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Complex dynamics and scale invariance of one-dimensional memristive networks

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Memristive systems, namely resistive systems with memory, are attracting considerable attention due to their ubiquity in several phenomena and technological applications. Here, we show that even the simplest one-dimensional network formed by the most common memristive elements with voltage threshold bears non-trivial physical properties. In particular, by taking into account the single element variability we find *i)* dynamical acceleration and slowing down of the total resistance in adiabatic processes, *ii)* dependence of the final state on the history of the input signal with same initial conditions, *iii)* existence of switching avalanches in memristive ladders, and *iv)* independence of the dynamics voltage threshold with respect to the number of memristive elements in the network (scale invariance). An important criterion for this scale invariance is the presence of memristive systems with very small threshold voltages in the ensemble. These results elucidate the role of memory in complex networks and are relevant to technological applications of these systems.

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I. INTRODUCTION

Resistors whose resistance depends on the past evolution of the system are quite common in both basic and applied science, and they have been known for at least several decades [1]. Recently, they have attracted considerable interest in the context of memory applications – where they are oftentimes referred to as memristive systems [2] – but their range of applicability spans various disciplines as diverse as non-traditional computing and biophysics [3–5]. In addition to their ubiquity, their theoretical description is quite simple: if a memristive system is subjected to a voltage $V(t)$, its resistance can be written as $R(x, V, t)$, namely it may depend on the voltage itself (which would simply make it a non-linear element), and, most importantly, it depends on some state variable(s), x , which could be, e.g., the spin polarization [6, 7] or the position of atomic defects [8], or any other physical property of the system that gives it memory.

While most of the research so far has focused on the properties of these single elements – both by identifying the physical mechanisms for memory, and by devising innovative ways to employ them in practice – very little is known about their response when they are organized into networks, with the associated (and inevitable) element variability. In other words, the statistical properties of networks of memristive systems are largely unknown and, as we demonstrate below, are different than those of traditionally studied networks [9–11].

Addressing this issue has several immediate benefits. It is not at all obvious how a network whose elements have

memory of past dynamics evolves collectively in time, or whether it possesses fundamental (and universal) characteristics that can be found in the natural world. In addition, a resistance with memory is just a particular case of a general class of response functions that depend on state variables [12] – memcapacitive and meminductive [12] systems are similarly defined. Moreover, networks with memory have been shown to be promising candidates for the solution of complex optimization problems (see, e.g., Ref. 13), and their circuit applications are likely to involve a combination of several elements with memory. Finally, it is worth mentioning that the human (and animal) brain is – in its most basic description – simply a network with memory. This type of research may thus have consequences in neuroscience.

With these motivations in mind, in this paper we set to study, both numerically and theoretically, the statistical properties of the simplest network of memristive systems: a finite 1D network (Fig. 1). In order to obtain its general features, we abstract as much as possible, and do not consider any particular physical mechanism for memory. However, due to physical constraints, memristive systems are commonly found with a threshold voltage, namely it takes a minimal voltage V_t to change their resistance (see, e.g., Refs. 1, 14–16). Therefore, in order to be as

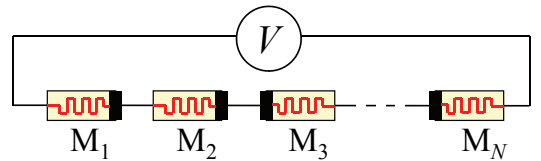


FIG. 1: (color online). One-dimensional network of N randomly oriented memristive systems M_i connected to a voltage source V .

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close as possible to experimental realizations, we focus on such type of elements. Moreover, it is generally observed that the memristance R changes between two limiting values, R_{on} and R_{off} . This experimental observation is also taken into account in our model. An extensive list of bipolar memristive devices with threshold can be found in the recent comprehensive review paper [1]. Specific examples of memristive systems with threshold that can be used to verify our predictions include different metal/oxide/metal memristive nanodevices [8, 14, 15].

Probability distribution functions $f_{V_t}(V)$, $f_R(R)$, $f_{R_{on}}(R)$, $f_{R_{off}}(R)$ satisfying the normalization conditions

$$\int_0^\infty f_i(y)dy = 1 \quad (1)$$

where $i = V_t, R, R_{on}$ or R_{off} are used to describe threshold voltages, memristances and limiting values of memristances in the network, respectively. To understand the meaning of these distribution functions, we note that, e.g., $f_{V_t}(V)dV$ is the probability to find a memristive device in the network with V_t in the interval between V and $V + dV$. All other distribution functions are introduced in a similar way. Moreover, we suppose that systems with required $f_{V_t}(V)$, $f_{R_{on}}(R)$ and $f_{R_{off}}(R)$ can be fabricated "on demand". (For example, one can extract a subset of devices from an ensemble with Gaussian distribution to form a different distribution. This can be performed, roughly speaking, by taking memristive devices one-by-one from the ensemble, measuring their characteristics and deciding if each of the devices should be kept.) Even in an ensemble of devices fabricated with specific values of V_t , R_{on} or R_{off} , an inevitable fluctuation of V_t , R_{on} and R_{off} around their average values can also be described by the above mentioned distribution functions. Finally, we note that the memristance distribution function $f_R(R)$ is a time-dependent function, unlike $f_{V_t}(V)$, $f_{R_{on}}(R)$ and $f_{R_{off}}(R)$ distribution functions. The distribution function $f_R(R)$ can be easily pre-set to the required form by applying appropriate pulse sequences to individual memristive devices [17]. Below, we give several examples of the evolution of $f_R(R)$.

