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Phys. Rev. E 86, 041306 — Published 16 October 2012
DOI: 10.1103/PhysRevE.86.041306
The Influence of Network Topology on Sound Propagation in Granular Materials

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(Dated: August 27, 2012)

Granular materials, whose features range from the particle scale to the force-chain scale to the bulk scale, are usually modeled as either particulate or continuum materials. In contrast with either of these approaches, network representations are natural for the simultaneous examination of microscopic, mesoscopic, and macroscopic features. In this paper, we treat granular materials as spatially-embedded networks in which the nodes (particles) are connected by weighted edges obtained from contact forces. We test a variety of network measures for their utility in helping to describe sound propagation in granular networks and find that network diagnostics can be used to probe particle-, curve-, domain-, and system-scale structures in granular media. In particular, diagnostics of meso-scale network structure are reproducible across experiments, are correlated with sound propagation in this medium, and can be used to identify potentially interesting size scales. We also demonstrate that the sensitivity of network diagnostics depends on the phase of sound propagation. In the injection phase, the signal propagates systemically, as indicated by correlations with the network diagnostic of global efficiency. In the scattering phase, however, the signal is better predicted by meso-scale community structure, suggesting that the acoustic signal scatters over local geographic neighborhoods. Collectively, our results demonstrate how the force network of a granular system is imprinted on transmitted waves.

PACS numbers: 64.60.aq, 43.25.+y, 81.05.Rm

During the past 15 years, techniques from areas of physics such as statistical mechanics and nonlinear dynamics have been used to make important advances in studying networks across myriad disciplines \([1]\). Conversely, the perspective of networks can also play important roles in physical problems, as there is a large class of heterogeneous systems such as foams, emulsions, and granular materials \([2, 3]\) for which the connectivity of the constituent elements is an important factor in the deviation of their behavior from continuum models. In fact, the discontinuous nature of granular materials led to the early idea of a fabric structure governing the anisotropic behavior of such materials \([4–6]\).

We investigate whether studying the rich and complex dynamics of granular materials \([7]\) using network analysis can provide new insights into the underlying physics. This treatment is a natural one, because granular materials can be represented as spatially-embedded networks \([8]\) composed of nodes (particles) and edges (contacts between particles) with definite locations in Euclidean space \([9, 10]\). In Fig. 1, we show a quasi-two-dimensional (quasi-2D) granular system composed of photoelastic disks that permits the determination of both the contact network and the interparticle forces. The forces between particles in these systems are non-homogeneous, and they form a network of chain-like structures that span the system (see Fig. 1B). This force chain network has the same topology as the contact network but contains edges that are weighted by the inter-particle forces (Fig. 1C). This is exciting from a networks perspective, as it allows us to study the influence of network topology on ‘network geometry’ in a spatially-embedded system. From the perspective of granular materials, earlier work suggests that force chains provide the main supporting structure for static and dynamic loading \([11, 12]\).

Because signal propagation in granular and heterogeneous materials \([13]\) is of considerable importance to numerous industrial and natural systems, it has been the topic of many investigations. A longstanding question is how to reconcile the failure of continuum models of granular sound propagation \([14–17]\), as such models fail to quantitatively describe important heterogeneous and nonlinear features of acoustic speed \([18–21]\). The presence of force chains has been suggested as a potential confounding phenomenon that might underlie the failure of previous physical models of sound propagation \([18, 22]\). Ultimately, it would be beneficial to quantify how the pressure or strain state of a system is imprinted on transmitted waves, and to understand how to use these waves to accurately detect buried objects or reservoirs of oil.

An increasing body of work has used tools from areas like network science and computational homology to obtain insights on the structural properties of granular materials \([9, 10, 23]\) and other continuous media \([24]\). Indeed, a networks perspective provides a valuable complement to the standard ways of studying granular materials. In the present paper, we analyze experimental data using network analysis to investigate the role of force-weighted contact networks in sound propagation. The use of photoelastic particles combined with high-speed
imarily exponential decay as a function of distance from the source. The use of such soft, dissipative particles differs from previous work [25–29], where much harder particles have been used. Further details about the apparatus are described in Ref. [22].

We excite acoustic waves from the bottom of the system by sending pulses of five 750 Hz sinusoidal waves with a voice coil driver attached to a 20 mm wide platform; maximum particle displacements are on the order of 5 μm. To assess the reliability of network diagnostics, we repeat the experiments for 17 different particle configurations, each of which is obtained by manual rearrangement. We restrict our analysis to a region of the system that contains just over \( N = 400 \) particles. This subsystem corresponds to a region in which vertical force gradients are minimized due to the Janssen effect [30].

We compute particle positions and forces using two high-resolution pictures of the static system and one high-speed movie that captures the system dynamics. We took one static image without the polarizer (see Fig. 1A) and used it to determine particle positions and contacts [22, 31]. We take a second static image using polarizers (see Fig. 1B), and we use this image to estimate the normal forces at each contact [22]. In the vicinity of each contact, we use a combination of the light intensity \( I \), the square of the mean intensity gradient \((\nabla I)^2\), and the position of the photoelastic fringes to estimate the contact forces, by comparing them to calibration images with known forces. We measure the amplitude and location of sound propagation using a high-speed camera operating at 4000 Hz; the camera records 80 frames of data (20 ms) containing both the injection of the signal \((0 < t < 40)\) and its dissipation \((40 < t < 80)\). For each particle in each frame, we compute \( \Delta I(x, y, t) = I(x, y, t) - I(x, y, t_0) \), which measures how much the brightness \( I \) of the particle changes with respect to its unperturbed brightness. In earlier work [22], we used piezoelectric sensors embedded in a subset of the particles to determine that \( \Delta I \) is proportional to the change in stress on that particle. Using \( \Delta I \) allows us to follow the propagating signal through all particles in the measurement region.

To determine which particles are in contact, we use the positions of the particle centers, which are determined from the static image of the system using a Hough transform. If the distance between two particle centers is less than 1.05 times the sum of their radii, we treat the particles as being in contact. This method overcounts the number of true contacts. However, the effect of such over-counting is minimized by the fact that false contacts are assigned a force value of almost zero when we apply our image-processing techniques to find the contact forces. Accordingly, they do not contribute to the structure of the weighted network.

