Mixed convolved action for classical and fractional-derivative dissipative dynamical systems

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The principle of mixed convolved action provides a new, rigorous weak variational formalism for a broad range of initial value problems in mathematical physics and mechanics. Here the focus is initially on classical single-degree-of-freedom oscillators incorporating either Kelvin-Voigt or Maxwell dissipative elements and then subsequently on systems that utilize fractional derivative constitutive models. In each case, an appropriate mixed convolved action is formulated and a corresponding weak form is discretized in time using temporal shape functions to produce an algorithm suitable for numerical solution. Several examples are considered to validate the mixed convolved action principles and to investigate the performance of the numerical algorithms. For undamped systems, the algorithm is found to be symplectic and unconditionally stable with respect to time step. In the case of dissipative systems, the approach is shown to be robust and accurate with good convergence characteristics for both classical and fractional-derivative based models. As part of the derivations, some interesting results in the calculus of Caputo fractional derivatives also are presented.

I. INTRODUCTION

Previously, a new stationary principle is established in Ref. [1] for the classical Kelvin-Voigt single-degree-of-freedom (SDOF) damped oscillator, by formulating a real scalar action using mixed variables, fractional derivatives and the temporal convolution of convolutions. This principle of mixed convolved action is then generalized in Ref. [2] to multi-degree-of-freedom spatially discrete dynamical systems and to elastodynamic continua undergoing infinitesimal deformation. The primary advantage of the mixed convolved action approach is that all of the governing differential equations and initial conditions for a general class of dissipative

dynamical systems can be derived from this scalar functional as its Euler-Lagrange equations, using mathematically sound operations of variational calculus.

Although Hamilton's principle [3, 4] has been with us for a long time and has found great importance throughout mathematical physics, it suffers from two main difficulties. The first is related to its inability to incorporate dissipation. Hamilton's principle applies only to conservative systems. The second difficulty is more subtle, relating to the handling of initial conditions and the restrictions on the function variations. In particular, Hamilton's principle requires that the variations vanish at the endpoints of the time interval, which for consistency implies that the functions are known at these two instants. Of course, when attempting to solve a dynamical problem, under normal circumstances, one does not know the solution at the end of the time interval. Rather, this is often the main objective of the analysis, which in turn suggests that there may be a serious philosophical or mathematical inconsistency associated with the application of Hamilton's principle. Furthermore, the initial conditions are not at all part of the variational statement and must be imposed outside of the process.

Tonti [5] was perhaps the first to identify these shortcomings and to suggest that convolution should replace the inner product as the operator of choice in establishing variational methods for initial value problems. Somewhat earlier, Gurtin [6, 7] devised a method based upon convolution to address dissipation within a variational approach. However, with his method, one never can recover the original strong form of the problem.

More recently, Riewe [8, 9] proposed the use of fractional derivatives to permit the development of a scalar functional for dissipative dynamical systems. This is an attractive idea and, subsequently, many other researchers including Agrawal [10-12], Atanackovic *et al*. [13], Baleanu and Muslih [14], Cresson [15], Dreisigmeyer and Young [16, 17], El-Nabulsi and Torres [18] and Abreu and Godinho [19] have followed similar approaches. Interestingly, none of these works include an analytical or numerical demonstration verifying that their approach can reproduce the many well-known closed-form solutions for classical linear mass-spring-damper systems. In the present paper, discrete time formulations are developed, based upon the concept of mixed convolved action, and then applied to solve example problems of free and forced

vibration. Initially, the classical Kelvin-Voigt and Maxwell models are considered and, as will be seen, excellent correlations are obtained.

Subsequently, to investigate the further potential of the mixed convolved action approach, non-classical constitutive models involving fractional-derivatives are considered. Such models have a long history, beginning with the exploratory work by Nutting [20-22], Schofield and Scott Blair [23-26], Gemant [27, 28] and Scott Blair [29] in the first half of the twentieth century to understand the behavior of a range of viscoelastic materials, including flour dough, cheese and clay. Subsequent efforts to characterize viscoelastic response in terms of fractional derivatives include the work by Caputo [30], Caputo and Mainardi [31, 32], Bagley and Torvik [33, 34] and Glockle and Nonnenmacher [35]. Alternatively, one can formulate these models equivalently in terms of weakly singular kernel functions. This latter approach was taken by Gross [36], Rabotnov [37, 38], Gerasimov [39], and Meshkov *et al.* [40]. Rossikhin [41] provides a recent comprehensive review. Bagley and Torvik [42] also developed finite element formulations, while Makris *et al.* [43, 44] formulated corresponding boundary integral approaches. Through the years, many other applications have appeared. For example, in the area of seismic response characterization, one may mention the work by Makris *et al.* [43-46], Koh and Kelly [47] and Singh and co-workers [48, 49] as illustrations. Furthermore, the related concept of fractional oscillators has been developed by Mainardi [50], Achar *et al.* [51] and Stanislavsky [52]. In the present work, a variational statement for a single-degree-of-freedom oscillator with a fractional-derivative Maxwell element is formulated, discretized in time and then solved numerically. In order to develop this new mixed convolved action variational formulation, a few results in fractional calculus also are established.

The remainder of the paper is organized as follows. In Sec. II, a weak form is developed, based upon the mixed convolved action defined in Ref. [1], for a classically damped Kelvin-Voigt single-degree-of-freedom dynamical system. Next, in Sec. III, several results are derived in fractional calculus that enable an effective numerical implementation for the weak Kelvin-Voigt formulation in Sec. IV. Using similar ideas, an algorithm for the classical Maxwell dynamical system is the focus of the development in Sec. V. Afterwards, attention shifts to non-classical constitutive models for the single-degree-of-freedom system. In Sec. VI, some results in Caputo fractional calculus are derived and then used in Sec. VII to formulate a mixed convolved action

for a fractional-derivative Maxwell dynamical system. Included in Sec. VII is the development of a weak form, along with the details for a numerical algorithm. Numerical examples are then presented in Sec. VIII to validate all of these new formulations for problems of both free and forced vibration. Finally, some conclusions are provided in Sec. IX.

II. WEAK FORM FOR KELVIN-VOIGT MODEL

In this section, consideration is given to the development of a computational algorithm for transient dynamic analysis, based upon the weak form given in Ref. [1]. The focus is on the SDOF Kelvin-Voigt system displayed in Figure 1, having mass m, damping coefficient c and stiffness $k = 1/a$ with a representing the flexibility. Let $u(t)$ and $J(t)$ denote the displacement of the mass and the impulse of the internal force $F(t)$ in the spring, respectively. Meanwhile, $\bar{f}(t)$ is the known applied force, which in general varies with time.

The mixed convolved action associated with this problem can be written in the following form [1]

$$\begin{aligned} I_{C_K}(u,\breve{u},\dot{u},J,\breve{J},\dot{J};t) = {} & \tfrac{1}{2}\left(\dot{u} * m\,\dot{u}\right)(t) - \tfrac{1}{2}\left(\dot{J} * a\,\dot{J}\right)(t) + \left(\breve{J} * \breve{u}\right)(t) + \tfrac{1}{2}\left(\breve{u} * c\,\breve{u}\right)(t) \\ & - \left(u * \bar{f}\right)(t) - u(t)\bar{j}(0), \end{aligned} \tag{1}$$

where a superposed dot represents a derivative with respect to time, while $\breve{\varphi}$ is used to denote the temporal left Riemann-Liouville semi-derivative of φ. This is defined by [53, 54]

$$\breve{\varphi}(t) = \left(\mathscr{D}_{0^+}^{1/2}\varphi\right)(t) \equiv \frac{1}{\Gamma(1/2)}\frac{d}{dt}\int_0^t \frac{\varphi(\tau)}{(t-\tau)^{1/2}}d\tau, \tag{2}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Meanwhile, the symbol $*$ indicates the convolution of two functions over time, such that

$$\left(\varphi * \psi\right)(t) = \int_0^t \varphi(\tau)\psi(t-\tau)d\tau. \tag{3}$$

The finite lower and upper limits of 0 and t, respectively, in the convolution, are chosen to coincide with the limits of the left Riemann-Liouville semi-derivative in Eq. (2) and the temporal range of interest for the initial value problems under consideration here. The variable upper limit t, which appears in the integrals within both Eqs. (2) and (3), requires careful treatment using the Leibniz rule, when taking temporal derivatives and performing integration-by-parts operations. This altogether becomes quite important for the formulation and success of the mixed convolved action and a number of interesting results that depend in part upon the variable upper limit are developed in Ref. [1].

The last term in Eq. (1) involves $\overline{j}(0)$, representing the initial impulse corresponding to $\overline{f}(t)$. The superposed bar is used here and throughout the paper to indicate a known or specified quantity, which is not subject to variation.

For the sake of consistency, all of the impulses relevant to the SDOF system may be written in a common manner. Thus, let

$$u(t) = \int_{-\infty}^{t} v(\tau) d\tau, \tag{4a}$$

$$J(t) = \int_{-\infty}^{t} F(\tau) d\tau, \tag{4b}$$

$$\overline{j}(t) = \int_{-\infty}^{t} \overline{f}(\tau) d\tau, \tag{4c}$$

where, from this perspective, displacement can be recognized as the impulse of velocity $v(t)$.

Notice, in particular, that the scalar functional I_{C_K} in Eq. (1) incorporates both conservative and non-conservative components within a unified framework. Consequently, this resolves the long-standing limitation of variational approaches for dissipative dynamical systems, while also providing an action that encapsulates the governing differential equations, along with all of the initial conditions of the underlying initial value problem. In retrospect, the appearance of temporal convolutions, rather than inner products, in this action is quite natural [5]. A true Lagrangian state function cannot possibly define dissipative processes. Instead, the introduction

of the convolution operator provides an opportunity to capture the history-dependent character of such phenomena in a systematic manner.

For stationary mixed convolved action, the first variation of Eq. (1) must be zero, thus providing the following weak form in time

$$\begin{aligned}\delta I_{C_K} = &\left(\delta\dot{u} * m\,\dot{u}\right)(t) - \left(\delta\dot{J} * a\,\dot{J}\right)(t) + \left(\delta\breve{J} * \breve{u}\right)(t) + \left(\delta\breve{u} * \breve{J}\right)(t) \\ &+ \left(\delta\breve{u} * c\,\breve{u}\right)(t) - \left(\delta u * \bar{f}\right)(t) - \delta u(t)\bar{j}(0) = 0.\end{aligned} \tag{5}$$

Equation (5) will provide the basis for a novel temporal finite element method for dissipative systems. Further details on the numerical implementation will be provided in Section IV, but two points should be made here. First, the variations at the end of the time interval are unconstrained in the mixed convolved action principle. This permits the development of an energy-preserving finite element methodology with discrete variables at temporal nodes, time-based shape functions and exact integration over the time interval. On the other hand, in Hamilton's principle, the variations are constrained at both the beginning and end of the time interval, which precludes the possibility to use a temporal finite element approach. Secondly, notice that the highest order derivatives that appear in the weak form in Eq. (5) are of first order. As a result, the discretized temporal function approximations of displacement $u(t)$, force impulse $J(t)$, displacement variation $\delta u(t)$ and force impulse variation $\delta J(t)$ require only C^0 continuity in time. This will permit the use of simple linear temporal shape functions for the numerical implementation.