In what follows, we will show that despite their apparent simplicity, memristive networks indeed show a quite complex dynamical behavior, and some general scale-invariant properties that can be readily verified experimentally. This paper is organized as follows. In Sec. II we consider adiabatic processes and introduce two types of memristive dynamics. Sec. III studies the dependence of the final state on the history of applied voltage. Scale-invariance properties of memristive networks are demonstrated in Sec. IV. In Sec. V we consider a memristive ladder and demonstrate the existence of avalanches in its switching dynamics. Finally, in Sec. VI we report our conclusions.

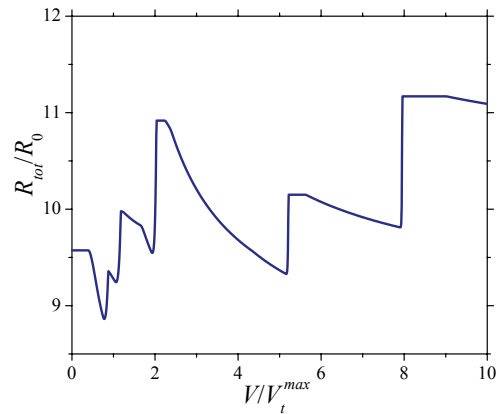


FIG. 2: (color online). Total memristance R_{tot} in a network of 10 randomly oriented devices as a function of a slowly ramped voltage V . In this simulation, we have used fixed values of $R_{on} = 0.05R_0$ and $R_{off} = 1.95R_0$. The threshold voltages and initial memristances of memristive systems have been randomly selected in the interval $[0, V_t^{max}]$ and $[R_{on}, R_{off}]$, respectively.

II. ACCELERATION AND SLOWING-DOWN OF THE RESISTANCE SWITCHING IN ADIABATIC PROCESSES

Let us first discuss the switching dynamics in a 1D chain of memristive systems. For the sake of definiteness let us consider a simplified version of a generic threshold voltage model of memristive system [5]

$$I = x^{-1}V_M, \quad (2)$$

$$\dot{x} = \beta (V_M - 0.5 [|V_M + V_t| - |V_M - V_t|]) \times [\theta(V_M)\theta(R_{off} - x) + \theta(-V_M)\theta(x - R_{on})] \quad (3)$$

where I and V_M are the current through and the voltage across the system, respectively, x is the internal state variable playing the role of memristance, $x \equiv R$, $\theta(\cdot)$ is the step function, β is a positive constant characterizing the rate of memristance change when $|V_M| > V_t$, V_t is the threshold voltage, and R_{on} and R_{off} are limiting values of the memristance R . In Eq. (3), the role of θ -functions is to confine the memristance change to the interval between R_{on} and R_{off} . In our numerical simulations, the value of $x = R$ is monitored at each time step and in the situations when $x < R_{on}$ or $x > R_{off}$, it is set equal to R_{on} or R_{off} , respectively. Importantly, the model given by Eqs. (2) and (3) describes a system whose memristance can only change when the applied voltage magnitude exceeds the threshold voltage. In addition, according to the sign of the applied voltage its resistance may increase or decrease, thus representing an asymmetric system, as indicated by the black thick line in the memristor symbols in Fig. 1.

For our numerical simulations, it is convenient to write Eq. (3) in dimensionless form. This can be done by expressing the voltage in units of V_t , the memristance

(and resistances) in units of R_0 and the time in units of $t_0 = R_0/\beta V_t$. Introducing $\tilde{V}_M = V_M/V_T$, $\tilde{x} = x/R_0$ and $\tilde{t} = t/t_0$, we rewrite Eq. (3) in the following form

$$\frac{d\tilde{x}}{d\tilde{t}} = \left(\tilde{V}_M - 0.5 \left[|\tilde{V}_M + 1| - |\tilde{V}_M - 1| \right] \right) \times \left[\theta(\tilde{V}_M) \theta(\tilde{R}_{off} - \tilde{x}) + \theta(-\tilde{V}_M) \theta(\tilde{x} - \tilde{R}_{on}) \right]. \quad (4)$$

Fig. 2 presents the total memristance of a network of such elements as a function of a slowly ramped voltage for a particular network realization. The initial resistances and the threshold voltages have been chosen randomly within uniform distributions. The polarity of the memristive elements in the network has also been randomly selected. The voltage is then increased following an *adiabatic process*: its increase rate is so slow that at any moment of time the network is in its equilibrium state. However, even in this simple adiabatic switch-on, one can easily notice two distinctive types of switching behavior. An abrupt – accelerated – switching, with well defined steps (see Fig. 2), occurs when the memristance R_j of the j -th memristor in the network increases at the given voltage polarity. In fact, as soon as the voltage drop across such element exceeds its threshold voltage, the increase in memristance increases the voltage drop across it, thus accelerating the switching. Instead, for memristive systems connected with opposite polarity, a decrease in memristance decreases the voltage drop across the system, thus decelerating the switching. This effect is clearly seen in long switching tails.