For each experimental run, we construct both an unweighted (binary) and a weighted network, which correspond respectively to an underlying contact network and a force-chain network (see Fig. 1). In each type of net-

**FIG. 1.** [Color online] (A) Image of a 2D vertical aggregate of photoelastic disks confined in a single layer. The driver position is marked with an arrow. Several particles are embedded with a piezoelectric sensor, for which wires are visible. (B) The internal stress pattern within the photoelastic particles manifests as a network of force chains. (C) The dark (blue) lines show a weighted graph, which is determined from image processing and overlaid on image (B). An edge between two particles (nodes) exists if the two particles are in physical contact with each other; the forces between particles give the weights of the edges.

imaging allows us to gain insight into internal force structures and particle-scale sound propagation that are not readily available in ordinary granular materials. We find that geographic community structure provides a fundamental constraint to sound propagation, illustrating that contact topology alone is insufficient to understand signal propagation in granular materials.

I. EXPERIMENTS

We perform experiments on a vertical 2D granular system of bidisperse disks confined between two sheets of Plexiglass, which have been slightly lubricated with baking powder to reduce friction with the container walls. The top of the container is open and the particles are confined exclusively by gravity. The particles are 6.35 mm thick and have diameters \( d_1 = 9 \) mm and \( d_2 = 11 \) mm, and are cut from Vishay PSM-4 photoelastic material to provide measurements of the internal forces. We show example images in Fig. 1. These particles have an elastic modulus of \( E = 4 \) MPa, and they are sufficiently dissipative that propagating sound waves experience an approx-
work, the nodes represent the particles in the system. In
the binary network \( \mathbf{A} \), an edge exists between node \( i \) and
\( j \) (i.e., \( A_{ij} = 1 \)) if node \( i \) is in contact with node \( j \); oth-
"erwise, \( A_{ij} = 0 \). The weighted network \( \mathbf{W} \) contains the
same edges, but each element \( W_{ij} \) now has a value that is
given by an estimate of the normal force \( f_{ij} \) between
particles \( i \) and \( j \), normalized by the mean force \( \bar{f} \) of all
contacts: \( W_{ij} = f_{ij}/\bar{f} \).

II. RESULTS

We assess the global organization of the networks using
21 candidate diagnostics for \( \mathbf{A} \) and 8 candidate diagnos-
tics for \( \mathbf{W} \). We define each diagnostic in Appendix A,
where we also include descriptions to provide intuition
about what each of them measures, as well as their pos-
sible physical significance to the granular system that we
study. We examine the reliability of these diagnostics
across experimental runs in Appendix B, and we com-
pare the binary-network diagnostics to those in a null
model constructed using an ensemble of random geo-
metric graphs (RGGs) [32] in Appendix C. We examine 4 di-
agnostics (clustering coefficient, geodesic node between-
ness, optimized modularity, and global efficiency) in fur-
ther detail. Each diagnostic can be defined for both bi-
nary and weighted networks, and each is helpful for ob-
taining insights into a particular type of spatial structure
in the system: particles (cluster coefficient), curves (be-
tweenness), meso-scale domains (via community struc-
ture determined from modularity optimization), and the
entire system (global efficiency). We describe our results
in the sections below.

A. Scale Sensitivity of Network Diagnostics

A key advantage of using network tools to study granu-
lar materials is that different network diagnostics (which
we define and discuss in detail in Appendix A) are sensi-
tive to different system scales, and this is especially help-
ful for spatially-embedded systems like granular packings
(see Fig. 2). Our results indicate that the global effi-
ciency \( E_w \) [see Eqs. (5) and (23)] is a system-level property
with smallest values along the perimeter of the system
and largest values in the center. Community structure
and its associated community label \( X \) [see Eqs. (15) and
(27)] and intracommunity strength \( z \)-score [see Eq. (3)]
is a meso-scale property and can be used to probe inter-
mediate structural features. We find that geodesic node
betweenness \( B_w \) [see Eqs. (6) and (25)] can be thought of
as a one dimensional property in these materials because
it is sensitive to curve-like structures in the network. Fi-
ally, we find that clustering coefficient \( C_w \) [see Eqs. (13
and 24)] is sensitive to particle-scale features of the net-
work. In a later section, we report how each of these
network diagnostics correlates with sound propagation
through the granular material.

FIG. 2. [Color online] Example distributions of several net-
work diagnostics for a sample granular packing. The net-
work characteristics that we examine include (A) global effi-
ciency \( E_w \), (B) community structure, which we visualize us-
ing the quantity \( X^2(z + 5) \), where \( X \) is the community label,
(C) geodesic node betweenness \( B_w \), and (D) clustering coef-
cient \( C_w \). This figure illustrates their respective sensitivities
to system-scale (2D), domain-scale (2D), curve-scale (1D),
and particle-scale (0D) structure, respectively. The quantity
\( X^2(z + 5) \) allows us to visualize both the community label
(\( X \)) and the intracommunity strength \( z \)-score \( z \) [see Eq. (3)]
simultaneously; we chose the constant \( 5 \) purely for visual clar-
ity.

B. Identifying a Characteristic Size Scale

An ongoing challenge in the study of granular systems
is identifying and measuring characteristic size scales
within granular materials, from the perspective of either
particles or force chains [2, 3, 33]. Network modularity
provides a novel means to measure such size scales via the
identification of community sizes. We find that the op-
timal value of modularity is a reliable diagnostic for the
structure of both the binary and weighted networks (see
Appendix B). To seek characteristic community sizes, we
also examine community structure as a function of a res-
olution parameter \( \gamma \) [34–36]. The modularity index is
\[ Q_w = \sum_{ij} [W_{ij} - \gamma P_{ij}] \delta(g_i, g_j), \]  
where node \( i \) is assigned to community \( g_i \), node \( j \) is as-
signed to community \( g_j \), \( \delta(g_i, g_j) = 1 \) if \( g_i = g_j \) and it
equals 0 otherwise, and \( P_{ij} \) is the expected weight of the
dge connecting node \( i \) and node \( j \) under a specified null
model. We used the usual Newman-Girvan null model, in
which the expected strength distribution of the network
is preserved but ends of edges are rewired uniformly at

random [37, 38]. We employed the Louvain locally greedy algorithm to optimize modularity [39], and we varied the resolution parameter $\gamma$ from 0.001 to 100. Low values of $\gamma$ probe large spatial scales, and high values probe small scales. When we increase $\gamma$, the number of communities increases (as expected), and the modularity decreases. See Fig. 3A-C.

One can think of the term $J_{ij}(\gamma) = W_{ij} - \gamma P_{ij}$ in Eq. (1) as a particular choice of interaction strength between a pair of spins in a Potts model [34, 37, 40]. We exploit this analogy with the Potts model to transform the resolution parameter $\gamma$ so that it measures the effective fraction of antiferromagnetic edges $\xi(\gamma)$. Error bars indicate the standard deviation over the 17 experimental runs.

