Exact solution for the self-induced motion of a vortex filament in the arc-length representation of the local induction approximation

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Exact solution for the self-induced motion of a vortex filament in the arclength representation of the LIA

Robert A. Van Gorder
Department of Mathematics, University of Central Florida,
Orlando, FL 32816-1364 USA
rav@knights.ucf.edu

We review two formulations of the fully nonlinear local induction equation approximating the self-induced motion of the vortex filament (in the local induction approximation), corresponding to the Cartesian and arclength coordinate systems, respectively. The arclength representation, put forth by Umeki, results in a type of 1+1 derivative nonlinear Schrödinger (NLS) equation describing the motion of such a vortex filament. We obtain exact stationary solutions to this derivative NLS equation; such exact solutions are a rarity. These solutions are periodic in space and we determine the nonlinear dependence of the period on the amplitude.
Keywords: Vortex dynamics; Vortex filament; Local induction equation

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The self-induced velocity of a vortex filament has been described by the approximation $\mathbf{v} = \gamma\kappa\mathbf{t}\times\mathbf{n}$ (Da Rios [1], Arms and Hama [2]), where $\mathbf{t}$ and $\mathbf{n}$ are unit tangent and unit normal vectors to the vortex filament, respectively, κ is the curvature and γ is the strength of the vortex filament. The Da Rios equations have an interesting history stretching back over the last century; for an interesting account of the history of the Da Rios equations, see Ricca [3]. A discussion of the mathematical formulation of the problem governing the self-induced motion of a vortex filament can be found in Widnall [4], where applications to vortices trailing aircraft are also discussed.

A number of methods have been employed to study the Da Rios equations, particularly the local induction approximation (LIA). Shivamoggi and van Heijst [5] recently reformulated the Da Rios equations in the extrinsic vortex filament coordinate space and were able to find an exact solutions to an approximate equation governing a localized stationary solution in the LIA. Exact stationary solutions to the LIA in extrinsic coordinate space have been found by Kida [6] in the case of torus knots, and these solutions were given in terms of elliptic integrals. By re-writing the LIA in cylindrical-polar coordinates, Ricca also obtained torus knot solutions - which were asymptotically equivalent to Kida's solutions - in explicit analytic form and derived a stability criterion (see, e.g., [7], [8], [9], [10]). Static solutions to the LIA have also been found by Lipniacki [11]. See also Ricca [12] for a discussion of the physical invariants obtained under LIA.

While solutions under various approximations to the LIA are indeed useful for certain applications, the study of the fully nonlinear equations governing the self-induced motion of a vortex filament in the LIA is itself with merit. The fully nonlinear equation governing the self-induced motion of a vortex filament in the LIA was previously derived in Van Gorder [13, 14] in the Cartesian coordinate space. To this end, consider the vortex filament essentially aligned along the x-axis: $\mathbf{r} = x\mathbf{i}_x + y(x,t)\mathbf{i}_y + z(x,t)\mathbf{i}_z$. We then have that

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dx}\frac{dx}{ds} = \left(\mathbf{i}_x + y_x\mathbf{i}_y + z_x\mathbf{i}_z\right)\frac{dx}{ds}$$

and $\mathbf{v} = y_t\mathbf{i}_y + z_t\mathbf{i}_z$, where $\frac{dx}{ds} = 1/\sqrt{1+y_x^2+z_x^2}$. From the governing equation $\mathbf{v} = \gamma\kappa\mathbf{t}\times\mathbf{n}$, we compute the quantities

$$\begin{aligned} y_t &= -\gamma z_{xx}\left(\frac{dx}{ds}\right)^3 = -\gamma z_{xx}\left(1+y_x^2+z_x^2\right)^{-3/2}, \\ z_t &= \gamma y_{xx}\left(\frac{dx}{ds}\right)^3 = \gamma y_{xx}\left(1+y_x^2+z_x^2\right)^{-3/2}, \end{aligned} \tag{1}$$

and, upon defining $\Phi(x,t) = y(x,t) + iz(x,t)$, it was shown in Van Gorder [14] that the coupled system of real partial differential equations (1) reduces to the single complex partial differential equation

