamma(1-\nu)}(4\lambda^2 - X)^{-\nu} e^{-i\pi\nu/2}(4\lambda\tau)^{-\nu}\varepsilon^\nu\\
&\times \exp\left\{-i\frac{2\lambda x_0 + \tilde{\theta}_0}{\varepsilon} + I(\lambda)\right\} + O(|Y|^{-1}),\\
\phi_{2,0}(Y) = {}& (4\lambda^2 - X)^\nu e^{-i\pi\nu}(4\lambda)^{-2\nu} e^{I(\lambda)} + O(|Y|^{-1}).
\end{aligned}$$

For $4\lambda^2 \gg X \gg 4c^2$, perturbation theory and matching to $\phi_1 \to \tilde{b}(\lambda)e^{8i\lambda^3 t/\varepsilon}$ and $\phi_2 \to \tilde{a}(\lambda)$ as $x \to -12c^2 t$ gives

$$\phi_{1,0}(X) \sim \tilde{b}(\lambda)e^{i8\lambda^3 t/\varepsilon + J(X;\lambda)} \quad \text{and} \quad \phi_{2,0}(X) \sim \tilde{a}(\lambda)e^{-J(X;\lambda)},$$

where

$$J(X;\lambda) \equiv \frac{i}{144\lambda^2\varepsilon^2}\int_{4c^2}^X A^2(z)\sqrt{z}\left(\tfrac{1}{\theta_x(z)+\eta_x} - \tfrac{1}{\theta_x(z)-\eta_x}\right)\mathrm{d}z.$$

Matching the limits of ϕ_2 as $X \to 4\lambda^2$ and as $Y \to -\infty$ gives

$$\begin{aligned}
\tilde{a}(\lambda) \sim \exp\Bigg\{&\frac{c^2}{\lambda^2}\frac{iA^2(4c^2)}{18\varepsilon}\log\left(\frac{c-\lambda}{c+\lambda}\right)\\
&+ \frac{i}{18\lambda^2\varepsilon}\int_c^\infty \left(\xi^2 A^2(4\xi^2)\right)_\xi \log\left|\frac{\xi-\lambda}{\xi+\lambda}\right|\mathrm{d}\xi\Bigg\}.
\end{aligned}$$

So, after contour integration,

$$A^2(X) \sim \frac{9\varepsilon}{\pi}\log\left|\tilde{a}(\sqrt{X}/2)\right|^2 = -\frac{9\varepsilon}{\pi}\log\left(1 - \left|R\left(\sqrt{X}/2\right)\right|^2\right),$$

since $X = 4\lambda^2$, $\tilde{r}(\lambda, t) \equiv \tilde{b}(\lambda, t)/\tilde{a}(\lambda)$ and $|\tilde{r}(\lambda, t)| = |R(\lambda, t)|$. Likewise, matching the limits of ϕ_1 as $X \to 4\lambda^2$ and as $Y \to -\infty$ gives

$$\begin{aligned}
\tilde{b}(\lambda) \sim \frac{\sqrt{2\pi\nu}}{\Gamma(1-\nu)}e^{i\pi\nu/2}\varepsilon^\nu \exp\Bigg\{&-i\frac{\theta_0}{\varepsilon} - 3\nu\log 2\\
&- \frac{c^2}{\lambda^2}\frac{iA^2(4c^2)}{18\varepsilon}\log\left(\frac{c-\lambda}{c+\lambda}\right)\\
&+ \frac{i}{18\lambda^2\varepsilon}\left(-\int_c^\lambda + \int_\lambda^\infty\right)\left(\xi^2 A^2(4\xi^2)\right)_\xi \log\left(\frac{\xi-\lambda}{\xi+\lambda}\right)\mathrm{d}\xi\Bigg\}.
\end{aligned}$$

Using that $\tilde{r} \equiv \tilde{b}/\tilde{a}$ gives (27).

This matches the DSW's left boundary since taking the limits $x \to -12c^2 t$ and $r_2 \sim 2(10c^2 - \chi)/3 \sim 2AX^{1/4}\tau^{-1/2}$ gives $u \sim 2\sqrt{2c/(3t)}\cos[16c^3 t/\varepsilon + O(\log t)]$; see [44] for details.

[1] R. J. Taylor, D. R. Baker, and H. Ikezi, Phys Rev Lett **24**, 206 (1970).