We examine community structure as a function of the transformed resolution parameter $\xi(\gamma)$, which we vary between 0 and 1. The optimized modularity, the mean size of communities, and the variance in community size all change gradually for most of the $\xi(\gamma)$ range (see Fig. 3B,F,H), although abrupt changes are evident for very low and very high values of $\xi(\gamma)$. The gradual change hints at an interesting size scale, which occurs in partitions that contain about 50–250 communities (with a characteristic size of roughly 2–8 particles). One possibility is that this size corresponds to the width of a shear band, which arise in a variety of materials with particulate structure [42]. Another possibility is that this size corresponds to the ‘cutting’ length scale $\ell^*$ [43], which is set by a community size at which the excess (over-constrained) number of contacts in the bulk of a region is equal to the number of contacts around the perimeter. If this latter association is correct, then the mean number of particles per community would scale with the confining pressure. Future experiments can test this hypothesis.

C. Geography of Community Structures

Using modularity optimization [36–38, 44], we find that the force-chain network exhibits geographically-constrained community structure: groups of particles in close spatial proximity are more likely to be a part of the same community (i.e., to contact one another with a large force) than particles that are farther apart. We examine this local neighborhood structure over a variety of size scales by varying the resolution parameter $\gamma$. We show representative results for large spatial scales in Fig. 4A,B,C. We also note that the communities that we identify in granular force networks resemble those in spatial entities like states or countries, whose borders are determined in part by physical boundaries between neighboring geographic domains.

Importantly, because the optimization of $Q$ is NP-hard [45], one does not expect an optimization algorithm to give a global optimum of $Q$. Instead, we harness numerous near-degeneracies [46] among good local optima of $Q$ by estimating $Q$ 100 times. We find that the these 100 values of $Q$ for a given run at a given $\gamma$ vary by approximately $1 \times 10^{-14}$, and the similarity in particle assignments to communities is approximately 0.98. We quantify this using partition similarity [47, 48], which ranges from 0 (not similar at all) to 1 (identical). These results indicate that the local geographic structures that we are identifying in the 2D granular system are robust, suggesting the potential for identifying reproducible 2D ‘geographic’ regions.

To probe the role of each particle in the community structure of a force-chain network (see Fig. 4D), we use the NEWintracommunity strength $z$-score $z_i$ to measure how well connected a node is to other particles in its community and the participation coefficient $P_i$ to measure how the connections emanating from a particle are...
spread among particles in the different communities [49].

The intracommunity strength z-score is

\[ z_i = \frac{S_{gi} - \bar{S}_{gi}}{\sigma_{S_{gi}}}, \]

where \( S_{gi} \) denotes the strength (i.e., total edge weight) of node \( i \)’s edges to other nodes in its own community \( g_i \), the quantity \( \bar{S}_{gi} \) is the mean of \( S_{gi} \) over all of the nodes in \( g_i \), and \( \sigma_{S_{gi}} \) is the standard deviation of \( S_{gi} \) in \( g_i \). The strength of node \( i \) is denoted by \( S_i \) and gives the total force of all contacts on particle \( i \).

The participation coefficient is [49]

\[ P_i = 1 - \sum_{g=1}^{N_m} \left( \frac{S_{ig}}{S_i} \right)^2, \]

where \( S_{ig} \) is the strength of edges of node \( i \) to nodes in community \( g \) [49].

In Fig. 4E,F, we show the intracommunity strength z-score and participation coefficient for the community structure depicted in Fig. 4A. Particles that are geographically central to a community tend to have higher values of \( z_i \) and lower values of \( P_i \) than particles at the geographic periphery of a community. From a physical perspective, \( z_i \) tends to be highest in particles with many force chains passing through them (compare, e.g., Figs. 4D and 4G) and high values of \( P_i \) are associated with the boundaries between communities (where there are few force chains).

We test whether the observed properties of community structure in our granular systems are related statistically to the inter-particle forces that constitute the force-chain structure (see Fig. 4D,G) by examining the relationship between intracommunity strength z-score \( z_i \), participation coefficient \( P_i \), the normalized node strength \( S_i' = S_i/N, \) (i.e., the mean force of all edges emanating from a node) and the amplitude of the acoustic signal \( \Delta I \). We use the Spearman rank correlation coefficient \( P \), which is defined as the Pearson correlation coefficient between ranked variables. We use the Spearman coefficient rather than the Pearson coefficient due to the non-normal distributions of \( \Delta I \) values over particles.

We find that the mean force of all contacts on a particle (i.e., \( S_i' \)) is significantly positively correlated with \( z_i \) (see Fig. 4H). The mean value of the Spearman rank correlation coefficient \( P \) over experimental runs and resolution-parameter values is \( P \approx 0.71 \pm 0.02 \) (where 0.02 gives the standard deviation over experimental runs). This strong positive correlation indicates that particles at the centers of communities are likely to have more or stronger force chains running through them. We also find that \( S_i' \) is negatively correlated with \( P \) (Fig. 4I). The mean force over experimental runs and values of the resolution parameter is \( P \approx -0.05 \pm 0.04 \), where we again take the standard deviation only over experimental runs. This negative correlation indicates that inter-community boundaries occur at particles with fewer or weaker force chains. Note additionally that a large fraction of the particles have \( P = 0 \).

This is a consequence of the fact that the communities are geographically constrained such that the majority of particles have contacts only within their own community.

The relationship between \( z, P, \) and \( S' \) is expected mathematically. For example, if the edges of node \( i \) all lie within its own community, then \( S' \) and \( z \) are related linearly according to the following equation:

\[ S_i = z_i \times \sigma_{S_{gi}} + \bar{S}_{gi}, \]

where \( S_{gi} \) is the strength of edges of node \( i \) to other nodes in its community \( g_i \), and \( \bar{S}_{gi} \) is the mean of \( S_{gi} \) over all of the nodes in \( g_i \). This linear relationship is evident for the four communities that we show in Fig. 4H. Nodes whose connections span more than one community (so-called ‘boundary nodes’, for which the value of \( P \) is greater than 0) are not so simply related.