$$i\Phi_t + \gamma\left(1+|\Phi_x|^2\right)^{-3/2}\Phi_{xx} = 0\,. \tag{2}$$

Dmitriyev [15] considered the approximation $i\Phi+\gamma\Phi_{xx} = 0$, while Shivamoggi and van Heijst [5] considered a quadratic approximation to the nonlinearity in (2). The full nonlinear equation was obtained in [13]. In order to recover y and z once a solution Φ to (2) is known, note that $y = \mathcal{R}e\ \Phi$ and $z = \mathcal{I}m\ \Phi$. Some mathematical properties of equation (2) were discussed in Van Gorder [14] in the case where periodic stationary solutions are possible, though a systematic study of all such stationary solutions was not considered. In Van Gorder [16] a more systematic approach was taken to classify all such stationary solutions $\Phi(x,t) = e^{-i\gamma t}\psi(x)$ to (2). Spatially-periodic solutions (2) were shown to be governed by an implicit relation involving the sum of elliptic integrals of differing kinds. The amplitude of such periodic solutions was shown to obey $|\psi| < \sqrt{2}$.

The formulation (2), corresponding to the Cartesian coordinate system, is one possible way to describe the fully nonlinear self-induced motion of a vortex filament in the LIA. Umeki [17] obtained an alternate formulation, applying an arclength-based coordinate system as opposed to a Cartesian coordinate system. Umeki defines $\mathbf{r} = \mathbf{t} \times \mathbf{t}_s$, where s is the arclength element. Now, $\mathbf{t}_t = \mathbf{t} \times \mathbf{t}_{ss}$. Let us write $\mathbf{t} = (\tau_x, \tau_y, \tau_z)$. Then Umeki defines the complex field v by

$$\tau_x + i\tau_y = \frac{2v}{1+|v|^2}\,, \qquad \tau_z = \frac{1-|v|^2}{1+|v|^2}\,. \tag{3}$$

The relation $\mathbf{t}_t = \mathbf{t} \times \mathbf{t}_{ss}$ then implies

$$(\tau_x + i\tau_y)_t = i((\tau_x + i\tau_y)_{ss}\tau_z - (\tau_x + i\tau_y)\tau_{z\,ss})\,,$$

$$2\tau_{zt} = i((\tau_x^* + i\tau_y^*)_{ss}(\tau_x + i\tau_y) - (\tau_x^* + i\tau_y^*)(\tau_x + i\tau_y)_{ss})\,.$$

From here, Umeki [17] then found

$$iv_t + v_{ss} - 2v^* v_s^2/(1+|v|^2) = 0\,, \tag{4}$$

where v denotes directly the tangential vector of the filament. While the Cartesian and arc-kength formulations are obtained through different derivations, both formulations are equivalent to the localized induction equation (LIE). Umeki [18] showed that there exists a transformation between solutions to (2) and solutions to (4). A plane wave solutions to (4) exists [18], and Umeki [18] was also able to show that the famous 1-soliton solution of Hasimoto [19] is given by

$$v(s,t) = \frac{\nu\,\mathrm{sech}\,(k(s-ct))}{\nu\,\mathrm{sech}^2\,(k(s-ct)) - 2}\left(\tanh\,(k(s-ct)) - \frac{ic}{2k}\right),$$

$\nu = 2k^2/(4k^2+c^2)$, $0 < \nu < 1/2$ in the arclength representation.

We now turn our attention to obtaining stationary solutions, which has not been done for the local induction equation in the arclength representation. Let us consider the ansatz

$$v(s,t) = e^{-i\alpha^2 t} q(\alpha s)\,, \tag{5}$$

where q is assumed to be a real-valued function, which puts (4) into the form

$$q + q_{ss} - \frac{2qq_s^2}{1+q^2} = 0\,. \tag{6}$$

Hence, the solution (5) is invariant under $\alpha \in \mathbb{R}$, so without loss of generality we shall consider $\alpha = 1$ henceforth. We should remark that a factor of $e^{+i\alpha^2 t}$ in (5) results in unstable solutions, so the '$-$' case in the exponent is what we limit our attention to. Also note that (6) is essentially a nonlinear oscillator provided $2q_s^2 < 1 + q^2$.