Returning now to analytical aspects of the formulation, one may rewrite Eq. (5) after performing classical and fractional integration by parts on the appropriate terms as follows [1]

$$\begin{aligned}&\left(\delta u * \left\{m\,\ddot{u} + c\,\dot{u} + \dot{J} - \bar{f}\right\}\right)(t) + \left(\delta J * \left\{-a\,\ddot{J} + \dot{u}\right\}\right)(t) \\ &\quad + \delta u(t)\left\{m\,\dot{u}(0) + c\,u(0) + J(0) - \bar{j}(0)\right\} + \delta u(0)\left\{m\,\dot{u}(t)\right\} \\ &\quad + \delta J(t)\left\{-a\,\dot{J}(0) + u(0)\right\} - \delta J(0)\left\{-a\,\dot{J}(t)\right\} = 0.\end{aligned} \tag{6}$$

For arbitrary variations, this produces the Euler-Lagrange equations, representing the governing balance laws for the mixed formulation

$$m\,\ddot{u} + c\,\dot{u} + \dot{J} = \bar{f}, \tag{7a}$$

$$-a\,\ddot{J} + \dot{u} = 0, \tag{7b}$$

along with the initial conditions

$$m\,\dot{u}(0) + c\,u(0) + J(0) = \bar{j}(0), \tag{8a}$$

$$-a\,\dot{J}(0) + u(0) = 0. \tag{8b}$$

In addition, the variations at the initial instant must be taken as zero. Thus,

$$\delta u(0) = 0\,, \qquad \delta J(0) = 0, \tag{9a,b}$$

which from Eq. (6) then allows complete freedom in $\dot{u}$ and $\dot{J}$, as well as δu and δJ, at time t.

Clearly, all of this development based upon the convolution operator is perfectly consistent with the definition of an initial value problem [5, 1]. On the other hand, as noted above, Hamilton's principle with an inner product-based action requires zero variations at the beginning and end of the time interval [3, 4], which is not consistent with the known and unknown conditions associated with an initial value problem [5, 1].

III. FRACTIONAL DERIVATIVES AND THE CONVOLUTION OPERATOR

One of the primary objectives of the present work is to develop a numerical approach for the weak form of the mixed convolved action. The key novel elements in Eq. (5) are terms involving the convolution of semi-derivatives. If one envisions a numerical algorithm that employs polynomial-based temporal shape functions, then it will be necessary to evaluate the convolution of fractional derivatives of power functions. Some new results in fractional calculus are developed in this section to enable that evaluation.

Notice that semi-derivatives appear in the coupling terms and in the dissipation integrals within the mixed convolved action of Eq. (1) and its weak form Eq. (5). As mentioned previously, Eq. (2) is the left Riemann-Liouville semi-derivative of φ. More generally, the left Riemann-Liouville fractional derivative of arbitrary order α may be written as follows [53, 54]

$$\left(\mathscr{D}_{0^+}^{\alpha}\varphi\right)(t) \equiv \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\varphi(\tau)}{(t-\tau)^{\alpha}}d\tau, \tag{10}$$

for $0<\alpha<1$.

From Eqs. (2) and (10), the Riemann-Liouville fractional derivative may be viewed as the first derivative of a fractional integral. Thus,

$$\left(\mathscr{D}_{0^+}^{\alpha}\varphi\right)(t) = \frac{d}{dt}\left(\mathscr{I}_{0^+}^{1-\alpha}\varphi\right)(t), \tag{11}$$

where

$$\left(\mathscr{I}_{0^+}^{1-\alpha}\varphi\right)(t) \equiv \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\varphi(\tau)}{(t-\tau)^{\alpha}}d\tau \quad \text{for } t>0,\ 1-\alpha>0. \tag{12}$$

Then, from Eq. (12), it is clear that a fractional integral is simply the convolution of the original function $\varphi(t)$ with a singular power function $t^{-\alpha}$.

For some special functions, the required integrals and subsequent derivatives may be expressed in closed form. In particular, the fractional derivative of power functions t^m may be evaluated quite simply in closed form in terms of the Gamma function, such that

$$\left(\mathscr{D}_{0^+}^{\alpha}t^m\right)(t) = \frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)}t^{m-\alpha}, \tag{13}$$

for integer powers $m=0,1,2,\ldots$ and fractional derivatives with order $0<\alpha<1$ [53].

Now consider the convolution of the fractional derivatives of power functions, which can be expressed as

$$\left(\left(\mathscr{D}_{0^+}^{\alpha}t^m\right)*\left(\mathscr{D}_{0^+}^{\beta}t^n\right)\right)(t) = \frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)}\frac{\Gamma(1+n)}{\Gamma(1+n-\beta)}\int_0^t \tau^{m-\alpha}(t-\tau)^{n-\beta}d\tau. \tag{14}$$

Through a change of variable, this can be reformulated as follows

$$\left(\left(\mathscr{D}_{0^+}^{\alpha} t^m\right) * \left(\mathscr{D}_{0^+}^{\beta} t^n\right)\right)(t) = \frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)} \frac{\Gamma(1+n)}{\Gamma(1+n-\beta)} B(1+m-\alpha, 1+n-\beta) t^{1+m+n-\alpha-\beta}, \tag{15}$$

by introducing the Beta function $B(\cdot,\cdot)$, where

$$B(x,y) = \int_0^1 \xi^{x-1}(1-\xi)^{y-1} d\xi. \tag{16}$$

Notice that $B(\cdot,\cdot)$ itself is a convolution, which ultimately can be expressed in terms of the Gamma function as

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{17}$$

Now substituting Eq. (17) into Eq. (15), one obtains the very interesting result

$$\left(\left(\mathscr{D}_{0^+}^{\alpha} t^m\right) * \left(\mathscr{D}_{0^+}^{\beta} t^n\right)\right)(t) = \frac{\Gamma(1+m)\,\Gamma(1+n)}{\Gamma(2+m+n-\alpha-\beta)} t^{1+m+n-\alpha-\beta}, \tag{18}$$

with $0<\alpha,\beta<1$ and $m,n=0,1,2,\ldots$. Notice, in particular, Eq. (18) specifies that the convolution of fractional derivatives of power functions depends not on the individual fractional orders α and β, but only on their sum $\alpha+\beta$. This result will have some significance for the numerical implementation of the weak form of the mixed convolved action and for the range of fractional derivatives that may be included in constitutive model formulations.

Although not required for the algorithm development here, the result in Eq. (18) can be generalized easily to non-integer powers. In that case,

$$\left(\left(\mathscr{D}_{0^+}^{\alpha} t^{\mu}\right) * \left(\mathscr{D}_{0^+}^{\beta} t^{\nu}\right)\right)(t) = \frac{\Gamma(1+\mu)\,\Gamma(1+\nu)}{\Gamma(2+\mu+\nu-\alpha-\beta)} t^{1+\mu+\nu-\alpha-\beta}, \tag{19}$$

for $0<\alpha,\beta<1$ and $\mu,\nu>-1$.

IV. COMPUTATIONAL METHOD FOR UNDAMPED AND KELVIN-VOIGT MODELS

There are many different numerical approaches that can be used to solve second order undamped systems. In deciding on a particular approach, one can consider several key criteria, including short-term accuracy, long-term stability, energy conservation, symplecticity and time reversibility [55, 56]. For example, one may write the problem in state-space form and then integrate the set of first order evolution equations with an adaptive Runge-Kutta method or one of a class of predictor-corrector algorithms [57]. Generally, these methods may provide high quality short-term solutions, but suffer from long-term energy drift and the lack of symplecticity and time reversibility. Alternatively, the undamped second order differential equation can be discretized directly with the Newmark-beta method [58] or the related Hilber-Hughes-Taylor algorithm [59]. Under certain parameter settings, these can be shown to be unconditionally stable and energy conserving. Another approach, for the undamped case, invokes ideas relating to Liouville's theorem to create a family of symplectic numerical methods, including the second order Verlet algorithms, which tend to provide modest short-term accuracy, but maintain very good long-term performance [60-62].

Methods based on Hamilton's principle also are possible using the discrete calculus of variations [63-65]. An analogous approach could be adopted here for the principle of mixed convolved action. However, this new principle affords the opportunity to develop instead an energy-preserving temporal finite element method. An initial version of this approach using low order temporal shape functions is examined below for the single-degree-of-freedom dynamical system with a Kelvin-Voigt element.

Recall from Section II that the weak form of Eq. (5) includes at most first derivatives of the primary variables $u(t)$ and $J(t)$, as well as the variations $\delta u(t)$ and $\delta J(t)$. Consequently, only C^0 continuity is required for the discretized representations. As an initial approach to discretize Eq. (5), consider a single step numerical algorithm from time 0 to time Δt and introduce linear temporal shape functions, as follows

$$u(t) = N_0(t)u_0 + N_1(t)u_1, \tag{20a}$$

$$J(t) = N_0(t)J_0 + N_1(t)J_1, \tag{20b}$$

for $0 \le t \le \Delta t$ and similarly for $\delta u(t)$ and $\delta J(t)$. Then, with linear variations through the time step, the usual shape functions can be invoked, such that

$$N_0(t) = 1 - \frac{t}{\Delta t}, \tag{21a}$$

$$N_1(t) = \frac{t}{\Delta t}. \tag{21b}$$

Required also for the weak form are the first derivatives

$$\dot{N}_0(t) = -\frac{1}{\Delta t}, \tag{22a}$$

$$\dot{N}_1(t) = +\frac{1}{\Delta t}, \tag{22b}$$

and semi-derivatives [53]

$$\begin{aligned} \breve{N}_0(t) &= \left(\mathscr{D}_{0^+}^{1/2} t^0\right)(t) - \frac{1}{\Delta t}\left(\mathscr{D}_{0^+}^{1/2} t\right)(t) \\ &= +\frac{\Gamma(1)}{\Gamma(1/2)} t^{-1/2} - \frac{\Gamma(2)}{\Delta t\, \Gamma(3/2)} t^{1/2} \\ &= +\frac{1}{\sqrt{\pi}} t^{-1/2} - \frac{2}{\Delta t\, \sqrt{\pi}} t^{1/2}, \end{aligned} \tag{23a}$$

$$\begin{aligned} \breve{N}_1(t) &= \frac{1}{\Delta t}\left(\mathscr{D}_{0^+}^{1/2} t\right)(t) \\ &= \frac{\Gamma(2)}{\Delta t\, \Gamma(3/2)} t^{1/2} = +\frac{2}{\Delta t \sqrt{\pi}} t^{1/2}. \end{aligned} \tag{23b}$$

Next, consider each term in order of appearance in Eq. (5). For the first term

$$\begin{aligned}\delta\dot{u} * m\ \dot{u} &= \begin{bmatrix}\delta u_0 & \delta u_1\end{bmatrix}\begin{bmatrix}\dot{N}_0 \\ \dot{N}_1\end{bmatrix} * m\begin{bmatrix}\dot{N}_0 & \dot{N}_1\end{bmatrix}\begin{bmatrix}u_0 \\ u_1\end{bmatrix} \\ &= \begin{bmatrix}\delta u_0 & \delta u_1\end{bmatrix}\begin{bmatrix}\dot{N}_0 * m\dot{N}_0 & \dot{N}_0 * m\dot{N}_1 \\ \dot{N}_1 * m\dot{N}_0 & \dot{N}_1 * m\dot{N}_1\end{bmatrix}\begin{bmatrix}u_0 \\ u_1\end{bmatrix} \\ &= \begin{bmatrix}\delta u_0 & \delta u_1\end{bmatrix}\frac{m}{\Delta t}\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix}\begin{bmatrix}u_0 \\ u_1\end{bmatrix}.\end{aligned} \tag{24}$$

Similarly,

$$\delta\dot{J} * a\ \dot{J} = \begin{bmatrix}\delta J_0 & \delta J_1\end{bmatrix}\frac{a}{\Delta t}\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix}\begin{bmatrix}J_0 \\ J_1\end{bmatrix}. \tag{25}$$