D. Signal Propagation on Force-Weighted Contact Networks

Previous work in Ref. [22] has shown that the propagation of acoustic signals is facilitated along strong force chains in granular materials, via the increased contact area at strong contacts. With this in mind, we test whether the geographic community structure of force-chain networks is related to signal propagation. As the example shown in Fig. 5A indicates, we found that \( z \), which we measured over a range of size scales associated with resolution-parameter values \( \gamma \in [0.001, 100] \), is significantly correlated with the signal amplitude \( \Delta I \). (For this example run, \( \rho \approx 0.57 \) and the p-value is \( p \approx 2.1 \times 10^{-45} \).) The statistical correlation between network structure and signal amplitude exists not only in the highly heterogeneous signal injection phase, in which sound propagates from the driver to nearby particles, but also in the more homogeneous scattering phase, in which sound reverberates throughout the system. In Fig. 5B, we show how the statistical correlation between network structure and signal propagation exists both for large \( \gamma \) values, and time is 1.34 ± 0.06. This suggests that similar dynamic principles underlie sound propagation in both injection and scattering phases. We discuss the dynamics within the two phases in more detail in the next section.

D. Signal Propagation on Force-Weighted Contact Networks
FIG. 4. [Color online] The geographic sizes of communities tend to decrease as the resolution parameter $\gamma$ is increased from (A) low ($\gamma = 0.1$) to (C) high $\gamma = 1$ values. We color particles according to their community label. One can examine community structure of the (D) force-chain network using geographic location, (E) intracommunity strength $z$-score $z_i$, (F) participation coefficient $P_i$, and (G) mean force per particle $S'_i$. Example scatterplots for a single experimental run showing that mean force $S'_i$ on particle $i$ is (H) positively correlated with the intracommunity strength $z$-score (the Spearman correlation coefficient for this run is $\rho \approx 0.96$ and the p-value is $p \approx 1.9 \times 10^{-285}$) and (I) negatively correlated with the participation coefficient ($\rho \approx -0.18$ and $p \approx 1.8 \times 10^{-16}$). In panel (H), nodes assigned to communities 1 through 4 and whose participation coefficients are equal to 0 are displayed using different colored markers. Nodes in any of the 4 communities whose participation coefficients are greater than 0 (so-called ‘boundary nodes’) are displayed using dark (purple) markers. The mean correlations for $z$ and $P$ over experimental runs and values of the resolution parameter are $\rho \approx 0.87 \pm 0.08$ and $\rho \approx -0.16 \pm 0.05$, respectively. We show the results for one experimental run (run #2) in this figure, and the results for the other runs are similar.

strong correlation between intracommunity strength $z$-score and the mean force (normalized strength) of a particle (see Fig. 4H), the latter of which is a particlescale measurement and is independent of spatial resolution. The relationship between the meso-scale (community structure) and particle-scale (mean force on a particle, which is equal to a node’s normalized strength) network properties stems from the physical embedding of the granular system in $\mathbb{R}^2$. A particle that is located geographically inside of a community has all of its connections to other particles in its community because it is constrained to connect only to its geographic neighbors (i.e., there can be no long-range contacts). This is unlike most investigated real-world networks [36, 37], in which communities tend to be highly interconnected and most nodes have at least some connections to nodes in other communities.

To assess whether community structure is unique in its ability to predict signal amplitude, we also examine other weighted network diagnostics that are sensitive to different system dimensionalities (see Fig. 2). Our results suggest that community structure (see Fig. 2B) is a better predictor of signal propagation than system-scale (e.g., global efficiency; see Fig. 2A), curve-scale (e.g., geodesic node betweenness; see Fig. 2C), and particle-scale (e.g., clustering coefficient; see Fig. 2D) network diagnostics. See Appendix A for mathematical definitions and intuitive descriptions. In particular, clustering coefficient and $\Delta J$ are not strongly correlated, so triangles of contacts do not appear to be important for signal propagation. See the comparison in Fig. 5D.
FIG. 5. [Color online] (A) An example scatterplot between logarithms of the intracommunity strength z-score \( \log_{10}(z+5) \) and the amplitude of the acoustic signal \( \log_{10}(\Delta I) \) at \( t = 1 \) for a single experimental run \( j = 2 \) and resolution-parameter value \( \gamma = 0.1 \). The constant 5 was added to \( z \) to ensure that all values were positive prior to taking the logarithm. The Spearman rank correlation coefficient is \( \rho \approx 0.57 \) and the \( p \)-value is \( p \approx 2.1 \times 10^{-45} \). (B) Correlation between \( \log_{10}(z+5) \) and \( \log_{10}(\Delta I) \) for all 17 runs as a function of time: \( t < 40 \) is the acoustic signal injection phase, and \( t > 40 \) is the acoustic signal scattering phase. In the bottom part of panel (B), we show a trace of the voltage \( V \) of a piezo particle; the injected signal has an amplitude of 0.5V. (C) Correlation between \( \log_{10}(z+5) \) and \( \log_{10}(\Delta I) \) as a function of \( \gamma \), where \( \gamma \in [0, 2] \) is increased in increments of 0.1. We averaged the correlation over all 80 time points at which \( \Delta I \) was measured, and the mean \( \rho \) over runs \( \gamma \) values, and time was 0.34 ± 0.06. Box plots show the variability over experimental runs. (D) Correlation between \( \log_{10}(\Delta I) \) and a variety of network diagnostics: intracommunity strength z-score \( z = z(\gamma) \) for \( \gamma = 0.001 \), \( \gamma = 10 \), and \( \gamma = 100 \); and weighted (white background, left of the figure) and unweighted (light gray background, middle of the figure) versions of global efficiency \( [E_w(i) \text{ and } E(i)] \), geodesic node betweenness \( [B_w(i) \text{ and } B(i)] \), and clustering coefficient \( [C_w(i) \text{ and } C(i)] \). For completeness, we also show results for the mean force \( S' \) (left of the figure), which was the variable previously reported to be correlated with \( \Delta I \) [22].

E. Phase Sensitivity of Network Diagnostics

Although the correlation between \( z \) and signal amplitude is strong in both injection and scattering phases for small \( \gamma \) (i.e., large community size), it is higher in the scattering phase than in the injection phase when averaged over all resolutions (\( \gamma \in [0.001, 100] \); see Fig. 6B,E) We do not observe such sensitivity to phase for the mean force per particle (see Fig. 6A,D). Interestingly, the signal propagation during the injection phase is more strongly correlated with the global efficiency than it is during the scattering phase (see Fig. 6C,F), suggesting that the acoustic signal propagates over the shortest weighted paths during the injection phase. These results illustrate insights from network analysis that one cannot obtain from particle-scale measurements: signal propagation during injection is well characterized by shortest paths that span the system, whereas it is characterized by local neighborhood structure during scattering. An interesting question is whether the amplitude of the injected signal affects the size of the geographic neighborhood through it propagates.