Our goal is to obtain an exact solution for (6), and defining a conserved quantity will greatly help in constructing a second integral. To this end, let us define the quantity

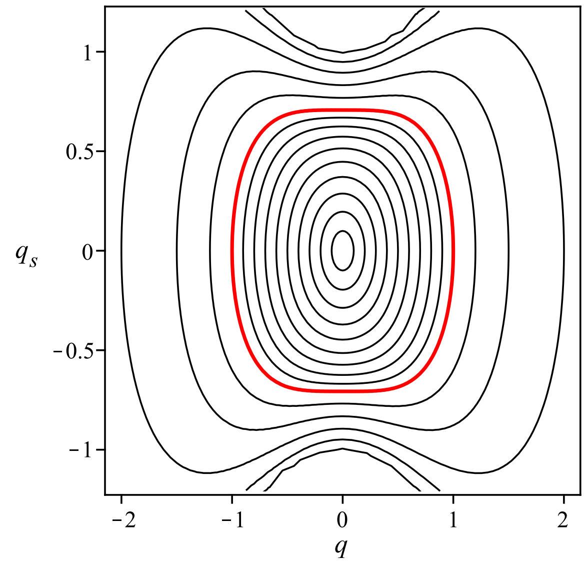

FIG. 1: (Color online) Phase portrait in (q, q_s) for the solution to the fully nonlinear oscillator equation modelling the local induction equation under the arclength representation.

$$E = -\frac{q_s^2 - q^2 - 1}{(1+q^2)^2}\,, \tag{7}$$

$E \in (0,1)$. Observe that the quantity is conserved:

$$\frac{dE}{ds} = -\frac{2q_s}{(1+q^2)}\left(q + q_{ss} - \frac{2qq_s^2}{1+q^2}\right) = 0\,. \tag{8}$$

For a fixed value of E, we find that

$$q_s^2 = (1+q^2)\left(1 - (1+q^2)E\right)\,, \tag{9}$$

and, upon separating variables,

$$\int_{q_0}^{q} \frac{d\xi}{\sqrt{(1+\xi^2)\,(1-(1+\xi^2)E)}} = \pm(s - s_0)\,, \tag{10}$$

where $q_0 = q(s_0)$ is a second arbitrary constant. Performing the required integration, we obtain the expression

$$\frac{1}{\sqrt{E}} F\left(\frac{\sqrt{E}}{\sqrt{1-E}}q, \frac{\sqrt{1-E}}{\sqrt{E}}i\right) = \pm(s - \hat{s})\,, \tag{11}$$

where $\hat{s}$ is a constant involving s_0 and q_0. Here, F is the elliptic integral of the first kind.

Inverting (11) to obtain $q(s)$, we find that

$$q(s) = \frac{\sqrt{1-E}}{\sqrt{E}}\,\mathrm{sn}\left(\pm\sqrt{E}(s-\hat{s}), \frac{\sqrt{1-E}}{\sqrt{E}}i\right)\,, \tag{12}$$

where $\mathrm{sn}(a,b)$ denotes the Jacobi elliptic function. While (12) is a closed form expression, it involves the conserved

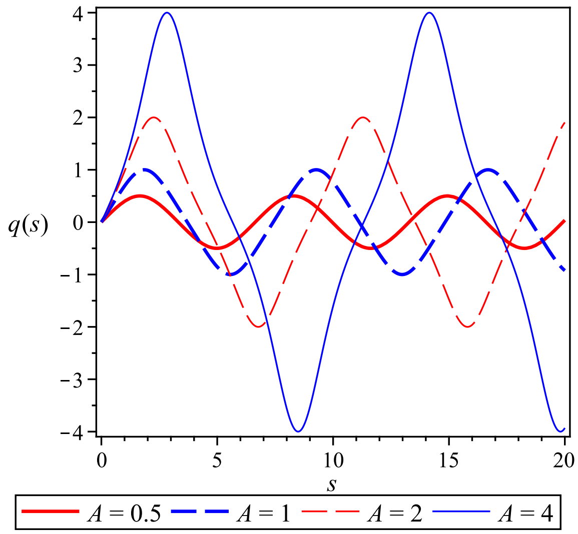

FIG. 2: (Color online) Plots of the solution $q(s)$ given in (14) for various values of the amplitude A. Note that the period of the solutions is strongly influenced by the amplitude. The nonlinear dependence of the period T with the amplitude A is shown graphically in Fig. 3.