The dependence of the switching rate on the device polarity in 2D networks of ideal memristors [18] was previously reported in Ref. [19]. This previous result, however, does not take into account the threshold-type dynamics of realistic memristive devices [1]. Consequently, in adiabatic experiments, instead of sharp steps and long tails predicted by us, this previous work suggests unrealistic switching of memristors into their limiting states already at very small applied voltages and the absence of any subsequent evolution of R_{tot} as shown in Fig. 2.

The accelerated/decelerated switching behavior described above reflects also on the evolution of the resistance distribution function, $f_R(x)$, of the network. This is shown in Fig. 3 at several moments of time when the applied voltage V_0 adiabatically ramps up from 0 to $10V_t^{max}$. In fact, the memristance distribution function dynamics (Fig. 3) indicates a faster switching of memristive systems with initially higher memristances (thus with voltage drops exceeding their threshold voltages). Moreover, while the switching into the "off" state is always complete (since it is accelerated), memristive systems connected with opposite polarity may only partially switch toward the "on" state because of the decelerating switching effect discussed above. Higher values of applied voltage would further promote their switching as well as initiate switching of memristive systems with higher V_t and lower initial resistances, requiring higher voltage drops to induce their dynamics.

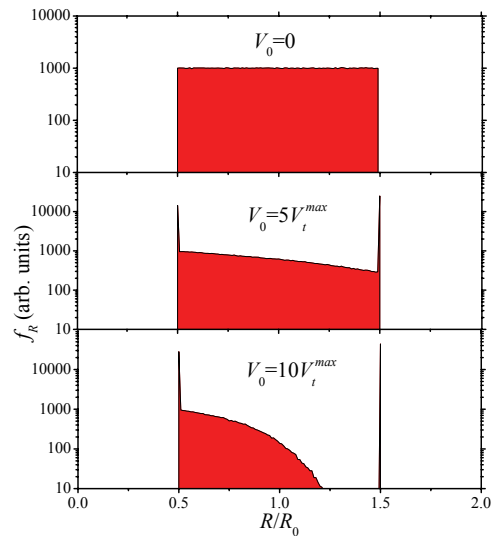


FIG. 3: (color online). Evolution of memristance distribution function f_R in a network of 10 randomly oriented memristive systems subjected to an adiabatically switched voltage V_0 . In this simulation, we have used fixed values of $R_{on} = 0.5R_0$ and $R_{off} = 1.5R_0$, the threshold voltages and initial memristances of memristive systems have been randomly selected (with uniform distribution) in the interval $[0, V_t^{max}]$ and $[R_{on}, R_{off}]$, respectively. The memristance distribution function f_R has been obtained by averaging over 10^4 random realizations of the network.

Fig. 4 presents the evolution of the resistance distribution function in the case of normally distributed threshold voltages and limiting values of memristances. The overall behavior of $f_R(x)$ is very similar to that shown in Fig. 3 for the case of uniformly distributed device parameters. Specifically, in both cases, memristive devices switch into the limiting R_{off} states faster than into R_{on} ones. Moreover, the dynamics of total memristance, $R_{tot}(t)$ in the network with normally distributed device characteristics has exactly the same features as $R_{tot}(t)$ depicted in Fig. 2 (acceleration and slowing-down of the switching).

III. DIFFERENT FINAL STATES WITH SAME INITIAL CONDITION

Another consequence of memory is that the memristance at the final moment of time is determined both by the initial conditions *and* by the actual shape of the applied signal, so that if the initial conditions are the same, the final memristance could be different according to the way the input signal is applied. This effect is even more pronounced in memristive networks, where all elements are *collectively* coupled by current conservation. As an example we compare the evolution of memristances in a network subject to both a slowly ramped-up voltage (from 0 to $10V_t^{max}$) and a step voltage of $10V_t^{max}$ applied

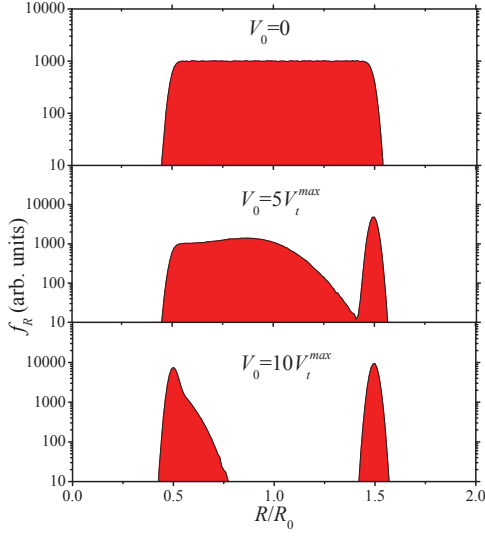


FIG. 4: (color online). Evolution of memristance distribution function f_R in a network of 10 randomly oriented memristive systems subjected to an adiabatically switched voltage V_0 . In this simulation, we have used normally distributed R_{on} , R_{off} , and V_t with means $0.5R_0$, $1.5R_0$ and V_t^0 and standard deviations $0.02R_0$, $0.02R_0$ and $0.05V_t^0$, respectively. The memristance distribution function f_R has been obtained by averaging over 10^5 random realizations of the network.

at $t = 0$. The initial conditions (initial parameters of memristive systems in the network) are chosen to be the same. Fig. 5 shows results of our simulations for a particular network realization of ten elements. One can clearly notice that in this particular situation the main difference of final states is associated with the fourth memristor from the left. While in the case of slowly ramped voltage $R_4(t = t_{final}) = R_4(t = 0)$, the sudden voltage pulse switches the fourth memristive element into the "off" state R_{off} . According to our observations, such a significant difference occurs in $\sim 8\%$ of network realizations at the selected simulation conditions.