As a typical term involving the convolution of semi-derivatives, consider

$$\delta\breve{J} * \breve{u} = \begin{bmatrix}\delta J_0 & \delta J_1\end{bmatrix}\begin{bmatrix}\breve{N}_0 * \breve{N}_0 & \breve{N}_0 * \breve{N}_1 \\ \breve{N}_1 * \breve{N}_0 & \breve{N}_1 * \breve{N}_1\end{bmatrix}\begin{bmatrix}u_0 \\ u_1\end{bmatrix}. \tag{26}$$

Each component in the 2×2 matrix can be evaluated directly using Eqs. (23a) and (23b), along with the relation Eq. (18). Thus,

$$\begin{aligned}\left(\breve{N}_0 * \breve{N}_0\right)(\Delta t) &= \left(\left(\mathscr{D}_{0^+}^{1/2} t^0\right) - \frac{1}{\Delta t}\left(\mathscr{D}_{0^+}^{1/2} t\right)\right) * \left(\left(\mathscr{D}_{0^+}^{1/2} t^0\right) - \frac{1}{\Delta t}\left(\mathscr{D}_{0^+}^{1/2} t\right)\right)(\Delta t) \\ &= \frac{\Gamma^2(1)}{\Gamma(1)} - \frac{2}{\Delta t}\frac{\Gamma(1)\Gamma(2)}{\Gamma(2)}\Delta t + \frac{1}{\Delta t^2}\frac{\Gamma^2(2)}{\Gamma(3)}\Delta t^2 \\ &= 1 - 2 + \tfrac{1}{2} = -\tfrac{1}{2},\end{aligned} \tag{27a}$$

$$\begin{aligned}\left(\breve{N}_0 * \breve{N}_1\right)(\Delta t) &= \left(\left(\mathscr{D}_{0^+}^{1/2} t^0\right) - \frac{1}{\Delta t}\left(\mathscr{D}_{0^+}^{1/2} t\right)\right) * \left(\frac{1}{\Delta t}\left(\mathscr{D}_{0^+}^{1/2} t\right)\right)(\Delta t) \\ &= \frac{1}{\Delta t}\frac{\Gamma(1)\Gamma(2)}{\Gamma(2)}\Delta t - \frac{1}{\Delta t^2}\frac{\Gamma^2(2)}{\Gamma(3)}\Delta t^2 \\ &= 1 - \tfrac{1}{2} = \tfrac{1}{2},\end{aligned} \tag{27b}$$

$$\left(\breve{N}_1 * \breve{N}_0\right)(\Delta t) = \left(\breve{N}_0 * \breve{N}_1\right)(\Delta t) = \tfrac{1}{2}, \tag{27c}$$

$$\begin{aligned}\left(\breve{N}_1 * \breve{N}_1\right)(\Delta t) &= \left(\frac{1}{\Delta t}\left(\mathscr{D}_{0^+}^{1/2} t\right)\right) * \left(\frac{1}{\Delta t}\left(\mathscr{D}_{0^+}^{1/2} t\right)\right)(\Delta t) \\ &= \frac{1}{\Delta t^2}\frac{\Gamma^2(2)}{\Gamma(3)}\Delta t^2 = \tfrac{1}{2}.\end{aligned} \tag{27d}$$

Consequently, substituting Eqs. (27a-d) into Eq. (26), one finds

$$\delta\breve{J} * \breve{u} = \begin{bmatrix}\delta J_0 & \delta J_1\end{bmatrix}\begin{bmatrix}-\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}u_0 \\ u_1\end{bmatrix}. \tag{28a}$$

In a similar way,

$$\delta\breve{u} * \breve{J} = \begin{bmatrix}\delta u_0 & \delta u_1\end{bmatrix}\begin{bmatrix}-\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}J_0 \\ J_1\end{bmatrix}, \tag{28b}$$

and for the viscous dissipation term

$$\delta\breve{u} * c\,\breve{u} = \begin{bmatrix}\delta u_0 & \delta u_1\end{bmatrix}\begin{bmatrix}-\frac{c}{2} & \frac{c}{2} \\ \frac{c}{2} & \frac{c}{2}\end{bmatrix}\begin{bmatrix}u_0 \\ u_1\end{bmatrix}. \tag{29}$$

Before moving on to the applied loading contributions, let us consider the fractional derivative orders employed for the action definition in Eqs. (1) and (5), which ultimately lead to the final matrix expressions in Eqs. (28a), (28b) and (29). In Ref. [1], there was no compelling reason to write these action contributions strictly in terms of semi-derivatives. Any other fractional derivatives could have been selected, as long as the two orders, say α_1 and α_2, involved in each term are both positive and complementary; that is, the two orders in any term must sum to unity [1]. Now, based upon the property in Eq. (18), one can understand that the specific orders do not matter. Any complementary pair α_1 and $\alpha_2 = 1 - \alpha_1$ will yield exactly the same matrices defined in Eqs. (28a), (28b) and (29).

Finally, turning to the loading term in Eq. (5), let

$$\bar{f}(t) = N_0(t)\bar{f}_0 + N_1(t)\bar{f}_1. \tag{30}$$

Then,

$$\begin{aligned} \delta u * \bar{f} &= \begin{bmatrix} \delta u_0 & \delta u_1 \end{bmatrix} \begin{bmatrix} N_0 * N_0 & N_0 * N_1 \\ N_1 * N_0 & N_1 * N_1 \end{bmatrix} \begin{bmatrix} \bar{f}_0 \\ \bar{f}_1 \end{bmatrix} \\ &= \begin{bmatrix} \delta u_0 & \delta u_1 \end{bmatrix} \begin{bmatrix} \frac{\Delta t}{6} & \frac{\Delta t}{3} \\ \frac{\Delta t}{3} & \frac{\Delta t}{6} \end{bmatrix} \begin{bmatrix} \bar{f}_0 \\ \bar{f}_1 \end{bmatrix}. \end{aligned} \tag{31}$$

Substituting Eqs. (24), (25), (28a), (28b), (29) and (31) into Eq. (5) yields the following discretized weak form

$$\begin{aligned} &\begin{bmatrix} \delta u_0 & \delta u_1 \end{bmatrix} \begin{bmatrix} \frac{m}{\Delta t} & -\frac{m}{\Delta t} \\ -\frac{m}{\Delta t} & \frac{m}{\Delta t} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} - \begin{bmatrix} \delta J_0 & \delta J_1 \end{bmatrix} \begin{bmatrix} \frac{a}{\Delta t} & -\frac{a}{\Delta t} \\ -\frac{a}{\Delta t} & \frac{a}{\Delta t} \end{bmatrix} \begin{bmatrix} J_0 \\ J_1 \end{bmatrix} \\ &+ \begin{bmatrix} \delta J_0 & \delta J_1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \begin{bmatrix} \delta u_0 & \delta u_1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} J_0 \\ J_1 \end{bmatrix} \\ &+ \begin{bmatrix} \delta u_0 & \delta u_1 \end{bmatrix} \begin{bmatrix} -\frac{c}{2} & \frac{c}{2} \\ \frac{c}{2} & \frac{c}{2} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} - \begin{bmatrix} \delta u_0 & \delta u_1 \end{bmatrix} \begin{bmatrix} \frac{\Delta t}{6} & \frac{\Delta t}{3} \\ \frac{\Delta t}{3} & \frac{\Delta t}{6} \end{bmatrix} \begin{bmatrix} \bar{f}_0 \\ \bar{f}_1 \end{bmatrix} \\ &- \begin{bmatrix} \delta u_1 \end{bmatrix} \begin{bmatrix} \bar{j}_0 \end{bmatrix} = 0. \end{aligned} \tag{32}$$

With u_0 and J_0 known from the initial conditions at $t = 0$, the variations δu_0 and δJ_0 are taken as zero. Therefore, the weak form reduces to the following

$$\begin{aligned} &\begin{bmatrix} \delta u_1 \end{bmatrix} \begin{bmatrix} -\frac{m}{\Delta t} & \frac{m}{\Delta t} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} - \begin{bmatrix} \delta J_1 \end{bmatrix} \begin{bmatrix} -\frac{a}{\Delta t} & \frac{a}{\Delta t} \end{bmatrix} \begin{bmatrix} J_0 \\ J_1 \end{bmatrix} + \begin{bmatrix} \delta J_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \\ &+ \begin{bmatrix} \delta u_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} J_0 \\ J_1 \end{bmatrix} + \begin{bmatrix} \delta u_1 \end{bmatrix} \begin{bmatrix} \frac{c}{2} & \frac{c}{2} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} - \begin{bmatrix} \delta u_1 \end{bmatrix} \begin{bmatrix} \frac{\Delta t}{3} & \frac{\Delta t}{6} \end{bmatrix} \begin{bmatrix} \bar{f}_0 \\ \bar{f}_1 \end{bmatrix} - \begin{bmatrix} \delta u_1 \end{bmatrix} \begin{bmatrix} \bar{j}_0 \end{bmatrix} = 0. \end{aligned} \tag{33}$$

Then, grouping the terms according to the variations, one obtains

$$
\begin{aligned}
&[\delta u_1]\left(\left[-\tfrac{m}{\Delta t} \quad \tfrac{m}{\Delta t}\right]\begin{bmatrix}u_0\\ u_1\end{bmatrix}+\left[\tfrac{1}{2} \quad \tfrac{1}{2}\right]\begin{bmatrix}J_0\\ J_1\end{bmatrix}+\left[\tfrac{c}{2} \quad \tfrac{c}{2}\right]\begin{bmatrix}u_0\\ u_1\end{bmatrix}-\left[\tfrac{\Delta t}{3} \quad \tfrac{\Delta t}{6}\right]\begin{bmatrix}\bar{f}_0\\ \bar{f}_1\end{bmatrix}-\left[\bar{j}_0\right]\right)\\
&+[\delta J_1]\left(\left[\tfrac{a}{\Delta t} \quad -\tfrac{a}{\Delta t}\right]\begin{bmatrix}J_0\\ J_1\end{bmatrix}+\left[\tfrac{1}{2} \quad \tfrac{1}{2}\right]\begin{bmatrix}u_0\\ u_1\end{bmatrix}\right)=0.
\end{aligned}
\tag{34}
$$

Consequently, for arbitrary variations δu_1 and δJ_1, Eq. (34) provides the following pair of equations for u_1 and J_1

$$
\begin{bmatrix}\frac{m}{\Delta t}+\frac{c}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{a}{\Delta t}\end{bmatrix}\begin{bmatrix}u_1\\ J_1\end{bmatrix}=\begin{bmatrix}\frac{m}{\Delta t}-\frac{c}{2} & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{a}{\Delta t}\end{bmatrix}\begin{bmatrix}u_0\\ J_0\end{bmatrix}+\begin{bmatrix}\bar{J}_1\\ 0\end{bmatrix},
\tag{35}
$$

where

$$
\bar{J}_1=\tfrac{\Delta t}{3}\bar{f}_0+\tfrac{\Delta t}{6}\bar{f}_1+\bar{j}_0.
\tag{36}
$$

More generally, for the n^{th} time step with $t_n=n\,\Delta t$, one may write

$$
\begin{bmatrix}\frac{m}{\Delta t}+\frac{c}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{a}{\Delta t}\end{bmatrix}\begin{bmatrix}u_n\\ J_n\end{bmatrix}=\begin{bmatrix}\frac{m}{\Delta t}-\frac{c}{2} & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{a}{\Delta t}\end{bmatrix}\begin{bmatrix}u_{n-1}\\ J_{n-1}\end{bmatrix}+\begin{bmatrix}\bar{J}_n\\ 0\end{bmatrix},
\tag{37}
$$

where

$$
\bar{J}_n=\tfrac{\Delta t}{3}\bar{f}_{n-1}+\tfrac{\Delta t}{6}\bar{f}_n+\bar{j}_{n-1}.
\tag{38}
$$

For the undamped case with no external forcing, Eq. (37) reduces to

$$
\begin{bmatrix}\frac{m}{\Delta t} & \frac{1}{2}\\ \frac{1}{2} & -\frac{a}{\Delta t}\end{bmatrix}\begin{bmatrix}u_n\\ J_n\end{bmatrix}=\begin{bmatrix}\frac{m}{\Delta t} & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{a}{\Delta t}\end{bmatrix}\begin{bmatrix}u_{n-1}\\ J_{n-1}\end{bmatrix}.
\tag{39}
$$

Notice that Eq. (39) is time reversible. One can recover exactly the state $n-1$ from state n by setting $\Delta t \to -\Delta t$.