quantity E, which is perhaps not so satisfying. Note that the amplitude of q may be found from (9); setting $q_s = 0$, we find that the amplitude $A = A(E)$ is given by

$$A = \max_s |q(s)| = \frac{\sqrt{1-E}}{\sqrt{E}}. \tag{13}$$

It follows that $E = 1/(1+A^2)$, hence (12) becomes

$$q(s) = A \,\mathrm{sn}\left(\pm\frac{1}{\sqrt{1+A^2}}(s-\hat{s}), Ai\right). \tag{14}$$

With this we have obtained an exact stationary solution $q(s)$ in terms of amplitude A. In Fig. 1 we plot the phase portrait for q versus q_s, which demonstrates the exact periodic solutions. In Fig. 2, we display solution profiles for various values of the amplitude A. We should remark that in the Cartesian case, solutions to models which are low-order approximations to the fully nonlinear model agree well for small amplitudes [16], and we expect the same will hold here (though we omit the details of any approximating models here).

A similar exact solution was obtained by Hasimoto [20], through a different derivation, for a two-dimensional model (recall that our model is three-dimensions). Hasimoto's derivation started with $\mathbf{v} = Y\mathbf{i}_y$, as opposed to $\mathbf{v} = y_t\mathbf{i}_y + z_t\mathbf{i}_z$. Assuming a stationary solution, Hasimoto's assumption leads to an equation $Y_{xx} + \frac{\Omega}{\gamma}\left(1+Y_x^2\right)^{3/2} Y = 0$. Hasimoto finds a solution $Y = A\mathrm{cn}(\xi,k)$ (where $x = x(\xi)$, ξ is a parametrization linking Y and x implicitly), which has initial conditions $Y(0,k) = A$ and $Y'(0) = 0$. Hence, Hasimoto's solution for the two-dimensional problem is a direct analogy to the solution for the three-dimensional problem we've found here under the arclength representation.

Observe the nonlinear dependence of the period on the amplitude. From this exact relation, we see that the period $T = T(A)$ obeys the relation

$$\begin{aligned} T(A) &= 4\sqrt{1+A^2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+A^2\sin^2\theta}} \\ &= 4K\left(\frac{A}{\sqrt{1+A^2}}\right), \end{aligned} \tag{15}$$

where K is the elliptic quarter period. Recalling the asymptotic expansion

$$K(m) \approx \frac{\pi}{2} + \frac{\pi}{8}\frac{m^2}{1-m^2} - \frac{\pi}{16}\frac{m^4}{1-m^2} \tag{16}$$

which is a good approximation for $m < 1/2$, we have

$$T(A) \approx 2\pi + \frac{\pi}{2}A^2 - \frac{\pi}{4}\frac{A^4}{1+A^2}, \tag{17}$$

which in turn is a good approximation for the small-amplitude regime $A < 1/\sqrt{3}$. The large amplitude asymptotics are slightly less standard. For $m > 2$, there exists an accurate asymptotic expansion

$$4K\left(1-\frac{1}{m}\right) \approx J(m), \tag{18}$$

where

$$\begin{aligned} J(m) =& 4\left(1+\frac{1}{m}+\frac{5}{16m^2}+\frac{7}{32m^3}\right)\ln\left(2\sqrt{2m}\right) \\ &-\left(\frac{1}{m}+\frac{7}{8m^2}+\frac{17}{24m^3}\right). \end{aligned} \tag{19}$$

When $m > 2$, the argument of K is less than or equal to $1/2$. Thus,

$$T(A) \approx J\left(\frac{\sqrt{1+A^2}}{\sqrt{1+A^2}-A}\right) \tag{20}$$

is a good approximation for $A > 1/\sqrt{3}$.

In Fig. 3, we plot the the period $T(A)$ of the solution (14) versus the amplitude A. The approximate asymptotic solutions are also included in their valid regions. Then, in Fig. 4, we plot the relative error in these approximations, showing the agreement between the exact and asymptotic solutions. For the $A > 1/\sqrt{3}$ asymptotics, only retaining the logarithmic term (as a lowest order approximation) is not completely sufficient, as demonstrated in Fig. 4.