We also examine the sensitivity of the relationship between \( z \) and \( \Delta I \) to the injection and scattering phases as a function of the resolution parameter (see Fig. 7A). The correlation between \( z \) and signal amplitude is consistently higher in the scattering phase than in the injection phase throughout \( \gamma \in [0.001, 100] \). Furthermore, the largest difference in the Spearman correlation between \( z \) and \( \Delta I \) for the scattering versus injection phases occurs for partitions with approximately 50 – 100 communities, corresponding to community sizes of roughly 5 – 8 particles (see Fig. 7B). This is similar to the size scale that we identified previously when using the transformed resolution parameter.

III. DISCUSSION

A networks perspective provides a useful framework in which to study the material and dynamic properties of granular materials. Network diagnostics vary in their sensitivity to scales of the granular system: the particle-scale can be probed with a clustering coefficient, the curve-scale can be probed with geodesic node betweenness, the domain-scale can be probed with community structure, and the system-scale can be probed with global efficiency. Moreover, one can identify potentially interesting length/size scales in the system using meso-scale network features such as community structure. As we show in Appendix C, one can also obtain physical insights into the geographic organization of the material by comparing the features of the actual networks to a null model consisting of an ensemble of random geometric graphs.

The dynamics of signal propagation on a network are best characterized by weighted diagnostics derived from the granular force-chain network, suggesting that the topology of the underlying (unweighted) contact network alone is not sufficient to explain signal propagation. In other words, one must also consider network geometry. This result underscores the important relationship between signal propagation and force-chain organization (see Fig. 5D). Similar phenomena are likely relevant for a variety of energy transport problems (e.g., in sound, heat, and electricity) in a broad class of amorphous materials. Although real 3D granular systems are not photoelastic, recent advances in tomography [50, 51] have
FIG. 6. [Color online] Spearman correlations between signal amplitude $\Delta I$ and (A) the mean force per particle $S'_\gamma$, (B) the intracommunity strength z-score $z$ for $\gamma = 2$, and (C) the global efficiency $E_{\mathrm{w}}$. Our data encompasses all experimental runs and for all times, including both injection (left) and scattering (right) phases. We show the Spearman correlations between signal amplitude $\Delta I$ and the three diagnostics shown in panels (A)-(C) in box plots: (D) mean force per particle ($\rho_{\mathrm{w}}$), (E) intracommunity strength z-score $z$ ($\rho_z$), and (F) global efficiency ($\rho_{E_{\mathrm{w}}}$). We have averaged the correlations that we show in the box plots over the injection (left) and scattering (right) phases. For (E), note that we also average the correlations over the resolution parameter $\gamma$. Using MATLAB notation, the precise values of $\gamma$ that we considered are $0.001 : 0.001 : 0.009, 0.01 : 0.1 : 1, 2 : 3, 4 : 0.1 : 20, 30 : 10 : 100, 200 : 100 : 1000$. The reported p-values indicate the results of 2-sample t-tests.

The presence of correlated regions such as geographic communities in a granular material is reminiscent of shear transformation zones (STZs [54]), in which local-
ized regions throughout a sheared material have a higher propensity to deform under shear. Importantly, however, the community structure that we compute spans the system, whereas STZs are relatively small structures dispersed throughout the system. Also of interest is a comparison with the results of Ref. [55], which illustrated that vibrational modes can identify soft spots in sheared systems.

In conclusion, using network analysis to study granular materials can be extremely useful, as it can help characterize particle, curve, domain, and system-scale properties of such materials. In particular, the algorithmic detection of communities provides a means to identify potentially interesting characteristic size scales in such systems. When combined with time-resolved acoustic measurements [22], such a networks perspective can illuminate the meso-scale structures within which sound travels preferentially. We found that particles that are well connected to their community have larger-amplitude signals passing through them. Our results also suggest that signals scatter in local geographic neighborhoods but propagate more systemically during signal injection. Investigation of both weighted and unweighted networks demonstrates that a weighted network is a better predictor of sound propagation, suggesting that the force-network structure of the granular material is an important component in sound propagation. Our results demonstrate that one cannot examine only system-scale or local-scale network features to understand how sound travels through a granular material. Importantly, one achieves a better description of sound propagation when one includes how the particles relate to their neighbors in a network.

Acknowledgments We thank Lee C. Bassett for helpful insights and Jean Carlson, Aaron Clauset, Wolfgang Losert, Peter Mucha, Colin McDiarmid, Zohar Nussinov, and Marta Sarzynska for useful comments. D.S.B. was supported by the David and Lucile Packard Foundation, Public Health Service Grant NS44393, the Institute for Collaborative Biotechnologies through Contract W911NF-09-D-0001 from the US Army Research Office, and the National Science Foundation (Division of Mathematical Sciences-0645369). E.T.O and K.E.D were supported by a NSF CAREER award DMR-0644743. M.A.P. acknowledges the Statistical and Applied Mathematical Sciences Institute, the Kavli Institute for Theoretical Physics, and a research award (#220020177) from the James S. McDonnell Foundation.

IV. APPENDIX A: DEFINITIONS OF NETWORK DIAGNOSTICS

Diagnostics Applied to Unweighted Contact Networks

To characterize structure of the binary (contact) networks, we examined 21 diagnostics: number of nodes, number of edges, global efficiency [56], geodesic node betweenness centrality [57], random-walk node betweenness [59], geodesic edge betweenness [60], eigenvector centrality [61], closeness centrality [62], subgraph centrality [63], communicability [64], clustering coefficient [65], local efficiency [56], modularity optimized using two different algorithms [37–39, 44], hierarchy [66], synchronizability [67], degree assortativity [68], robustness to targeted and random attacks [69], the Rent exponent [70], and mean connection distance [71].

In our descriptions below, we give for each diagnostic (i) a mathematical definition, (ii) an intuitive description of the term, and (iii) a comment on its possible physical significance for the granular system that we study. We also computed node-specific values for the following diagnostics: geodesic betweenness, global efficiency, and clustering coefficient. (See the discussions below.)

1. Number of nodes \( N \): (i) The diagnostic \( N \) is defined as the number of nodes in a network. (ii) It is used as a measure of the size of a system. (iii) In this study, \( N \) is the number of particles in the system. It provides a consistent but dynamically uninteresting characterization of the network because it is identical at all points in time.