Equation (39) can be rewritten in matrix form as

$$
\mathbf{A_1 x_n = A_0 x_{n-1}},
\tag{40}
$$

or simply

$$\mathbf{x_n} = \mathbf{A}\mathbf{x_{n-1}}, \tag{41}$$

where

$$\mathbf{A} = \mathbf{A_1}^{-1}\mathbf{A_0}. \tag{42}$$

The stability and dissipative character of the computational method can be determined by considering the eigenvalues of $\mathbf{A}$. For the undamped case, the two eigenvalues of $\mathbf{A}$ become complex conjugate pairs with

$$\lambda_{1,2} = \frac{4\mathrm{T}^2 - 1 \pm \mathrm{i}\,4\mathrm{T}}{4\mathrm{T}^2 + 1}, \tag{43}$$

where $\mathrm{T} = (ma)^{1/2}/\Delta t$ represents a non-dimensional time. Notice that

$$|\lambda_1| = |\lambda_2| = 1. \tag{44}$$

Consequently, in addition to being time reversible, the present temporal finite element mixed convolved action algorithm is also symplectic, energy conserving and unconditionally stable for the undamped case.

V. WEAK FORM FOR CLASSICAL MAXWELL MODEL

In the previous section, a numerical formulation has been provided for the principle of mixed convolved action applied to a dissipative single degree-of-freedom dynamical system represented in terms of a Kelvin-Voigt element. This is just one particular application. A broad range of classical models can be formulated in terms of mixed convolved action. As an additional example, in this section, the dynamical system displayed in Figure 2, involving a classical Maxwell element is considered.

The mixed convolved action associated with this problem can be written:

$$\begin{aligned} I_{C_M}(u,\breve{u},\dot{u},J,\breve{J},\dot{J};t) = {} & \tfrac{1}{2}(\dot{u} * m\,\dot{u})(t) - \tfrac{1}{2}(\dot{J} * a\,\dot{J})(t) + (\breve{J} * \breve{u})(t) - \tfrac{1}{2}(\breve{J} * d_M\,\breve{J})(t) \\ & - (u * \bar{f})(t) - u(t)\bar{j}(0), \end{aligned} \tag{45}$$

where $d_M = a/\tau_M$ with τ_M as the relaxation time.

For stationary mixed convolved action with the Maxwell model, the first variation of Eq. (45) must be zero, which leads to the following weak form in time suitable for numerical implementation

$$\begin{aligned}\delta I_{C_M} = &\left(\delta\dot{u} * m\,\dot{u}\right)(t) - \left(\delta\dot{J} * a\,\dot{J}\right)(t) + \left(\delta\breve{J} * \breve{u}\right)(t) + \left(\delta\breve{u} * \breve{J}\right)(t) \\ &- \left(\delta\breve{J} * d_M\,\breve{J}\right)(t) - \left(\delta u * \bar{f}\right)(t) - \delta u(t)\bar{j}(0) = 0.\end{aligned} \tag{46}$$

In order to recover the Euler-Lagrange equations associated with the strong form, one can perform classical and fractional integration by parts on the appropriate terms. The resulting statement becomes

$$\begin{aligned}&\left(\delta u * \left\{m\,\ddot{u} + \dot{J} - \bar{f}\right\}\right)(t) + \left(\delta J * \left\{-a\,\ddot{J} - d_M\,\dot{J} + \dot{u}\right\}\right)(t) \\ &+ \delta u(t)\left\{m\,\dot{u}(0) + J(0) - \bar{j}(0)\right\} + \delta u(0)\left\{m\,\dot{u}(t)\right\} \\ &+ \delta J(t)\left\{-a\,\dot{J}(0) - d_M\,J(0) + u(0)\right\} - \delta J(0)\left\{-a\,\dot{J}(t)\right\} = 0.\end{aligned} \tag{47}$$

For arbitrary variations, this produces the appropriate Euler-Lagrange equations, representing the governing balance laws for the mixed formulation with a Maxwell element

$$m\,\ddot{u} + \dot{J} = \bar{f}, \tag{48a}$$

$$-a\,\ddot{J} - d_M\,\dot{J} + \dot{u} = 0, \tag{48b}$$

along with the initial conditions

$$m\,\dot{u}(0) + J(0) = \bar{j}(0), \tag{49a}$$

$$-a\,\dot{J}(0) - d_M\,J(0) + u(0) = 0. \tag{49b}$$

In addition, as with the Kelvin-Voigt model, the variations at the initial instant must be taken as zero. Thus,

$$\delta u(0) = 0\,, \qquad \delta J(0) = 0. \tag{50a,b}$$

After discretization of Eq. (46) using the linear temporal shape functions Eq. (20a) and (20b), one finds for the n^{th} time step with $t_n = n\,\Delta t$

$$\begin{bmatrix} \frac{m}{\Delta t} & \frac{1}{2} \\ \frac{1}{2} & -\frac{a}{\Delta t} - \frac{d_M}{2} \end{bmatrix} \begin{bmatrix} u_n \\ J_n \end{bmatrix} = \begin{bmatrix} \frac{m}{\Delta t} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{a}{\Delta t} + \frac{d_M}{2} \end{bmatrix} \begin{bmatrix} u_{n-1} \\ J_{n-1} \end{bmatrix} + \begin{bmatrix} \overline{J}_n \\ 0 \end{bmatrix}, \tag{51}$$

where again

$$\overline{J}_n = \tfrac{\Delta t}{3}\,\overline{f}_{n-1} + \tfrac{\Delta t}{6}\,\overline{f}_n + \overline{j}_{n-1}. \tag{52}$$

VI. CAPUTO FRACTIONAL CALCULUS

The following alternative expression for the left Riemann-Liouville fractional derivative may be developed by using the commutativity property of the convolution operator in Eq. (10), along with the Leibniz rule [54]

$$\left(\mathscr{D}^{\alpha}_{0^+} u\right)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{u(0)}{t^{\alpha}} + \left(\mathscr{I}^{1-\alpha}_{0^+} \dot{u}\right)(t). \tag{53}$$

From this it is clear that the left Riemann-Liouville fractional derivative can easily become unbounded as $t \to 0$. In particular, from Eqs. (13) and (53), one finds that the α-order fractional derivative of the constant function produces a $t^{-\alpha}$ singularity. Because these are weak (*i.e.*, integrable) singularities, this causes no difficulty within mixed convolved action principles.

On the other hand, the application of Riemann-Liouville fractional derivatives in constitutive modeling is problematic due to this singular character. Instead, Caputo fractional derivatives [30] may be more appropriate, where

$$\left({}^{C}\mathscr{D}^{\alpha}_{0^+} u\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{u}(\xi)}{(t-\xi)^{\alpha}} d\xi, \tag{54}$$

or from Eq. (12), one may write

$$\left({}^{C}\mathscr{D}^{\alpha}_{0^+} u\right)(t) = \left(\mathscr{I}^{1-\alpha}_{0^+} \dot{u}\right)(t). \tag{55}$$

Notice, then, in comparing Eq. (55) with Eq. (53), one finds that by definition the Caputo fractional derivative simply excludes the term containing the $t \to 0$ singularity. In fact, Caputo introduced this idea within the context of constitutive modeling of dissipative media [30].

However, this difference has consequences for a number of fractional calculus operations, which are developed in the following.

Consider, first, the composition of left Caputo fractional derivatives and Riemann-Liouville integrals of order α operating on any summable function $u(\tau)$, with $0<\alpha<1$. The results can be written as

$$\left({}^{C}\mathscr{D}_{0^+}^{\alpha}\left(\mathscr{I}_{0^+}^{\alpha}u\right)\right)(\tau)=u(\tau)-\frac{1}{\Gamma(1-\alpha)}\frac{\left(\mathscr{I}_{0^+}^{\alpha}u\right)(0)}{\tau^{\alpha}}, \tag{56a}$$

while by reversing the order

$$\left(\mathscr{I}_{0^+}^{\alpha}\left({}^{C}\mathscr{D}_{0^+}^{\alpha}u\right)\right)(\tau)=u(\tau). \tag{56b}$$

In a similar manner, one finds for the composition of right fractional operators

$$\left({}^{C}\mathscr{D}_{t^-}^{\alpha}\left(\mathscr{I}_{t^-}^{\alpha}u\right)\right)(\tau)=u(\tau)-\frac{1}{\Gamma(1-\alpha)}\frac{\left(\mathscr{I}_{t^-}^{\alpha}u\right)(t)}{(t-\tau)^{\alpha}}, \tag{57a}$$

and

$$\left(\mathscr{I}_{t^-}^{\alpha}\left({}^{C}\mathscr{D}_{t^-}^{\alpha}u\right)\right)(\tau)=u(\tau). \tag{57b}$$

Equations (56) and (57) for Caputo fractional derivatives should be compared with Eqs. (31) and (32) in Ref. [1] for Riemann-Liouville derivatives. Cleary, the omission of the singular term in the Caputo definition affects significantly the composition of fractional operators.

Furthermore, integration-by-parts relations are also affected. Based upon an extension of the Love and Young result for fractional integration by parts [66], one may write for Riemann-Liouville fractional derivatives [1]

$$\int_0^t\varphi(\tau)\left(\mathscr{D}_{0^+}^{\alpha}\psi\right)(\tau)d\tau=\int_0^t\left(\mathscr{D}_{t^-}^{\alpha}\varphi\right)(\tau)\psi(\tau)d\tau+\left(\mathscr{I}_{t^-}^{1-\alpha}\varphi\right)(t)\psi(t)-\varphi(0)\left(\mathscr{I}_{0^+}^{1-\alpha}\psi\right)(0), \tag{58}$$

for $0<\alpha<1$, while for Caputo fractional derivatives this simplifies to the following

$$\int_0^t\varphi(\tau)\left({}^{C}\mathscr{D}_{0^+}^{\alpha}\psi\right)(\tau)d\tau=\int_0^t\left({}^{C}\mathscr{D}_{t^-}^{\alpha}\varphi\right)(\tau)\psi(\tau)d\tau. \tag{59}$$

Notice that, in both forms, the inner product of one function with the left fractional derivative of another converts to the inner product of the latter function with a right fractional derivative of the

former function. When operating with Riemann-Liouville fractional derivatives as in Eq. (58), boundary terms involving both left and right fractional integrals are released during this process; whereas these are absent in Eq. (59) when operating with Caputo derivatives. This difference is again due to the removal of the singular contributions in the Caputo definition.