We have found an exact stationary solution for the self-induced motion of a vortex filament in the arclength representation of the LIA. Such a formula is interesting in both it's simplicity and it's potential applications. Note

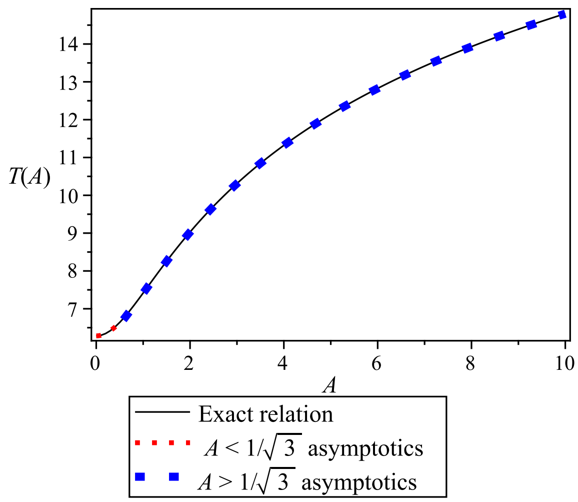

FIG. 3: (Color online) Plot of the period $T(A)$ of the solution (14) versus the amplitude A. The exact relation is found by numerically plotting (15). Note that both the $A < 1/\sqrt{3}$ and $A > 1/\sqrt{3}$ asymptotic expansions are excellent fits to the exact relation.

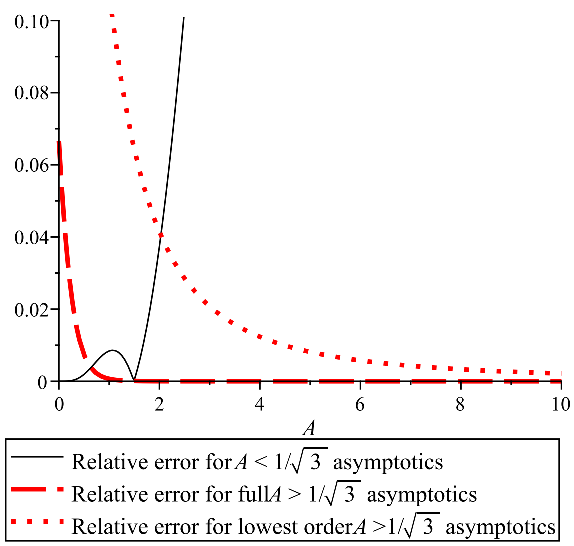

FIG. 4: (Color online) Relative error $|T(A) - T_{\text{approx}}|/|T(A)|$ of the approximations to $T(A)$. We also include the lowest order approximation $T(A) \approx 4\ln(2\sqrt{2m})$ for the $A > 1/\sqrt{3}$ case. We see the good agreement with the $A < 1/\sqrt{3}$ asymptotics and $A > 1/\sqrt{3}$ asymptotics where needed.

that this representation is simpler than that found in the Cartesian representation; in particular, the integral representation permits a clean inversion so that we may obtain solutions in the form (14). In the Cartesian case, however, the solutions were defined implicitly by a linear combination of elliptic integrals, which was then inverted numerically. Umeki [18] gives a relation between the arclength and Cartesian representations which can be used to map the arclength formula into a formula for the Cartesian representation. This involves complicated mathematical expressions and we omit the details of this inversion here.

We should remark that, while interesting, the physical scenario considered here is certainly not the only case of interest. The behavior of a vortex filament in a superfluid is another area of current research [21]-[29], since it grants us a model of superfluid turbulence. The nonlinear motion of a vortex filament in a superfluid has been previously studied in the case of a Cartesian coordinate system, in the form of a partially linearized model [30] and, more recently, a fully nonlinear model [31]. In developing these models, one begins with the local induction equation and adds terms due to the ambient superfluid; see [30]. It is possible that, in an arclength coordinate system, the solution representation for the fully nonlinear model can be simplified. The application of the present results to the study of the motion of a vortex filament in a superfluid, under the arclength formulation, is certainly possible. In particular, it becomes clear that the solution presented here would serve as the order-zero perturbation theory for the superfluid case, with higher-order corrections resulting from the superfluid friction parameters [31, 32]. Along these lines, see also [33]. Since we were able to obtain an explicit exact stationary solution in the present geometry, perhaps the arclength formulation will prove most useful int eh study of such superfluid models. This is one potential area of future work.