2. Number of edges \( D \): (i) The diagnostic \( D \) is defined as
\[
D = \sum_{i,j} A_{ij},
\]
where \( A \) is an unweighted (binary) network with components \( A_{ij} \). Nodes are particles, and an edge exists between particles \( i \) and \( j \) (i.e., \( A_{ij} = 1 \)) if and only if particles \( i \) and \( j \) are in contact with each other (otherwise, \( A_{ij} = 0 \)). (ii) The quantity \( D \) is simply the total number of edges in the system. (iii) The number of edges \( D \) is related to the mean contact number, which is denoted by \( z \) in the granular-materials community. The mean contact number of the system is equal to \( z = D/N \). The diagnostic \( D \) provides a consistent but uninteresting characterization of the network because the number of contacts scales with pressure [2] (which is the same for all experimental runs).

3. Global efficiency \( E \) [56]: (i) Let \( d_{ij} \) be the shortest (geodesic) number of steps necessary to get from node \( i \) to node \( j \). The global efficiency is then defined as
\[
E = \frac{1}{N(N-1)} \sum_{i \neq j} \frac{1}{d_{ij}}.
\]
(ii) Global efficiency can be interpreted as a measure of how well a signal is transmitted through a network. (iii) One can expect the global efficiency to be small in 2D granular packings because particles that are not geographically close to one another are separated by multiple contacts (edges) and therefore by a long path length (low efficiency). As one can see in Table III, this is indeed the case.
4. Geodesic node betweenness $B$ [57]: (i) Geodesic node betweenness is defined for the $i^{th}$ node in a network $\mathcal{G}$ as

$$B_i = \sum_{j,m \in \mathcal{G}} \frac{\psi_{j,m}(i)}{\psi_{j,m}}$$

where all three nodes ($j$, $m$, and $i$) must be different from each other, $\psi_{j,m}$ is the number of geodesic paths between $j$ and $m$, and $\psi_{j,m}(i)$ is the number of geodesic paths between $j$, $m$, and $i$ that pass through node $i$. The geodesic betweenness of an entire network $B$ is defined as the mean of $B_i$ over all nodes $i$ in the network. (ii) Geodesic betweenness can be interpreted as a measure of traffic flow on a network. (iii) One might expect the majority of geodesic paths that link any node of the packing to any other node to pass through the middle of the system. Indeed, we find that the largest values of betweenness occur in the center of the system and the smallest values along the edges of the packing.

5. Random-walk node betweenness $B_{rw}$ [59]: (i) For an adjacency matrix $A$ and diagonal matrix $D$, let $M_t = A_t \cdot D_t^{-1}$ be the matrix $M$ with the row and column $t$ removed (and $A_t$ and $D_t$ are defined analogously). The probability that a walk starts at $s$, takes $n$ steps, and ends up at some node $i$ (which cannot be $t$ because $t$ has been removed) is given by element $is$ of $M^n_s$; denote this element by $[M^n_s]_{is}$. Walks end up at $v$ and $w$ with probabilities $[M^n_s]_{vs}$ and $[M^n_s]_{ws}$. Fractions $1/k_v$ and $1/k_w$ of these walks subsequently pass along the edge $(v, w)$ in one direction or the other, assuming that such an edge exists. (Note that $k_v$ is the degree of $v$ and $k_w$ is the degree of $w$.) Summing over all $n$ shows that the mean number of times that a walk of any length traverses the edge from $v$ to $w$ is $k_v^{-1}[(I - M_t)^{-1}]_{vs}$. The random-walk betweenness of a node is the mean of this quantity over all edges emanating from that node, and the random-walk betweenness of the entire network is the mean of the random-walk betweenness of all nodes in the network. (ii) Random-walk betweenness can be interpreted as a measure of information flow or signal flow in a network. (iii) Similar to geodesic node betweenness, one might expect the random-walk node betweenness to be highest in the center of the system and lowest on the edges of the system. This is indeed the case.

6. Geodesic edge betweenness $B_e$ [60]: (i) Inspired by Freeman’s geodesic node betweenness, the geodesic edge betweenness of an edge is defined as the number of shortest paths between pairs of nodes that run along it. For the edge connecting nodes $j$ and $m$, the geodesic edge betweenness is given by

$$B_e(j, m) = \sum_{i,k} \psi_{i,k}(j, m),$$

where $\psi_{i,k}(j, m)$ is the number of shortest paths between $i$ and $k$ that pass through the edge connecting nodes $j$ and $m$. (ii) One can interpret edge betweenness as a measure of the influence of an edge on traffic flow through a network. (iii) In a 2D granular packing, edge betweenness might indicate the influence of a contact on a hypothetical flow through the network. In our system, we find that edge betweenness is largest in the center of the system and smallest on the edges of the system. This is consistent with the results for the geodesic node betweenness.

7. Eigenvector centrality $C_e$ [72]: (i) The eigenvector centrality $C_e(i)$ of node $i$ is proportional to the sum of the centralities of the nodes connected to it:

$$C_e(i) = \frac{1}{\lambda} \sum_{j \in M(i)} C_e(j) = \frac{1}{\lambda} \sum_j A_{ij} C_e(j),$$

where $M(i)$ is the set of nodes that are neighbors of $i$ (i.e., which are connected to $i$ directly via an edge) and $\lambda$ is the largest eigenvalue of $A$. From Eq. (8), one can deduce that $C_e(i)$ is the $i^{th}$ component of the leading eigenvector (each entry of which is positive by the Perron-Frobenius theorem [1]) of the adjacency matrix. (ii) Eigenvector centrality can be used to measure the importance of a node in a network based on its direct connection to important nodes. (iii) In a 2D granular packing, one would expect eigenvector centrality to be large for a particle that has many contacts or for a particle whose immediate neighbors have many contacts. Indeed, we find that eigenvector centrality is highest in a local region of the system in which high-degree nodes are most concentrated.

8. Closeness centrality $C_c$ [62]: (i) We use a version of closeness centrality that is appropriate for both connected and disconnected graphs [73]. It is defined as

$$C_c(i) = \sum_{j \in V/i} 2^{-\psi_G(i,j)},$$

where $\psi_G(i,j)$ is the geodesic distance between nodes $i$ and $j$ (i.e., the length of the shortest path connecting $i$ and $j$) and the notation $V/i$ indicates that $V$ is the connected network component reachable from $i$ and does not include $i$. (ii) Closeness centrality can be used as a measure of the importance of a node in a network. (iii) For 2D granular systems, one might expect closeness centrality to be small given the lattice-like topology of a contact network. However, as shown in Table III, closeness
values are somewhat larger than those for random geometric graphs (RGGs; see the discussion in Appendix VI.