Finally, consider integration-by-parts for convolutions with fractional derivatives. The relations associated with left and right Riemann-Liouville fractional derivatives can be written, respectively, as [1]

$$\int_0^t \varphi(\tau)\left(\mathscr{D}_{0^+}^{\alpha}\psi\right)(t-\tau)d\tau = \int_0^t \left(\mathscr{D}_{0^+}^{\alpha}\varphi\right)(\tau)\psi(t-\tau)d\tau + \left(\mathscr{I}_{0^+}^{1-\alpha}\varphi\right)(0)\psi(t) - \varphi(t)\left(\mathscr{I}_{0^+}^{1-\alpha}\psi\right)(0), \tag{60a}$$

$$\int_0^t \varphi(\tau)\left(\mathscr{D}_{t^-}^{\alpha}\psi\right)(t-\tau)d\tau = \int_0^t \left(\mathscr{D}_{t^-}^{\alpha}\varphi\right)(\tau)\psi(t-\tau)d\tau + \left(\mathscr{I}_{t^-}^{1-\alpha}\varphi\right)(t)\psi(0) - \varphi(0)\left(\mathscr{I}_{t^-}^{1-\alpha}\psi\right)(t), \tag{60b}$$

whereas for Caputo fractional derivative operators, one finds the simple relations

$$\int_0^t \varphi(\tau)\left({}^{C}\mathscr{D}_{0^+}^{\alpha}\psi\right)(t-\tau)d\tau = \int_0^t \left({}^{C}\mathscr{D}_{0^+}^{\alpha}\varphi\right)(\tau)\psi(t-\tau)d\tau, \tag{61a}$$

$$\int_0^t \varphi(\tau)\left({}^{C}\mathscr{D}_{t^-}^{\alpha}\psi\right)(t-\tau)d\tau = \int_0^t \left({}^{C}\mathscr{D}_{t^-}^{\alpha}\varphi\right)(\tau)\psi(t-\tau)d\tau, \tag{61b}$$

without the released end point terms.

VII. WEAK FORM FOR FRACTIONAL-DERIVATIVE MAXWELL MODEL

Many possibilities exist for constructing fractional derivative constitutive models. For a first example, the previous mixed convolved action formulation for the classical Maxwell element is generalized in this section to incorporate a fractional damper of order β with $0<\beta<1$, as illustrated in Figure 3. The governing differential equations for balance of momentum and compatibility of velocities, respectively, can be written

$$m\,\ddot{u} + \dot{J} = \bar{f}, \tag{62a}$$

$$-a\,\ddot{J} - \frac{a}{\tau_F^{\,1-\beta}}\left({}^{C}\mathscr{D}_{0^+}^{\beta}\dot{J}\right) + \dot{u} = 0, \tag{62b}$$

along with the initial conditions

$$m\,\dot{u}(0)+J(0)=\overline{j}(0), \tag{63a}$$

$$-a\,\dot{J}(0)-\overline{u}_F(0)+u(0)=0. \tag{63b}$$

Here, τ_F represents the relaxation time and $\overline{u}_F(0)$ is the displacement across the fractional damper at time zero, which is not subject to variation. Notice that the Caputo fractional derivative is used in Eq. (62b) to assure bounded velocities at the initial instant.

The objective now is to define a mixed convolved action that returns Eqs. (62) and (63) as its Euler-Lagrange equations. Let us consider the following

$$\begin{aligned} I_{C_F}(u,\breve{u},\dot{u},J,\breve{J},{}^{C}\mathscr{D}_{0^+}^{(1+\beta)/2}J,\dot{J};t) &= \tfrac{1}{2}\left(\dot{u}*m\,\dot{u}\right)(t)-\tfrac{1}{2}\left(\dot{J}*a\,\dot{J}\right)(t)+\left(\breve{J}*\breve{u}\right)(t) \\ &\quad -\tfrac{1}{2}\left(\left({}^{C}\mathscr{D}_{0^+}^{(1+\beta)/2}J\right)*\frac{a}{\tau_F{}^{1-\beta}}\left({}^{C}\mathscr{D}_{0^+}^{(1+\beta)/2}J\right)\right)(t)-J(t)\overline{u}_F(0) \\ &\quad -\left(u*\overline{f}\right)(t)-u(t)\overline{j}(0). \end{aligned} \tag{64}$$

Then, setting the first variation to zero produces the weak form

$$\begin{aligned} \delta I_{C_F} &= \left(\delta\dot{u}*m\,\dot{u}\right)(t)-\left(\delta\dot{J}*a\,\dot{J}\right)(t)+\left(\delta\breve{J}*\breve{u}\right)(t)+\left(\delta\breve{u}*\breve{J}\right)(t) \\ &\quad -\left(\left({}^{C}\mathscr{D}_{0^+}^{(1+\beta)/2}\delta J\right)*\frac{a}{\tau_F{}^{1-\beta}}\left({}^{C}\mathscr{D}_{0^+}^{(1+\beta)/2}J\right)\right)(t)-\delta J(t)\overline{u}_F(0) \\ &\quad -\left(\delta u*\overline{f}\right)(t)-\delta u(t)\overline{j}(0)=0, \end{aligned} \tag{65}$$

which is suitable for numerical solution. By performing classical, Riemann-Liouville fractional and Caputo fractional integration by parts on the appropriate terms, the variational statement becomes

$$\begin{aligned} &\left(\delta u*\left\{m\,\ddot{u}+\dot{J}-\overline{f}\right\}\right)(t)+\left(\delta J*\left\{-a\,\ddot{J}-\frac{a}{\tau_F{}^{1-\beta}}\left({}^{C}\mathscr{D}_{0^+}^{\beta}\dot{J}\right)+\dot{u}\right\}\right)(t) \\ &\quad +\delta u(t)\left\{m\,\dot{u}(0)+J(0)-\overline{j}(0)\right\}+\delta u(0)\left\{m\,\dot{u}(t)\right\} \\ &\quad +\delta J(t)\left\{-a\,\dot{J}(0)-\overline{u}_F(0)+u(0)\right\}-\delta J(0)\left\{-a\,\dot{J}(t)\right\}=0. \end{aligned} \tag{66}$$

With arbitrary variations δu and δJ throughout the time interval, Eq. (66) leads immediately to the governing differential equations Eqs. (62a) and (62b) and initial conditions Eqs. (63a) and

(63b), along with the requirement for zero variations $\delta u(0)=0$ and $\delta J(0)=0$ at the initial instant. Consequently, the functional I_{C_F} defined in Eq. (64) does afford a principle of stationary mixed convolved action for the fractional Maxwell dissipative system of Figure 3.

For the numerical implementation, it is perhaps most convenient to begin with the weak form given in Eq. (65). The discrete matrix formulations already have been defined for all of the individual terms, except for those on the second line of Eq. (65) associated with the fractional damper.

Let us begin by replacing the Caputo fractional derivatives with fractional integrals, where

$$\left(\left({}^{C}\mathscr{D}_{0^+}^{(1+\beta)/2}\delta J\right)*\frac{a}{\tau_F{}^{1-\beta}}\left({}^{C}\mathscr{D}_{0^+}^{(1+\beta)/2}J\right)\right)(t)=\left(\left(\mathscr{I}_{0^+}^{(1-\beta)/2}\delta\dot{J}\right)*\frac{a}{\tau_F{}^{1-\beta}}\left(\mathscr{I}_{0^+}^{(1-\beta)/2}\dot{J}\right)\right)(t). \tag{67}$$

Then, assuming linear temporal shape functions for J and δJ, the quantities $\dot{J}$ and $\delta\dot{J}$ are constant within each time step and the only necessary convolutions required in Eq. (67) reduce to the following

$$\begin{aligned}\left(\left(\mathscr{I}_{0^+}^{(1-\beta)/2}1\right)*\left(\mathscr{I}_{0^+}^{(1-\beta)/2}1\right)\right)(t)&=\frac{1}{\Gamma^2((3-\beta)/2)}\int_0^t\tau^{(1-\beta)/2}(t-\tau)^{(1-\beta)/2}d\tau\\&=\frac{1}{\Gamma^2((3-\beta)/2)}B((3-\beta)/2,(3-\beta)/2)t^{2-\beta}\\&=\frac{t^{2-\beta}}{\Gamma(3-\beta)}.\end{aligned} \tag{68}$$

Then, for a time step of duration Δt

$$\left(\left({}^{C}\mathscr{D}_{0^+}^{(1+\beta)/2}\delta J\right)*\frac{a}{\tau_F{}^{1-\beta}}\left({}^{C}\mathscr{D}_{0^+}^{(1+\beta)/2}J\right)\right)(\Delta t)=\begin{bmatrix}\delta J_0 & \delta J_1\end{bmatrix}\begin{bmatrix}\frac{d_F}{2} & -\frac{d_F}{2}\\ -\frac{d_F}{2} & \frac{d_F}{2}\end{bmatrix}\begin{bmatrix}J_0\\ J_1\end{bmatrix}, \tag{69}$$

where

$$d_F=\frac{2}{\Gamma(3-\beta)}\frac{a}{\tau_F{}^{1-\beta}\Delta t^{\beta}}. \tag{70}$$

Next, the total displacement across the fractional damper u_F must be evaluated at the beginning of each time step for use in the initial condition contribution in Eq. (65). In general,

$$u_F(t) = \frac{a}{\tau_F^{\,1-\beta}} \left(\mathscr{I}_{0^+}^{1-\beta} \dot{J} \right)(t) + \bar{u}_F(0), \tag{71}$$

which in discretized form becomes

$$u_F(t_n) = \frac{a}{\Delta t\, \Gamma(1-\beta)} \left(\frac{t_n}{\tau_F} \right)^{1-\beta} \sum_{l=1}^{n} \left(J_l - J_{l-1} \right) \left\{ B_{l/n}\left(1, 1-\beta\right) - B_{(l-1)/n}\left(1, 1-\beta\right) \right\} + \bar{u}_F(0), \tag{72}$$

where $B_x(\cdot,\cdot)$ is the incomplete Beta function, with the special case [67]

$$B_x(1, y) = \frac{1-(1-x)^y}{y}. \tag{73}$$

Consequently, the fractional damper displacement at time t_n can be written

$$u_F(t_n) = \frac{a}{\Delta t\, (1-\beta)\, \Gamma(1-\beta)} \left(\frac{t_n}{\tau_F} \right)^{1-\beta} \sum_{l=1}^{n} \left(J_l - J_{l-1} \right) \left\{ \left(\frac{n-l+1}{n} \right)^{1-\beta} - \left(\frac{n-l}{n} \right)^{1-\beta} \right\} + \bar{u}_F(0). \tag{74}$$

Finally, merging Eqs. (69) and (74) into Eq. (51) by replacing the classical Maxwell damper terms, one obtains a discretized mixed convolved action formulation for the fractional Maxwell dissipative dynamical system in the following form

$$\begin{bmatrix} \frac{m}{\Delta t} & \frac{1}{2} \\ \frac{1}{2} & -\frac{a}{\Delta t} - \frac{d_F}{2} \end{bmatrix} \begin{bmatrix} u_n \\ J_n \end{bmatrix} = \begin{bmatrix} \frac{m}{\Delta t} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{a}{\Delta t} - \frac{d_F}{2} \end{bmatrix} \begin{bmatrix} u_{n-1} \\ J_{n-1} \end{bmatrix} + \begin{bmatrix} \bar{J}_n \\ \bar{u}_{F(n-1)} \end{bmatrix}. \tag{75}$$

Here d_F and $\bar{J}_n$ are defined in Eqs. (70) and (52), respectively, while $\bar{u}_{Fn} = u_F(t_n)$ with $u_F(t_n)$ given in Eq. (74). Note that $\bar{u}_{F0} = \bar{u}_F(0)$ represents the initial displacement across the fractional damper, which must be specified at time zero.

VIII. NUMERICAL EXAMPLES

For all of the numerical examples considered here, with no loss of generality, the model parameters are taken in non-dimensional form. In particular, let $m = 1$ and $a = 1/\left(4\pi^2\right)$, thus providing a natural circular frequency $\omega = 2\pi$ and a natural period $T = 1$.

In the first example, let us consider the free vibration of an undamped linear oscillator. Thus, $c = 0$ and $\bar{f} = 0$ in the weak formulation defined in Eq. (37). For initial conditions, let $u(0) = 1$, $J(0) = 0$ and $\bar{j}(0) = 0$, which from Eq. (8a) implies that the initial velocity $\dot{u}(0) = 0$ for the undamped system, while from Eq. (8b) the initial force in the spring $\dot{J}(0) = u(0)/a = 1/a$.