In order to understand the signal-dependent switching in the memristive networks, let us consider the simplest nontrivial case of a network of two memristive elements, M_1 and M_2 , connected in series (see Fig. 1). According to Eqs. (2), (3), such a network, subjected to a non-negative voltage $V \geq 0$, is described by the following equations

$$\frac{dR_1}{dt} = \begin{cases} \pm\beta \left(\frac{R_1}{R_1+R_2}V - V_{t1} \right), & \text{when } \frac{R_1}{R_1+R_2}V > V_{t1}, \\ \text{and } R_1 < R_{off} \text{ for sign "+"}, \\ \text{or } R_1 > R_{on} \text{ for sign "-"}; \\ 0, & \text{in all other cases;} \end{cases} \quad (5)$$

$$\frac{dR_2}{dt} = \begin{cases} \pm\beta \left(\frac{R_2}{R_1+R_2}V - V_{t2} \right), & \text{when } \frac{R_2}{R_1+R_2}V > V_{t2}, \\ \text{and } R_2 < R_{off} \text{ for sign "+"}, \\ \text{or } R_2 > R_{on} \text{ for sign "-"}; \\ 0, & \text{in all other cases,} \end{cases} \quad (6)$$

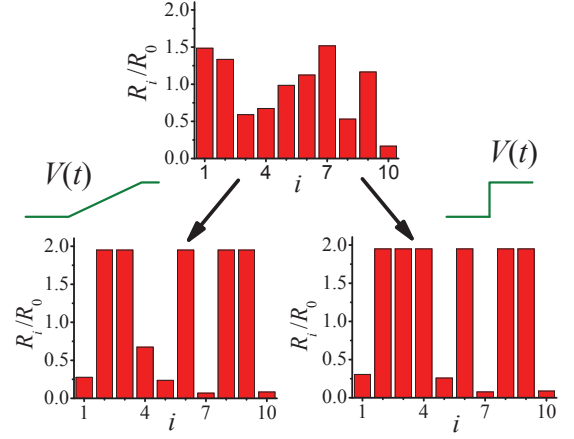


FIG. 5: (color online). Initial (top) and final (bottom) memristances in a network of 10 memristive elements (denoted by the index i) subjected to a slow ramp-up and a step-like applied voltage so that in both cases $V(t = t_{final}) = 10V_t^{max}$ (the respective voltage shapes are indicated in the middle). In this simulation, we have used fixed values of $R_{on} = 0.05R_0$ and $R_{off} = 1.95R_0$, the threshold voltages and initial memristances have been randomly selected (with uniform distribution) in the interval $[0, V_t^{max}]$ and $[0.05R_0, 1.95R_0]$, respectively, and the polarity of the memristive systems in the network has also been randomly selected. However, both simulations have been performed starting with the same initial randomly selected network realization.

where R_1 and R_2 , and V_{t1} and V_{t2} are the memristances and the threshold voltages of the first and second memristive systems, respectively. The signs " \pm " in the RHS of Eqs. (5), (6) take into account different possible polarities of the memristive systems in the network. Here we suppose that the parameter β determining the switching rate is positive and of the same magnitude for both memristive systems. Eqs. (5), (6) determine a system of nonlinear differential equations, which must be supplied with the initial conditions for memristances R_1 and R_2 at $t = 0$.

Let us consider a situation when a constant voltage V is applied at $t = 0$. In this case, Eqs. (5), (6) can be solved analytically. However, due to rather complex functional dependencies in the RHS of these equations, there are a lot of cases depending on the mutual relations between the initial memristances, threshold voltages, and applied voltage V . Below, we investigate several possible scenarios for signal-dependent switching.

According to Eqs. (5), (6), any changes in memristances occur when the voltage drop across at least one memristive system exceeds its threshold voltage. Let us consider then the case when the voltage drop $V_1 = R_1V/(R_1 + R_2)$ across M_1 exceeds V_{t1} , but the voltage drop $V_2 = R_2V/(R_1 + R_2)$ across M_2 is smaller than V_{t2} . We also suppose that $R_{on} < R_1 < R_{off}$. In this case (see Eq. (6)), the memristance R_2 does not change, $R_2 = \text{const}$. Eq. (5) can be integrated resulting

in the following algebraic equation for $R_1(t)$:

$$R_1 + \frac{VR_2}{V - V_{t1}} \ln \left[\frac{(V - V_{t1})R_1 - V_{t1}R_2}{(V - V_{t1})R_1(0) - V_{t1}R_2} \right] = R_1(0) \pm \beta(V - V_{t1})t. \quad (7)$$