9. Subgraph centrality $C_s$ [63]: (i) We first note that the number of closed walks of length $k$ starting and ending at node $i$ is given by the $k^{th}$ local spectral moment $\mu_k(i)$, which is defined as the $i^{th}$ diagonal entry of the $k^{th}$ power of the adjacency matrix $A$:

$$\mu_k(i) = [A^k]_{ii}.$$  \hfill (10)

The subgraph centrality of node $i$ is then defined as

$$C_s(i) = \sum_{k=0}^{\infty} \frac{\mu_k(i)}{k!}.$$ \hfill (11)

(ii) Subgraph centrality characterizes the participation of each node in all subgraphs in a network. (iii) For 2D granular systems, one might expect subgraph centrality to be small because nodes participate in few subgraphs other than their own, as their connectivity is strongly constrained to their local geographic neighborhood. Indeed, as indicated in Table III, the subgraph centrality has a value that is less than $1/3$ that of the value that we computed for a corresponding ensemble of RGGs (see our later discussion).

10. Communicability $Co$ [64]: Because of the direct relationship between the powers of the adjacency matrix $A$ and the number of walks in a network, one can define the communicability between nodes $i$ and $j$ as

$$Co_{ij} = \sum_{k=0}^{\infty} \frac{[A^k]_{ij}}{k!}.$$ \hfill (12)

The communicability $Co$ of a network is then the mean of the communicabilities of each pair of (non-identical) nodes. (ii) Communicability was developed to measure the ease of communication or transmission in terms of passage between different nodes in a network, and it is specifically based on walks rather than paths [64]. (iii) For 2D granular systems, one might expect the mean communicability to be small because the geographic nature of the contacts creates a lattice-like topology. Indeed, as indicated in Table III, its value is less than $1/4$ than the value that we computed for a corresponding ensemble of RGGs.

11. Clustering coefficient $C$ [65]: (i) The diagnostic $C$ is defined by supposing that a node $i$ has $k_i$ neighbors, so a maximum of $k_i(k_i - 1)/2$ edges can exist between these neighbors. The local clustering coefficient $C_i$ is the fraction of these possible edges that actually exist:

$$C_i = \frac{\sum_{m,j} A_{mj} A_{im} A_{ij}}{k_i(k_i - 1)}.$$ \hfill (13)

The clustering coefficient $C$ of an entire network is then defined as the mean of $C_i$ over all nodes $i$. (ii) The clustering coefficient $C$ can be interpreted as a measure of local clustering properties in a network. (iii) One can expect $C$ to be large in 2D granular packings because particles that are geographically close to one another are also near each other in a network. This ought to yield a large number of connected triples and hence a high value of $C$. As shown in Table III, we do indeed observe reasonably large values [74] for clustering coefficients in the 17 experimental runs (the mean value over all runs is $C \approx 0.26$), but interestingly the mean value of $C$ in the corresponding RGG ensemble is twice as high.

12. Local efficiency $E_l$ [56]: (i) The local efficiency of node $i$ is defined as

$$E_l(i) = \frac{1}{N_G(i)(N_G(i) - 1)} \sum_{j,k \in E_i} \frac{1}{d_{jk}},$$ \hfill (14)

where $G_i$ is the subgraph consisting of all nodes connected to node $i$ along with all of their edges between each other, and $d_{jk}$ is the minimum path length between nodes $j$ and $k$ in this subgraph. The local efficiency $E_l$ is the mean value of $E_l(i)$ over all nodes $i$. (ii) Local efficiency $E_l$ can be interpreted as a measure how well a signal is transmitted through a subgraph. (iii) One might expect local efficiency to be large in 2D granular packings because particles that are very close to each other geographically lie in one another’s subgraphs. However, as we show in Table III, we obtain values that are only about half of those for corresponding RGGs. The mean granular-network value of 0.33 is comparable in value to some communication networks [56].

13. Modularity index $Q$ [36–38]: (i) Networks can be partitioned into communities (or modules) in which nodes inside the same community are more densely connected to each other than they are to nodes in other communities. The modularity of a network partition is defined as

$$Q = \frac{1}{2D} \sum_{ij} \left[ A_{ij} - \frac{k_i k_j}{2D} \right] \delta_{g_i, g_j},$$ \hfill (15)

where $k_i$ is the degree of node $i$, $D$ is the total number of edges in the network, $\delta_{ij}$ is the Kronecker delta, and $g_i$ is the community to which node $i$ has been assigned. With the standard null model $P_{ij} = k_i k_j/(2D)$, Eq. (15) is sometimes called ‘Newman-Girvan modularity.’ One uses one of numerous possible computational heuristics to maximize $Q$ in the space of all network partitions, and one can then report the maximum value obtained for $Q$. However, it is important to note that the optimization of $Q$ is NP-hard [45], so one
cannot expect the output of an optimization algorithm to be a globally optimal partition of a network. In this light, we use two different computational heuristics to optimize $Q$: Newman’s spectral algorithm [44] (which yields a modularity value that we denote $Q_s$) and the Louvain locally greedy method [39] (yielding a modularity value that we denote $Q_L$). (ii) The optimal value of $Q$ is a measure of how well a network can be partitioned into cohesive communities. (iii) In a 2D granular system, one might expect communities to be localized geographically because connectivity between nodes in potential communities is constrained geographically. Indeed, as shown in Table III, the values of $Q_s$ and $Q_L$ are both extremely high [36, 37].

14. Hierarchy $h$ [66]: (i) A sense of hierarchical structure in a network can be characterized by the coefficient $h$, which is used to quantify a putative power-law relationship between clustering coefficient $C_i$ and the degree $k_i$ of all nodes in the network [66]:

$$C_i \sim k_i^{-h}. \quad (16)$$

(ii) Networks in which clustering coefficient has a power-law scaling with degree possess a hierarchy in which hubs (i.e., high-degree nodes) tend to have low local clustering and low-degree nodes tend to have high local clustering. The parameter $h$ gives a precise scaling of such effects when (16) holds, and it can perhaps indicate a looser sense of hierarchy in more general situations. (iii) It is not clear a priori whether 2D granular packings should display degree assortativity. Our calculations indicate that the degrees exhibit some mild positive assortativity ($a \approx 0.14$), but the corresponding RGG ensembles have a significantly higher positive assortativity of $a \approx 0.56$. (See Table III.)

15. Synchronizability $s$ [67]: (i) The synchronizability is defined as

$$s = \frac{\lambda_2}{\lambda_N}, \quad (17)$$

where $\lambda_2$ is the second smallest eigenvalue of the Laplacian $L$ of the adjacency matrix and $\lambda_N$ is the largest eigenvalue of $L$ [67]. (ii) The synchronizability of a network characterizes structural properties of a network that hypothetically enable it to synchronize rapidly. (iii) One might expect that the synchronizability of the contact network in a 2D granular packing is small due to the lattice-like nature of the network topology. Indeed, as shown in Table III, the value for $s$ for our system is tiny.