The numerical solutions for displacement $u(t)$ versus time t, based upon the mixed formulation of Eq. (37) using several different time steps Δt, are compared in Figure 4 with the analytical solution. The accuracy is quite good for sufficiently small time steps and convergence to the analytical solution is obtained. Meanwhile, Figure 5 provides the corresponding phase space plot of velocity $\dot{u}(t)$ versus displacement $u(t)$, which demonstrates the symplectic character of the mixed convolved action numerical formulation of Eq. (37). Figure 6 presents a plot of energy versus time, which confirms that the energy preserving nature of the algorithm is independent of time step.

As noted previously in Section IV, the two eigenvalues of the discrete update operator matrix $\mathbf{A}$ in Eq. (41) are complex conjugates with unit modulus. Consequently, the spectral radius of $\mathbf{A}$ is one and Eq. (37) represents an unconditionally stable, symplectic formulation for the undamped linear oscillator with no numerical dissipation [68, 69]. Interestingly, in numerical simulations, the total energy is conserved at time $t = 5000$ to at least nine significant digits for a range of time steps from $\Delta t = 0.001$ to $\Delta t = 100$. Notice that in the former case, this entails five million time steps, while in the latter simulation each time step is one hundred times the period of the oscillator. Of course, for such very large time steps, detailed accuracy in terms of displacement $u(t)$ or force impulse $J(t)$ cannot be maintained, but the solution continues to conserve overall energy.

Next, consider the damped free vibration of the same system, having mass $m = 1$ and spring with flexibility $a = 1/\left(4\pi^2\right)$ to provide an undamped natural circular frequency $\omega = 2\pi$, but now with a damper positioned in parallel to the spring. Assume that this Kelvin-Voigt system has a damper of coefficient c, sized to deliver a non-dimensional damping ratio $\zeta_K = c/c_{\text{cr}} = 0.05$

with $c_{\text{cr}} = 2m\omega$. For initial conditions, again let $u(0) = 1$, $J(0) = 0$ and $\bar{j}(0) = 0$. From Eq. (8a), this implies that the initial velocity $\dot{u}(0) = -cu(0)/m = -2\zeta_K \omega$, while the initial force in the spring $\dot{J}(0) = u(0)/a = 1/a$ from Eq. (8b). Figure 7 displays the numerical solution of displacement versus time, based upon the formulation of Eq. (37), in comparison with the well-known analytical solution. Notice once again that excellent correlation is obtained for sufficiently small time steps. As the time step is enlarged, the amplitude decay per cycle appears from Figure 7 to remain consistent with the analytical solution. The errors are primarily due to period elongation, as in the undamped case. This behavior of the numerical formulation is confirmed in the phase space plot of Figure 8, where the radius of the decaying spiral is correctly maintained, even for rather large time steps $\Delta t = 0.100$.

Interestingly, one can obtain exactly the same damped free vibration response from either a Kelvin-Voigt or Maxwell element. The only requirements are that the initial conditions in terms of displacement, velocity and applied force impulse must be the same, and consistent definitions of the non-dimensional damping ratio must be specified. As the third example, consider the same Kelvin-Voigt system from the previous case having $\zeta_K = 0.05$, but with $u(0) = 1$, $\dot{u}(0) = 0$ and $\bar{j}(0) = 0$ for the initial conditions. The numerical results are displayed in Figure 9, along with the corresponding analytical solution. In addition, the figure includes the numerical solution for the damped Maxwell mixed convolved action formulation of Eq. (51) with non-dimensional damping ratio $\zeta_M = d_M / d_{M,\text{cr}} = 0.05$, where $d_{M,\text{cr}} = 2a\omega$. Clearly, from Figure 9, the result is identical to that for the Kelvin-Voigt system with the same non-dimensional damping ratio. Lastly, the fractional Maxwell formulation of Eq. (75) can provide exactly the same free vibration response by setting $\beta = 0$, $\tau_F = 1/(2\zeta_F \omega)$ and $\bar{u}_{F0}$ with $\zeta_F = 0.05$, in which case $d_F = a/\tau_F$ in Eq. (75). For $\Delta t = 0.001$, all three models coincide to at least five significant digits.

Figure 10 provides the numerical solutions, based upon Eq. (75), for the fractional Maxwell model with three different values of β. In all three cases, the mass m, flexibility a, relaxation time τ_F and initial conditions are taken as the same values specified in the previous example.

The lines in the figure represent the numerical solutions with time step $\Delta t = 0.001$, while the discrete symbols denote the solutions using a time step $\Delta t = 0.020$. These results illustrate the good convergence characteristics of the present mixed convolved action formulation for fractional derivative Maxwell models.

The three algorithms for Kelvin-Voigt, Maxwell and fractional Maxwell damping also apply to cases of forced vibration. Figure 11 provides the displacement results for the previous Kelvin-Voigt model now subjected to an applied force in the form $\bar{f}(t) = f_0 \sin(\omega_0 t)$ with $f_0 = 100$ and $\omega_0 = 10$ in non-dimensional units. The numerical results are obtained with the algorithm of Eq. (37) with a time step $\Delta t = 0.001$. The correlation with the analytical solution is excellent throughout the time interval.

Finally, damped forced vibration of the fractional Maxwell model from above is examined under excitation by the sinusoidal load of the previous case. Results are shown in Figure 12 for three different values of β. Again, the lines represent the numerical solutions with time step $\Delta t = 0.001$, while the discrete symbols correspond to the results for a time step $\Delta t = 0.020$. The mixed convolved action weak form of Eq. (75) clearly provides converging numerical solutions for these fractional derivative Maxwell models.

IX. CONCLUSIONS

In recent work, the principle of mixed convolved action was formulated as a fully consistent variational statement for dynamical systems, including those with dissipation [1]. The formulation involves mixed variables, fractional derivatives and the temporal convolution of convolutions. Initially, in Ref. [1], the principle of mixed convolved action was developed in detail for the classical single-degree-of-freedom Kelvin-Voigt linear oscillator. Unlike previous variational approaches for this problem, with this new principle there is no need for an accompanying fictitious negatively damped system, a separate dissipation function with *ad hoc* rules for taking variations, external specification of initial conditions, or restrictions on the variations that are inconsistent with the nature of an initial value problem.

Here the principle is extended in a straightforward manner to the classical Maxwell linear oscillator, having the spring and viscous dashpot in series. In addition, a principle of mixed convolved action is developed for non-classically damped systems, incorporating fractional derivatives. As a result, such systems, which have significant advantages from a constitutive modeling perspective, can now be represented in terms of a consistent variational principle governing their dynamical response.

Furthermore, low-order discrete time formulations are developed for all three of the systems and numerical simulations are conducted to demonstrate the potential of this new approach. These numerical algorithms are shown to be symplectic and unconditionally stable for the undamped case and accurate and robust for cases involving classical and non-classical damping elements, under both free and forced vibrations.

The applicability of the mixed convolved action approach is quite broad, spanning many fields of mathematical physics and mechanics. In particular, future work will be directed toward development of higher-order time-stepping methods, fast convolution algorithms [70-72] for the fractional derivative models and the application of these concepts to multi-degree-of-freedom dynamical systems, continuum dynamical systems and coupled multi-physics problems, following the ideas in Refs. [65, 73-77, 2].

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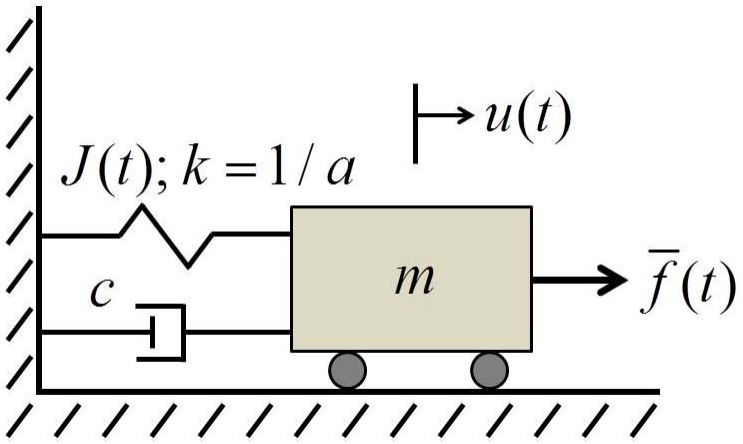

FIG. 1. Single-degree-of-freedom Kelvin-Voigt damped oscillator

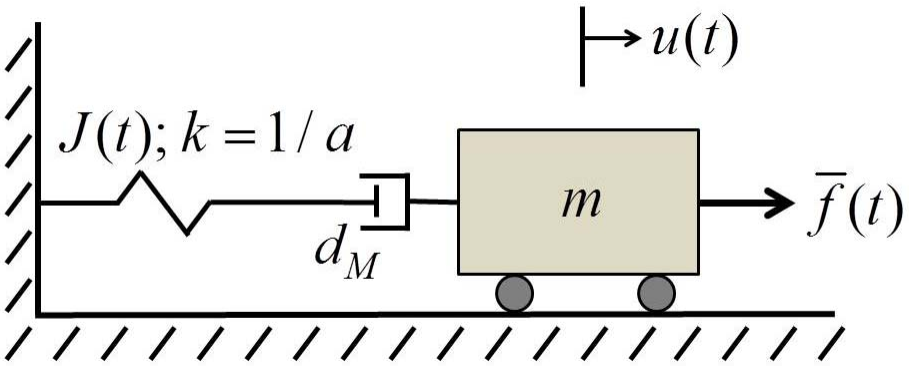

FIG. 2. Single-degree-of-freedom Maxwell damped oscillator

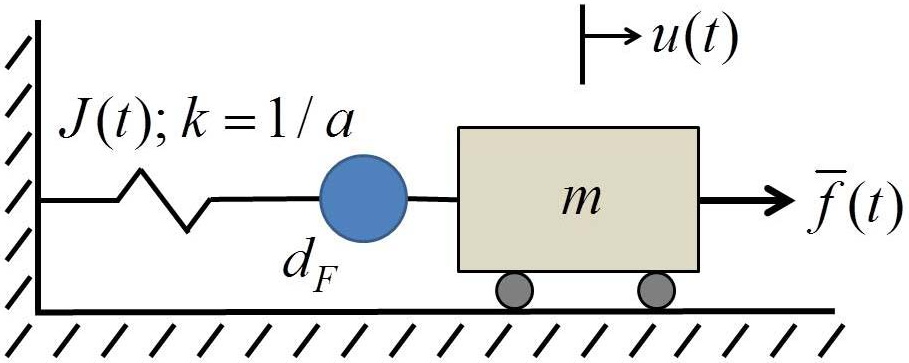

FIG. 3. (Color online) Single-degree-of-freedom fractional Maxwell damped oscillator