It should be noted that Eq. (7) is valid while the following inequalities are satisfied:

$$R_{on} < R_1 < R_{off}, \quad (8)$$

$$V_1 = \frac{R_1}{(R_1 + R_2)}V > V_{t1}, \quad (9)$$

$$V_2 = \frac{R_2}{(R_1 + R_2)}V < V_{t2}. \quad (10)$$

If we have a "positive" polarity with respect to applied external voltage V (sign "+" before β in the RHS of Eq. (5)), then as it follows from Eq. (7) or directly from Eq. (5), the memristance R_1 increases with time. In this case the voltage drop V_1 increases, but V_2 decreases, because the total voltage drop across both memristors is constant, $V_1 + V_2 = V$. Thus the memristance R_1 increases with an acceleration (as V_1 increases). When $R_1(t)$ reaches R_{off} (the corresponding moment of time t can be immediately calculated from Eq.(7)), the memristance R_1 stops increasing (according to Eq. (5)) and stays constant.

In the case of a "negative" polarity (sign "-" before β in the RHS of Eq. (5)), it follows from Eq. (7) (or directly from Eq. (5)) that the memristance R_1 decreases with time. In this case the voltage drop V_2 increases, and V_1 decreases. This evolution, as determined by Eq.(7) stops when memristance R_1 reaches the lowest possible value R_{on} , or inequality (9) does not hold, depending on what happens first. If inequality (10) becomes invalid first, while other inequalities (8), (9) are still valid, we encounter the situation when both memristances change (if, of course, $R_{on} < R_2 < R_{off}$). Let us now consider the case when both memristances are changed simultaneously. This is possible only when the following inequalities are valid:

$$R_{on} < R_{1,2} < R_{off}, \quad (11)$$

$$V_1 = \frac{R_1}{(R_1 + R_2)}V > V_{t1}, \quad (12)$$

$$V_2 = \frac{R_2}{(R_1 + R_2)}V > V_{t2}. \quad (13)$$

We limit ourselves to the cases when the memristive elements, M_1 and M_2 , are connected with the same polarity. Then by taking the sum of Eqs. (5) and (6), and integrating it with respect to time we find for the total memristance

$$R_1 + R_2 = \pm\beta(V - V_{t1} - V_{t2})t + R_1(0) + R_2(0). \quad (14)$$

By substituting this relation into Eq. (5) we get a linear differential equation which can be easily integrated.

As a result we find the following expressions which determine the time dependence of memristances for both memristors:

$$R_1(t) = P(R_1 + R_2)^{\frac{V}{V - V_{t1} - V_{t2}}} + \frac{V_{t1}}{V_{t1} + V_{t2}}(R_1 + R_2), \quad (15)$$

$$R_2(t) = -P(R_1 + R_2)^{\frac{V}{V - V_{t1} - V_{t2}}} + \frac{V_{t2}}{V_{t1} + V_{t2}}(R_1 + R_2), \quad (16)$$

where the parameter P is determined by the initial conditions

$$P = \frac{V_{t2}R_1(0) - V_{t1}R_2(0)}{V_{t1} + V_{t2}} (R_1(0) + R_2(0))^{-\frac{V}{V - V_{t1} - V_{t2}}}. \quad (17)$$

From Eqs. (14)-(17) it follows that the voltage drop V_1 increases with time for the "positive" polarity and when $V_{t2}R_1(0) > V_{t1}R_2(0)$, while the voltage drop V_2 decreases. Thus it leads to accelerating of the switching of memristor M_1 and to decelerating of the switching of memristor M_2 .

Having discussed the switching dynamics due to a constant applied voltage, let us compare the final states of memristive systems for two different bias application protocols. Specifically, by assuming the same initial states of the memristive systems and the same final voltage magnitude, we consider the situations of the step and slowly ramped voltages. Our consideration is based on the following model parameters: $R_1(0) = R_2(0) = 0.2R_0$, $R_{on} = 0.05R_0$, $R_{off} = 1.95R_0$, $V_{t1} = V/3$, $V_{t2} = 1.1V/3$, and the signs "+" in the RHS of Eqs. (5), (6).

For slowly ramped applied voltage, M_1 is in the regime of accelerated switching (see also discussion after Eq. (10)). In addition, one can easily notice that the voltage drop across M_2 never exceeds V_{t2} . Consequently, the final memristances are $R_1(t = t_{final}) = R_{off} = 1.95R_0$ and $R_2(t = t_{final}) = R_2(t = 0) = 0.2R_0$.

In the case of the step-like applied voltage V , with our specific model parameters the inequalities (11)-(13) are satisfied at the initial moment of time. As a result, both memristances are changing according to Eqs. (14)-(17). Because of the positivity of the parameter P , the voltage V_1 increases in time, but the voltage V_2 decreases (as it follows from Eqs. (14)-(16)). At the same time, both memristances R_1 and R_2 increase according to Eqs. (5), (6). At a certain moment of time the voltage drop V_2 across memristor M_2 becomes equal to the threshold voltage V_{t2} , and further change of M_2 becomes impossible. Further evolution of memristance R_1 is determined by Eq. (7) and leads to $R_1(t = t_{final}) = R_{off} = 1.95R_0$, while the memristance R_2 does not change, $R_2(t = t_{final}) = 0.33R_0$. Clearly, the final state of M_2 in this case is different than that for the case of the slowly ramped voltage.