[2] N. F. Smyth and P. E. Holloway, J Phys Oceanogr **18**, 947

(1988).
[3] J. Lighthill, *Waves in Fluids* (Cambridge University Press, 1978).
[4] Z. Dutton, M. Budde, C. Slowe, and L. V. Hau, Science **293**, 663 (2001).
[5] T. P. Simula, P. Engels, I. Coddington, V. Schweikhard, E. A. Cornell, and R. J. Ballagh, Phys Rev Lett **94**, 080404 (2005).
[6] M. A. Hoefer, M. J. Ablowitz, I. Coddington, E. A. Cornell, P. Engels, and V. Schweikhard, Phys Rev A **74**, 023623 (2006).
[7] J. J. Chang, P. Engels, and M. A. Hoefer, Phys Lett Rev **101**, 170404 (2008).
[8] W. Wan, S. Jia, and J. W. Fleischer, Nat Phys **3**, 46 (2007).
[9] S. Jia, W. Wan, and J. W. Fleischer, Phys Rev Lett **99**, 223901 (2007).
[10] N. Ghofraniha, C. Conti, G. Ruocco, and S. Trillo, Phys Rev Lett **99**, 043903 (2007).
[11] C. Conti, A. Fratalocchi, M. Peccianti, G. Ruocco, and S. Trillo, Phys Rev Lett **102**, 083902 (2009).
[12] M. J. Ablowitz, *Nonlinear Dispersive Waves, Asymptotic Analysis and Solitons* (Cambridge University Press, 2011).
[13] T. Grava and F.-R. Tian, Commun Pur Appl Math **55**, 1569 (2002).
[14] M. J. Ablowitz, D. E. Baldwin, and M. A. Hoefer, Phys Rev E **80**, 016603 (2009).
[15] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Studies in Applied Mathematics **53**, 249 (1974).
[16] P. Deift and E. Trubowitz, Communications on Pure and Applied Mathematics **32**, 121 (1979).
[17] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, 1981).
[18] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, 1991).
[19] M. J. Ablowitz and H. Segur, Studies in Applied Mathematics **57**, 13 (1977).
[20] H. Segur and M. J. Ablowitz, Physica D **3**, 165 (1981).
[21] P. Deift and X. Zhou, Annals of Mathematics **137**, 295 (1993).
[22] P. Deift, S. Venakides, and X. Zhou, International Mathematics Research Notices **6**, 285 (1997).
[23] R. Buckingham and S. Venakides, Commun Pur Appl Math **60**, 1349 (2007).
[24] E. J. Hruslov, Math. USSR Sbornik **28**, 229 (1976).
[25] A. Cohen, Commun Part Diff Eq **9**, 751 (1984).
[26] A. Cohen and T. Kappeler, Indiana University Mathematics Journal **34**, 127 (1985).
[27] V. S. Buslaev and V. N. Fomin, Vestnik Leningrad Univ **17**, 56 (1962).
[28] A. V. Gurevich and L. P. Pitaevskii, Sov Phys JETP **38**, 291 (1974).
[29] A. M. Kamchatnov, *Nonlinear Periodic Waves and Their Modulations* (World Scientific, River Edge, NJ, 2000).
[30] G. A. El and R. H. J. Grimshaw, Chaos **12**, 1015 (2002).
[31] G. A. El, Chaos **15**, 1 (2005).
[32] G. B. Whitham, Proc Roy Soc A **283**, 238 (1965).
[33] G. B. Whitham, *Linear and Nonlinear Waves* (John Wiley & Sons, New York, 1974).
[34] T. Grava and C. Klein, Proc Roy Soc A **464**, 733 (2008).
[35] T. Claeys and T. Grava, SIAM J Math Anal **42**, 2132 (2010).
[36] J. C. Luke, Proc Roy Soc A **292**, 403 (1966).
[37] R. H. J. Grimshaw, Proc Roy Soc A **368**, 359 (1979).
[38] M. J. Ablowitz and A. S. Fokas, *Complex Variables: Introduction and Applications*, Cambridge Texts in Applied Mathematics (Cambridge University Press, 2003), 2nd ed.
[39] N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals* (Dover, 1986).
[40] M. D. Kruskal, in *Mathematical Models in Physical Sciences*, edited by S. Drobot and P. A. Viebrock, University of Notre Dame (Prentice–Hall, Inc, Englewood Cliffs, NJ, 1962), pp. 17–48.
[41] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., *NIST Handbook of Mathematical Functions* (Cambridge University Press, 2010).
[42] D. E. Baldwin, Ü. Göktaş, W. Hereman, L. Hong, R. S. Martino, and J. Miller, Journal of Symbolic Computation **37**, 669 (2004).
[43] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists* (Springer–Verlag, 1971), 2nd ed.
[44] D. E. Baldwin, Ph.D. thesis, University of Colorado at Boulder (2013).