Acknowledgements

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[1] L.S. Da Rios, Rend. Circ. Mat. Palermo 22, 117 (1906).
[2] R.J. Arms and F.R. Hama, Phys. Fluids 8, 553 (1965).
[3] R.L. Ricca, Nature 352, 561 (1991).
[4] S.E. Widnall, J. Fluid Mech. 54, 641 (1972); S.E. Widnall, Ann. Rev. Fluid Mech. 7, 141 (1975).
[5] B.K. Shivamoggi and G.J.F. van Heijst, Physics Letters A 374, 1742 (2010).
[6] S. Kida, J. Fluid Mech. 112, 397 (1981).
[7] R.L. Ricca, Chaos 3, 83 (1993).
[8] R.L. Ricca, D.C. Samuels, and C.F. Barenghi, J. Fluid Mech. 391, 29 (1999).
[9] F. Maggioni, S.Z. Alamri, C.F. Barenghi, and R.L. Ricca, Nuovo Cimento C 32, 133 (2009).
[10] F. Maggioni, S.Z. Alamri, C.F. Barenghi, and R.L. Ricca, Phys. Rev. E 82, 026309 (2010).
[11] T. Lipniacki, J. Fluid Mech. 477, 321 (2003).
[12] R.L. Ricca, Physics of Fluids A 4, 938 (1992).
[13] R.A. Van Gorder, Theor. Comput. Fluid Dyn. 26, 161 (2012).
[14] R.A. Van Gorder, Theor. Comput. Fluid Dyn., in press (2011) doi: 10.1007/s00162-011-0244-8.
[15] V.P. Dmitriyev, Am. J. Phys. 73, 563 (2005).
[16] R.A. Van Gorder, Solution of derivative NLS equations governing the self-induced motion of a vortex filament (2012) preprint.

[17] M. Umeki, Theor. Comput. Fluid Dyn. 24, 383 (2010).
[18] M. Umeki, A real-space representation of a locally induced vortex filament (2012) preprint.
[19] H. Hasimoto, J. Fluid Mech. 51, 477 (1972).
[20] H. Hasimoto, J. Phys. Soc. Japan 31, 293 (1971).
[21] K.W. Schwarz, Phys. Rev. B 31, 5782 (1985).
[22] J. Koplik and H. Levine, Phys. Rev. Lett. 71, 1375 (1993).
[23] R.E. Packard, Rev. Mod. Phys. 70, 641 (1998).
[24] D. Kivotides, J.C. Vassilicos, D.C. Samuels and C.F. Barenghi, Phys. Rev. Lett. 86, 3080 (2001).
[25] W.F. Vinen, M. Tsubota, and A. Mitani, Phys. Rev. Lett. 91, 135301 (2003).
[26] M. Tsubota, T. Araki, and C.F. Barenghi, Phys. Rev. Lett. 90, 205301 (2003).
[27] R. Goto, S. Fujiyama, H. Yano, Y. Nago, N. Hashimoto, K. Obara, O. Ishikawa, M. Tsubota, T. Hata, Phys. Rev. Lett. 100, 45301 (2008).
[28] V.B. Eltsov, R. De Graaf, P.J. Heikkinen, J.J. Hosio, R. Hänninen, M. Krusius, and V.S. Lvov, Phys. Rev. Lett. 105, 125301 (2010).
[29] A.W. Baggaley and C.F. Barenghi, Phys. Rev. B 83, 134509 (2011).
[30] B.K. Shivamoggi, Phys. Rev. B 84, 012506 (2011).
[31] R.A. Van Gorder, J. Fluid Mech. 707, 585 (2012).
[32] H. Adachi, S. Fujiyama and M. Tsubota, Phys. Rev. B. 81, 104511 (2010).
[33] H. Wang, Phys. Rev. Lett. 80, 4665 (1998).