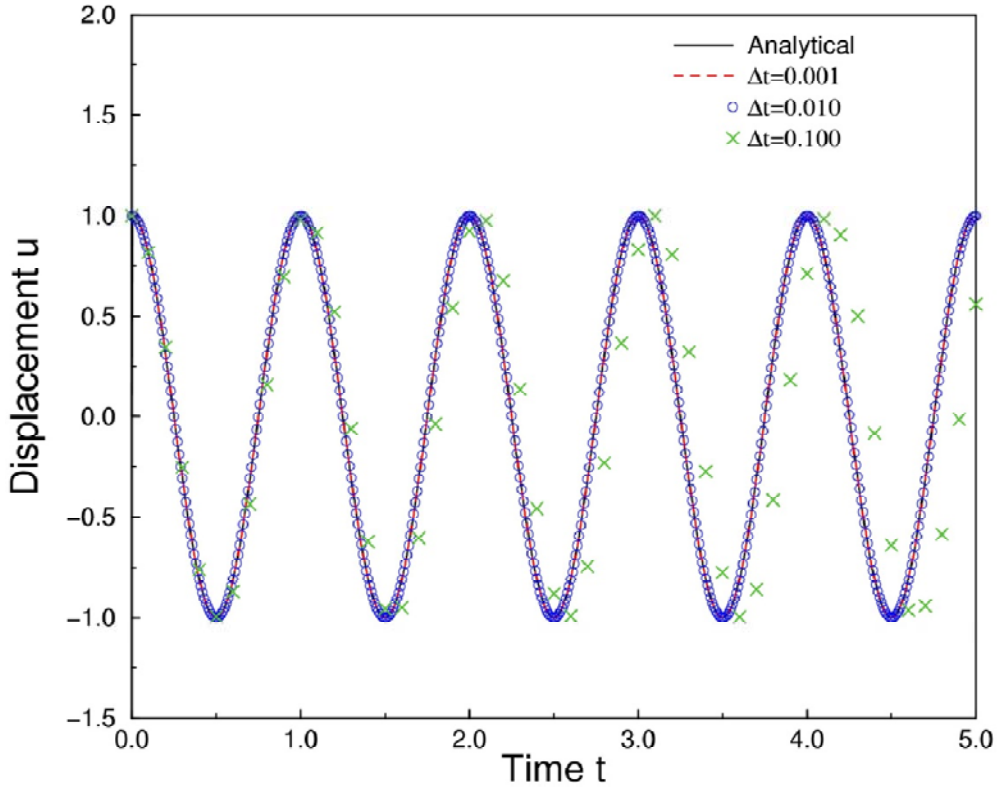

FIG. 4. (Color online) Free vibration displacement response for undamped oscillator

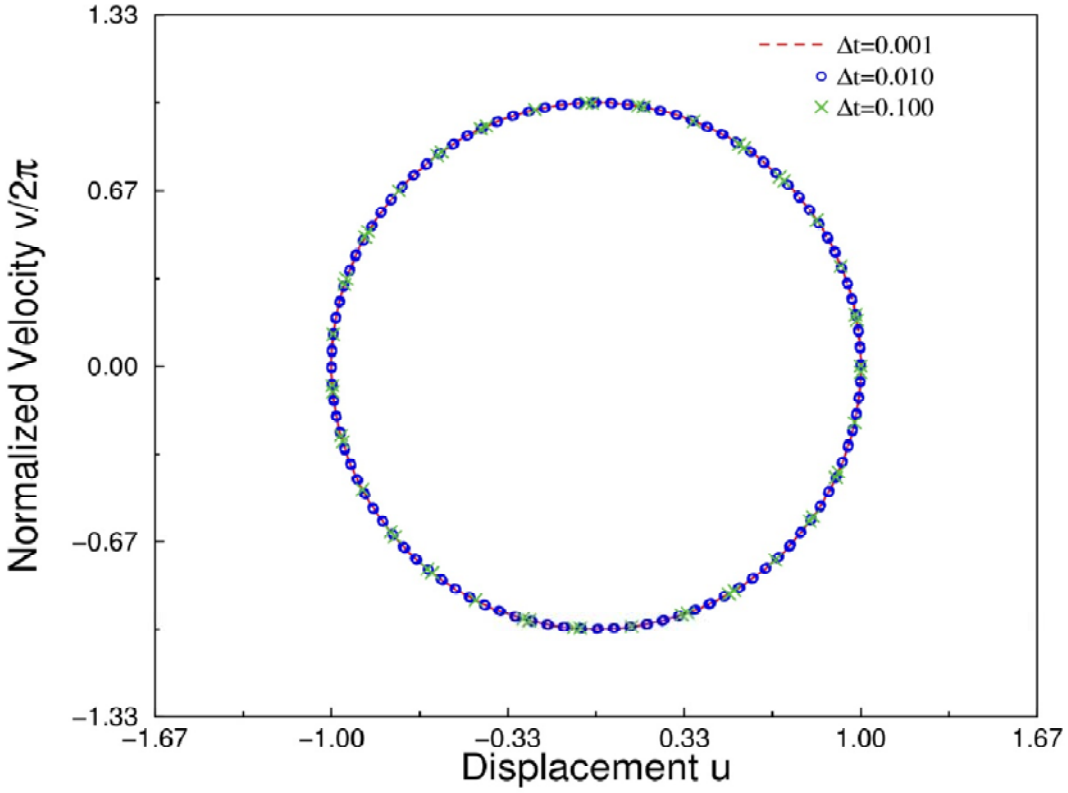

FIG. 5. (Color online) Free vibration phase space response for undamped oscillator

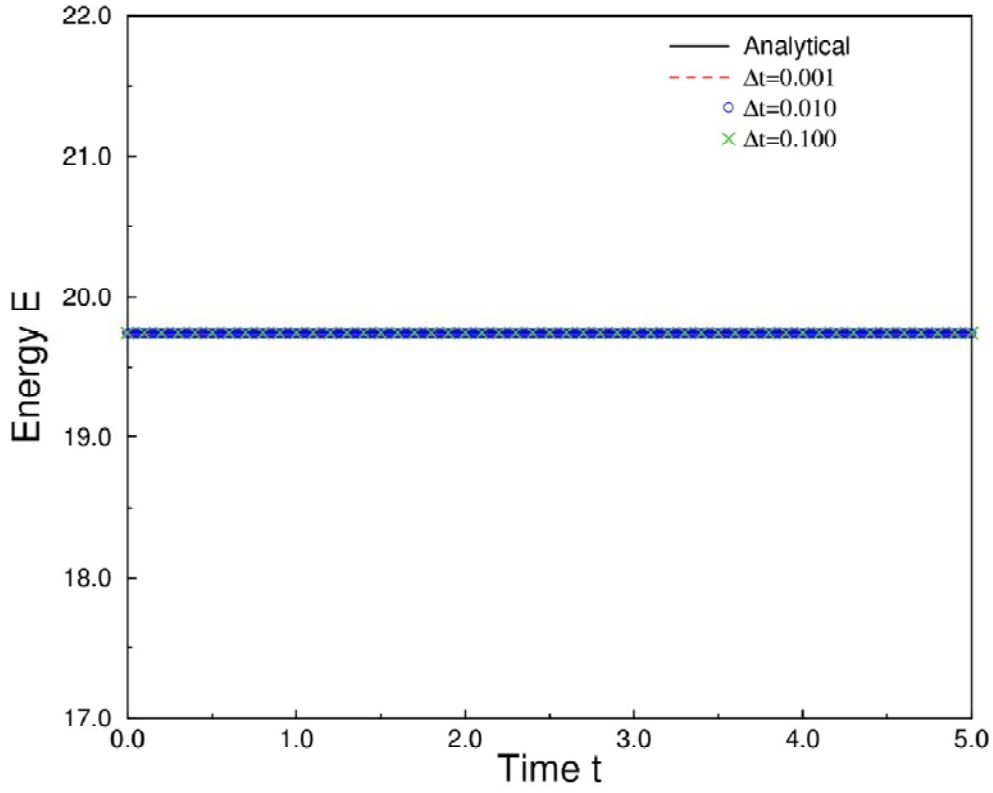

FIG. 6. (Color online) Free vibration energy response for undamped oscillator

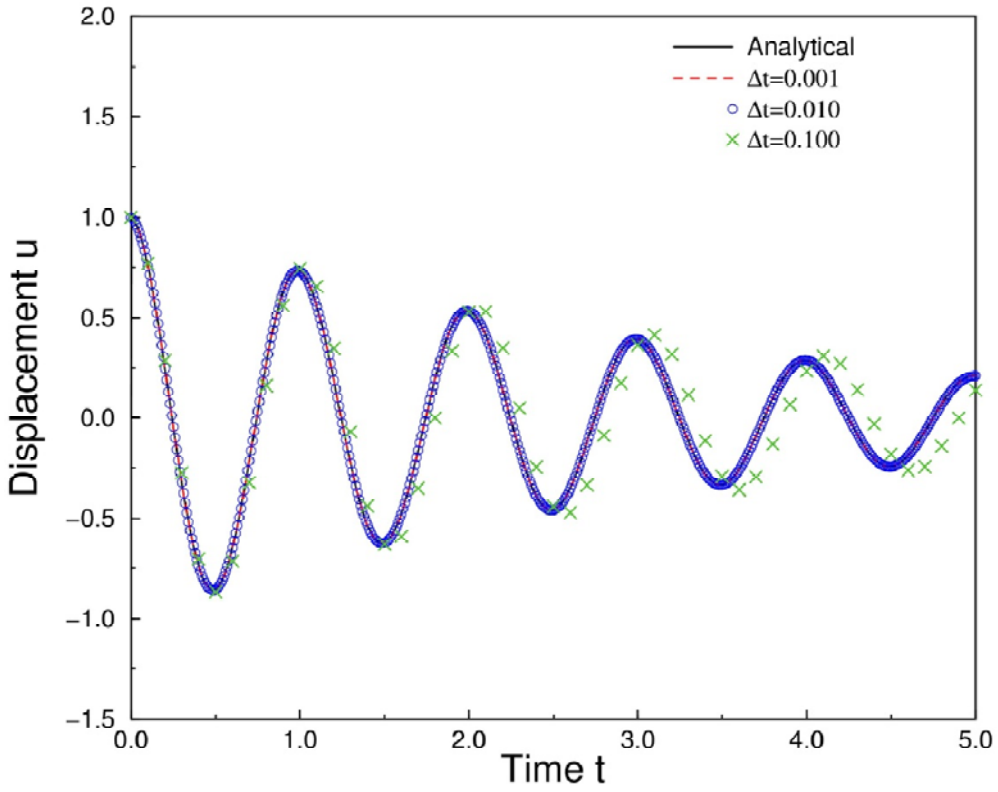

FIG. 7. (Color online) Free vibration displacement response for Kelvin-Voigt damped oscillator

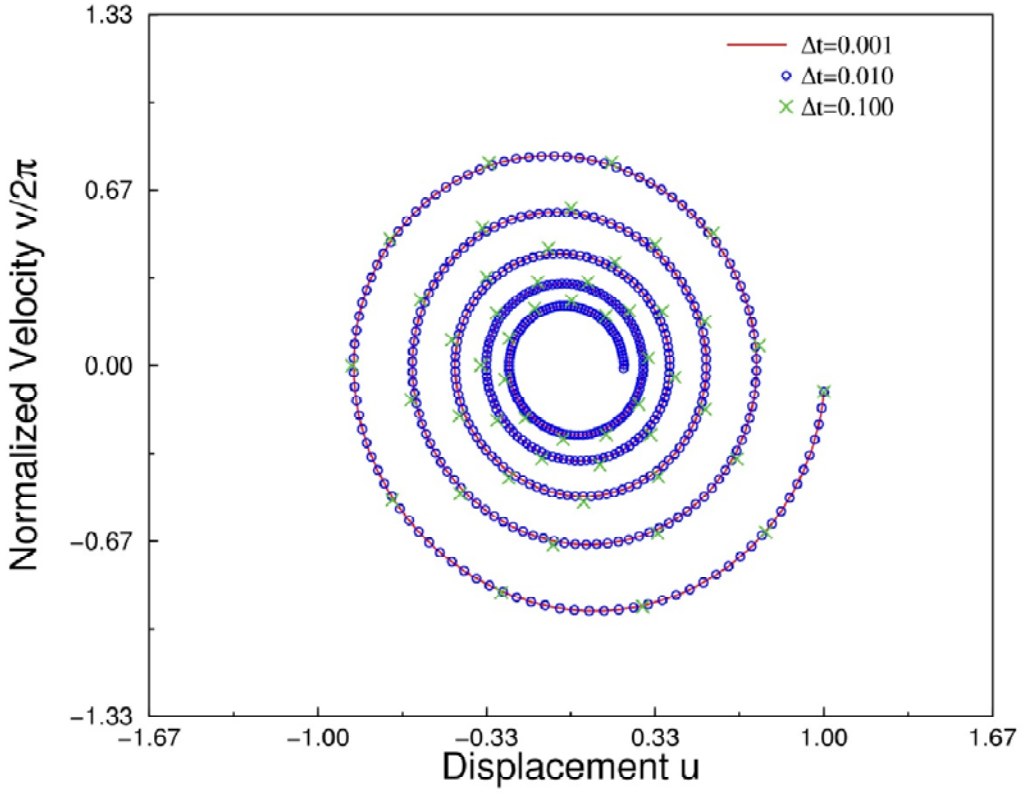

FIG. 8. (Color online) Free vibration phase space response for Kelvin-Voigt damped oscillator