We would like to emphasize that the different final states are obtained as a result of the *collective dynamics* of many memristive systems in the network. Such a

behavior cannot be observed considering a single memristive device described by Eqs. (2), (3) subjected to similar voltage patterns. The interaction among devices in the network is provided by the current that, at each moment of time, is determined by states of *all* devices in the network.

Our observation of different final states can find useful applications in electronics, albeit in a modified form. For example, the sensitivity of the final states to the applied pulse profile can be used in signal recognition. We would also like to note that the response of single memristive devices with more complex internal degrees of freedom can be richer than that predicted by Eqs. (2), (3). Even with such single memory devices, one can expect the dependence of the final states on the applied voltage protocol. Examples of such memory devices can be found in Refs. [20, 21].

IV. SCALE INVARIANCE OF DYNAMICS THRESHOLD VOLTAGE

In this Section, we ask the question of the probability of dynamics in the memristive network. Let us consider a random memristive network (Fig. 1) that is characterized by smooth distribution functions $f_{V_i}(V)$ and $f_R(R)$, which are the same for all memristive systems in the network. Moreover, the results of this section do not depend on the orientation (polarity) of memristive systems in the network. Clearly, if a very low voltage V is applied to the network then the probability of dynamics is close to zero as the voltage across each memristive system is likely below its threshold voltage. If a high voltage is applied, the probability of dynamics is close to 1 as it is highly probable that the voltage across at least one memristive system exceeds its threshold. To quantify this effect, let us find a *dynamics voltage threshold* V^* , which is the magnitude of the applied voltage when, with a probability of 1/2, the state of at least a single memristive system will change. Formally, it can be found from

$$p_d(V^*) = \int p_d(V^*|R_1, \dots, R_N) \prod_{i=1}^N f_R(R_i) dR_i = 1/2, \quad (18)$$

where $p_d(V^*|R_1, \dots, R_N)$ is the conditional probability of dynamics in the network at given memristances R_1, R_2, \dots, R_N . Alternatively, Eq. (18) can be rewritten as $1 - p_d(V^*) = 1/2$, where the left-hand-side is the probability of no dynamics. The latter occurs when the voltage across each memristive system in the network, V_i , does not exceed its threshold voltage, $V_{t,i}$. Consequently, the conditional probability of no dynamics $p_{nd}(V^*|R_1, \dots, R_N) = 1 - p_d(V^*|R_1, \dots, R_N)$ can be written as a product of single element conditional probabilities

$$p_{nd}(V^*|R_1, \dots, R_N) = \prod_{i=1}^N p(V_i \leq V_{t,i}|R_1, \dots, R_N), \quad (19)$$

where $V_i = R_i V^* / R_{tot}$ and $R_{tot} = \sum_{i=1}^N R_i$. Noticing that the probabilities in Eq. (19) can be expressed through the threshold voltage distribution function f_{V_t} as

$$p(V_i \leq V_{t,i}|R_1, \dots, R_N) = \int_{V_i}^{\infty} f_{V_t}(y) dy, \quad (20)$$

we finally arrive at the following equation

$$\int \left(\prod_{i=1}^N \int_{R_i V^* / R_{tot}}^{\infty} f_{V_t}(y_i) dy_i \right) \prod_{i=1}^N f_R(R_i) dR_i = 1/2 \quad (21)$$

determining the dynamics threshold voltage V^* .

A. General case, narrow distribution of initial memristances

Assuming that initial memristances are narrow distributed around a certain value \bar{R} (formally we can assume $f_R(R) = \delta(R - \bar{R})$) we use $V_i = V^* / N$ in Eq. (21) to derive

$$\left[\int_{V^*/N}^{\infty} f_{V_t}(x) dx \right]^N = 1/2. \quad (22)$$

Let us then find the dynamics threshold voltage V^* in the limit of large N . By using the normalization condition, and by taking the logarithm of both sides of Eq. (22) we find, for arbitrary N , the following relation

$$\ln \left[1 - \int_0^{V^*/N} f_{V_t}(x) dx \right] = -\frac{\ln 2}{N}. \quad (23)$$

From Eq. (23) we conclude that for large N the integral in Eq. (23) tends to zero. Because of the positivity of the distribution function $f_{V_t}(x)$ this means that the integration domain must tend to zero as N tends to infinity. Thus we see that $\lim_{N \rightarrow \infty} V^* / N = 0$. This means that we can use the asymptotic behavior of the distribution function to find the dynamics threshold voltage. Let us then consider a general situation when

$$f_{V_t}(x) = f_0 x^\gamma, \quad x \rightarrow 0, \quad (24)$$

with some positive parameters f_0 and $\gamma \geq 0$. In this case the integration in Eq. (23) gives the final result

$$V^* = \left[\frac{(1 + \gamma) \ln 2}{f_0} \right]^{1/(1+\gamma)} N^{\frac{\gamma}{1+\gamma}}, \quad N \rightarrow \infty. \quad (25)$$

When there is a finite probability of arbitrarily small thresholds ($\gamma = 0$ and $f_0 \neq 0$), we obtain a scale invariant expression

$$V^* = \frac{\ln 2}{f_{V_t}(0)}, \quad N \rightarrow \infty. \quad (26)$$

This is the main result of Section IV.