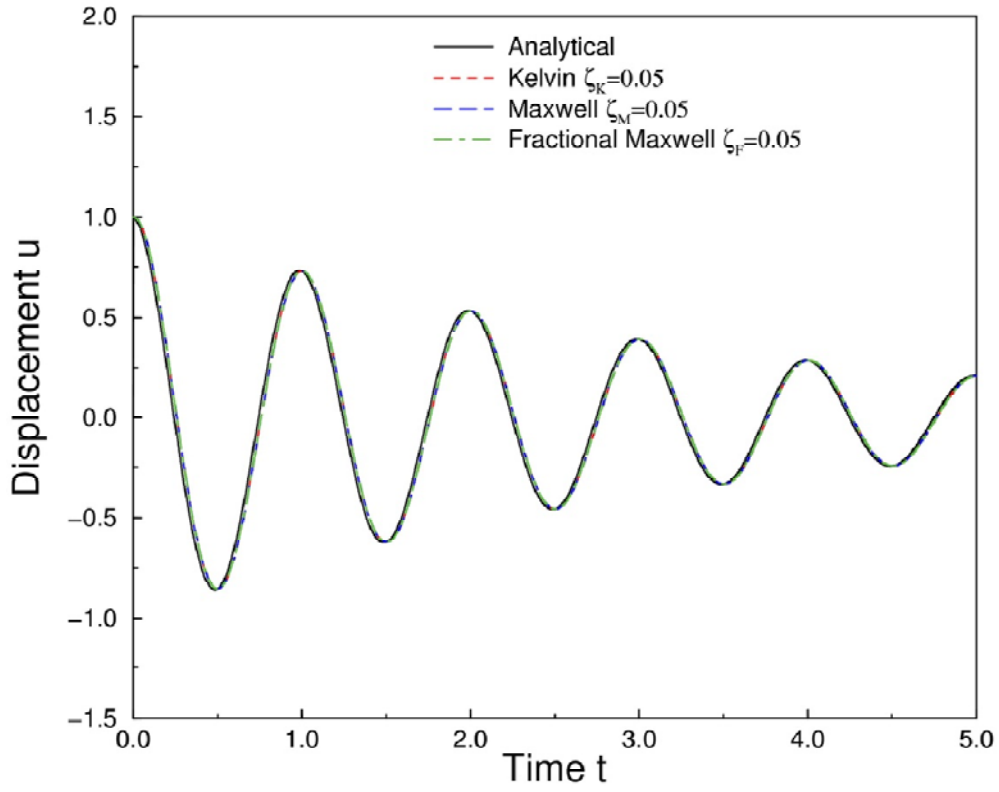

FIG. 9. (Color online) Free vibration displacement response for damped oscillators

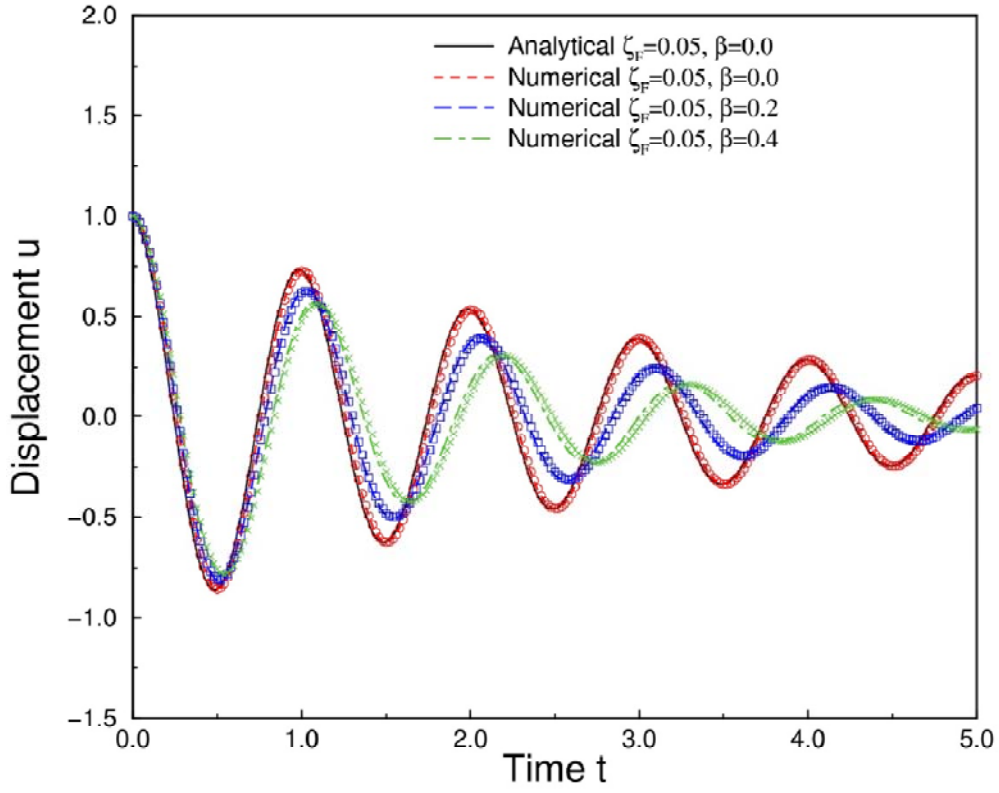

FIG. 10. (Color online) Free vibration displacement response for Fractional Maxwell damped oscillator

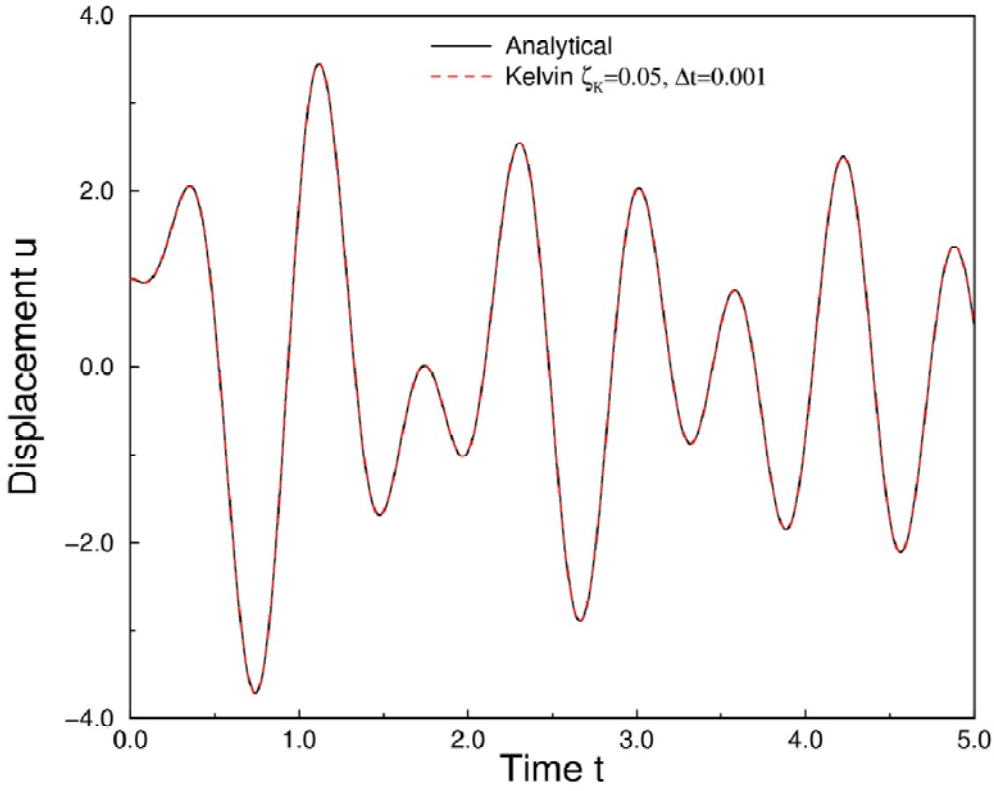

FIG. 11. (Color online) Forced vibration displacement response for Kelvin-Voigt damped oscillator

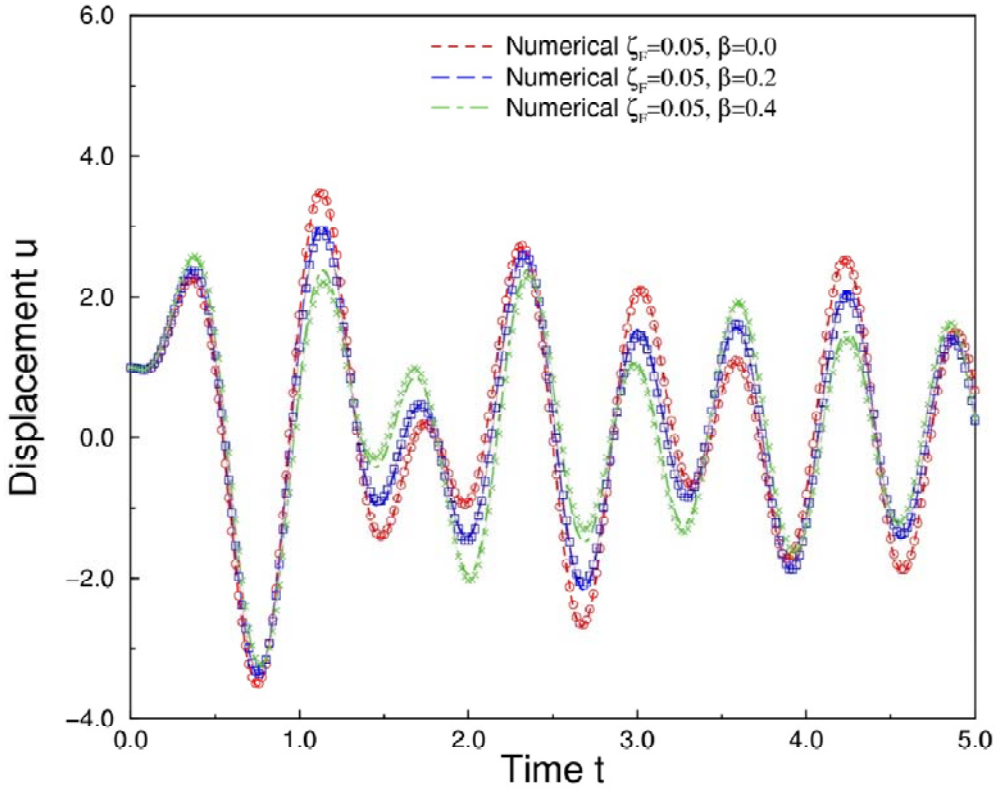

FIG. 12. (Color online) Forced vibration displacement response for Fractional Maxwell damped oscillator