B. Exponential distribution of threshold voltages

Note that the condition $R_i = \bar{R}$ is not always necessarily required to obtain Eq. (26). To see this, let us consider, e.g., an exponential distribution of threshold voltages, $f_{V_i}(y) = \exp(-y/V_0)/V_0$, where V_0 is a characteristic threshold voltage of memristive systems. It follows from Eq. (21) that

$$\int \left(\prod_{i=1}^N e^{-\frac{V_i}{V_0}} \right) \prod_{i=1}^N f_R(R_i) dR_i = 1/2. \quad (27)$$

Since $\sum V_i = V^*$, and taking into account normalization conditions (1) we find the following scale invariant expression for the dynamics threshold voltage:

$$V^* = \ln(2)V_0, \quad (28)$$

which is in full agreement with Eq. (26). Note, however, that Eq. (28) has been obtained without the assumption of narrow distribution of memristances.

C. Uniform distribution of threshold voltages

Let us now consider the uniform distribution of the threshold voltages, namely, $f_V(x) = 1/V_M$, if $0 \leq x \leq V_M$, and $f_V(x) = 0$ otherwise. Here, V_M is the maximum threshold voltage. For the sake of simplicity, we also assume a narrow distribution of initial memristances $R_i = \bar{R}$. Then, from Eq. (21), it can be found that

$$V^* = NV_M \left(1 - \sqrt[N]{\frac{1}{2}} \right) \xrightarrow{N \rightarrow \infty} \ln(2)V_M. \quad (29)$$

Clearly, the right hand side of Eq. (29) is again of the form of Eq. (26). Numerical calculations shown in Fig. 6 indicate that a random uniform distribution of initial memristances does not change the result predicted by Eq. (29).

D. $x \exp(-x)$ distribution of threshold voltages

Finally, we consider an ensemble of memristive systems described by the threshold voltage distribution function $f_V(x) = (x/V_0^2) \exp(-x/V_0)$. The difference with the previously considered cases is that now $f_V(0) = 0$ and Eq. (26) is not applicable. Assuming a narrow distribution of initial memristances $R_i = \bar{R}$, Eq. (21) is written as

$$\left(1 + \frac{V^*}{NV_0} \right)^N e^{-\frac{V^*}{V_0}} = 1/2. \quad (30)$$

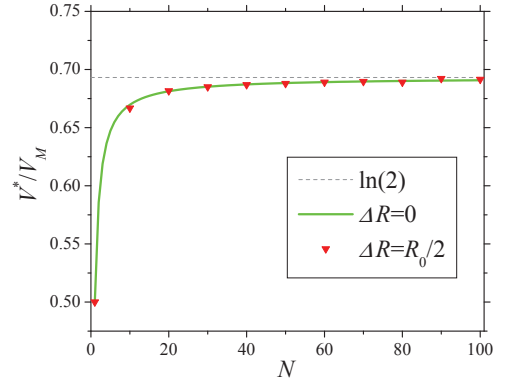


FIG. 6: (color online). Dynamics threshold voltage as a function of the number of memristive systems in the network. This plot has been obtained assuming uniform probability distributions of threshold voltages and initial memristances. Specifically, initial memristances are selected randomly (with uniform distribution) in the interval between $R_0 - \Delta R$ and $R_0 + \Delta R$, and threshold voltages are selected in the interval between 0 and V_M . The solid line was obtained using Eq. (29).

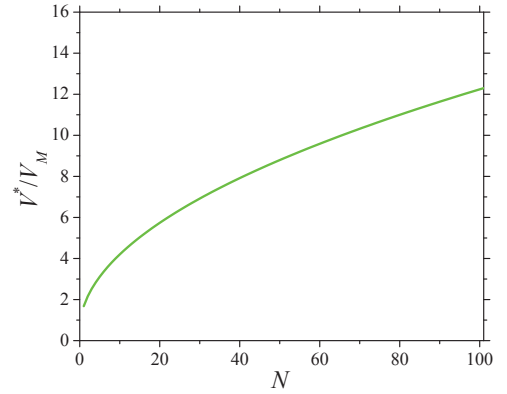


FIG. 7: (color online). Dynamics threshold voltage as a function of the number of memristive systems in the network. This plot has been obtained as a solution of Eq. (30) based on $x \exp(-x)$ distribution of threshold voltages.

Fig. 7 shows a numerical solution of Eq. (30). The asymptotic behavior of the dynamics threshold voltage V^* as N tends to infinity is described by Eq. (25) with $\gamma = 1$, and $f_0 = 1/V_0^2$, i.e. $V^* = V_0 \sqrt{2 \ln 2 N}$ as $N \rightarrow \infty$. Of course, the same result can be derived from Eq. (30) as well. It is clearly seen that the dynamic threshold voltage V^* does not saturate as in the case of previously considered distributions (see Eqs. (26), (28) and (29)). We relate this observation to the absence of memristive systems with zero threshold voltage, V_i , in the ensemble.

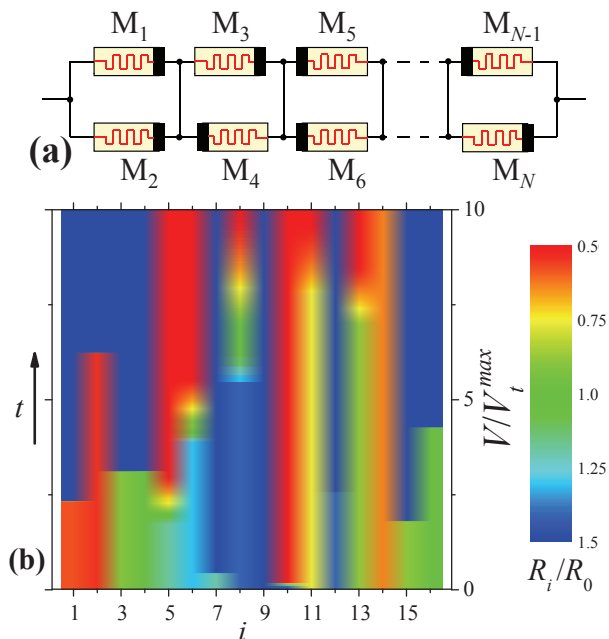


FIG. 8: (color online). (a) Memristive ladder consisting of N randomly oriented memristive systems. (b) Individual memristances R_i in a memristive ladder of 16 randomly oriented devices as a function of a slowly ramped voltage. The memristance of each device is represented by a color in the band in the vertical direction representing time evolution. In this simulation, we have used fixed values of $R_{on} = 0.5R_0$ and $R_{off} = 1.5R_0$. The threshold voltages and initial memristances of memristive systems have been randomly selected (with uniform distribution) in the interval $[0, V_t^{max}]$ and $[R_{on}, R_{off}]$, respectively. The polarities of memristive devices in (a) and (b) are not related to each other.

V. SWITCHING AVALANCHES IN MEMRISTIVE LADDERS

Finally, let us consider the possibility of avalanches in memristive networks. By an avalanche, we mean the situation where a single rapid switching event induces switchings in other memristive systems. It follows from our previous considerations (reported in Sec. II) that avalanches are not possible in purely 1D networks. Indeed, an accelerated switching can not induce an avalanche since the increase in memristance of the switching element reduces the voltages across all other elements in the network. During the slowing-down switching, the inducing switching event is not well characterized as being spread in time.

Avalanches, however, are possible in networks of higher dimensions. A memristive ladder (a quasi-one dimensional memristive network) presented in Fig. 8(a) is an example of such situation. We have performed a numerical simulation of the dynamics of a memristive ladder consisting of 2 chains of 8 memristive systems each (16 memristive devices in total) subjected to a slowly ramped

voltage. In the particular realization of memristive devices whose evolution is presented in Fig. 8(b), the simultaneous switching of 3-rd and 4-th memristive systems into the "off" state is an avalanche.

In the memristive ladder, an avalanche is possible in a pair of memristive systems connected in parallel (M_j and M_{j+1} , where j is an odd number) that are in the regime of accelerated switching. The switching of the memristive system with a lower V_t , say M_j , initiates the avalanche (the switching of M_{j+1}) if the voltage across these two memristive systems after the switching of M_j exceeds the threshold voltage of M_{j+1} . As the switching in the accelerated switching regime occurs fast, effectively, the switchings of M_j and M_{j+1} are observed at the same value of slowly ramped voltage (see Fig. 8(b) for an example). Moreover, in the circuit in Fig. 8(a) an avalanche can develop only in the vertical direction. Since there are only 2 memristive devices in the vertical direction in each cell, the maximum number of devices in avalanche is two. Horizontal avalanches are not possible since a memristive device switching into its "off" state reduces the voltage fall across all other cells of the ladder. We expect that the size and role of switching avalanches increase with the increase of the network complexity. In particular, avalanches involving several memristive devices are possible in ladders as in Fig. 8(a) extended in the vertical direction. Detailed studies of avalanches in 2D and 3D networks will be reported elsewhere.

VI. CONCLUSIONS

In conclusion, we have found that one-dimensional memristive networks exhibit several distinctive features that make them quite unlike traditionally studied dynamical networks [10, 11]. The essential difference is related to intrinsic non-local interactions coupled to memory features. Indeed, the order of memristive systems in 1D networks is completely irrelevant since the current through the network is conserved. Two possible types of switching dynamics – accelerating and decelerating – have been identified depending on the polarity of memristive elements in the network. We have also demonstrated that the final state of the network depends on the *protocol* of how the external voltage is applied, and that switching avalanches can occur in memristive ladders. Additionally, a scale invariance of dynamics threshold voltage has been demonstrated. Most of our results are universal in the sense that they do not depend on the selected distribution functions (e.g., the acceleration and slowing down of switching, different final states, existence of switching avalanches). The results reported in this paper are also relevant to networks of memcapacitive and meminductive systems [12], while specific features of switching dynamics with such elements may not coincide. In general, we expect some of these predictions to be shared by arbitrary complex networks with memory.

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