16. Degree assortativity $a$ [68]: (i) The degree assortativity of a network (which is often called simply ‘assortativity’) is defined as

$$a = \frac{E^{-1} \sum_i j_i k_i - \left[E^{-1} \sum_i \frac{1}{2} (j_i + k_i)^2 \right]}{E^{-1} \sum_i \frac{1}{2} (j_i^2 + k_i^2) - \left[E^{-1} \sum_i \frac{1}{2} (j_i + k_i)^2 \right]^2}, \quad (18)$$

where $j_i$ and $k_i$ are the degrees of the nodes at the two ends of the $i^{th}$ edge ($i \in \{1, \ldots, E\}$ [44]. (ii) Degree assortativity measures the preference of a node to connect to other nodes of similar degree (leading to an assortative network, for which $a > 0$) or to nodes of very different degree (leading to a disassortative network, for which $a < 0$). (iii) It is not clear a priori whether 2D granular packings should display degree assortativity. Our calculations indicate that the degrees exhibit some mild positive assortativity ($a \approx 0.14$), but the corresponding RGG ensembles have a significantly higher positive assortativity of $a \approx 0.56$. (See Table III.)

17. Robustness $R$ [69]: (i) One can define robustness in terms of different types of attacks on a network. In the most commonly studied type of targeted attack, nodes are removed (one by one) in descending order of their degree; in a random attack, nodes are removed in random order. Each time a node is removed from a network, we re-calculate the size $S$ (i.e., number of nodes) of the largest connected component. One can examine robustness by plotting $S$ versus the number of nodes removed $n$ [75–77]. One can then define a robustness parameter $R$ as the area under the curve in the plot of $S = S(n)$. More robust networks retain a larger connected component even when several nodes have been removed; this is represented by a larger area under the curve and hence by larger values of $R$. (ii) Robustness indicates network resilience to either targeted ($R_t$) or random ($R_r$) attacks. (iii) Robustness tends to be most interesting for networks with highly heterogeneous degree distributions, so it might not be very insightful for 2D granular packings. We note, however, that we find values of $R_t$ and $R_r$ for our granular networks that are more than 3 times as large as those for the corresponding RGG ensemble. (See Table III.)

18. Rent exponent $p$ [78]: (i) Rent’s rule, which was first discovered in relation to computer chip design, defines a scaling relationship between the number of external signal connections (edges) $e$ to a block of logic and the number of connected nodes $n$ in the block [78]:

$$e \sim n^p, \quad (19)$$

where $p \in [0, 1]$ is the Rent exponent. (ii) The Rent exponent measures the efficiency of the physical embedding of a topological structure. (iii) Due to the physical constraints of a 2D granular packing, we expect to observe Rentian scaling with a
relatively low Rent exponent (similar to that of a lattice). The theoretically expected minimum of the Rent’s exponent for a 2-dimensional physical lattice is \( p_r = 1 - 1/D_E \) [79–81] where \( D_E \) is the Euclidean dimension of the space (e.g., 2), and therefore \( p_r \approx 0.5 \), which is consistent with empirical results on memory circuits [82]. However, the expected value of the Rent exponent might further depend on the type of physical lattice under study (e.g., a rectangular or hexagonal lattice).

19. Mean connection distance \( \text{med} \): (i) An edge’s estimated connection distance \( L_{ij} \) is defined as the Euclidean distance between the centroids of particles \( i \) and \( j \). (ii) The mean connection distance \( \text{med} \) is defined as the mean of all \( L_{ij} \) values in the network. (iii) The mean connection distance in a 2D granular packing is related to the number of particles, the area of the system, and particle size.

Diagnostics Applied to Force-Weighted Contact Networks

To characterize the structure of the force-weighted networks, we used 8 diagnostics: normalized strength [83, 84], diversity [85], path length [86], geodesic node betweenness [57, 84], geodesic edge betweenness [60], clustering coefficient [84], transitivity [87], and optimized modularity [38].

1. Normalized Strength \( S' \) [83, 84]: (i) The strength of node \( i \) is given by the column sum of the weighted adjacency matrix:

\[
S_i = \sum_j W_{ij},
\]

and the strength of an entire weighted network \( \mathbf{W} \) is the mean of \( S_i \) over all \( i \). The normalized strength \( S'_i \) is

\[
S'_i = \frac{S_i}{N},
\]

where \( N \) is the total number of nodes. (ii) Strength is a measure of how strong the connections are in a network. (iii) In the present context, normalized strength provides a measure of the mean contact forces between particles, and we therefore expect this diagnostic to be correlated with sound propagation. Indeed, we observe this in our calculations.

2. Diversity \( V \) [85]: (i) The diversity of node \( i \) is defined as the variance of the edge weights for the set of all edges connected that are connected to it. It is given by

\[
V_i = \left[ \frac{1}{N} \sum_j (W_{ij} - \langle W_i \rangle)^2 \right]^{1/2}.
\]

The diversity of an entire weighted network is the mean of \( V_i \) over all \( i \). (ii) Diversity is a measure of the variance of connectivity strengths in a network. (iii) In the present context, diversity is a measure of the variance of contact forces between particles, and it has a high positive correlation with normalized strength.

3. Global efficiency \( E_w \) [56]: (i) Let \( d_{ij}^w = \max(W_{ij}) - W_{ij} \) be the weighted shortest path between nodes \( i \) and \( j \). The global efficiency of node \( i \) is then defined as

\[
E_w(i) = \frac{1}{N-1} \sum_{j \neq i} \frac{1}{d_{ij}^w}.
\]

The global efficiency \( E_w \) is the mean value of \( E_w(i) \) over all nodes \( i \). (ii) One can interpret global efficiency as a measure of how efficiently a signal is transmitted through a network. (iii) We expect the global efficiency of the force-weighted contact network to be large in the center of the packing and small on the edges of the packing because particles that are not geographically close to each other do not exert forces on one another. Indeed, this is what we observe.

4. Clustering coefficient \( C_w \) [84]: (i) One can define a weighted clustering coefficient \( C_w(i) \) of node \( i \) using the formula

\[
C_w(i) = \frac{1}{S_i(k_i-1)} \sum_{j,k} \left( \frac{W_{ij} + W_{ik}}{2} \right) A_{ij} A_{ik} A_{jk},
\]

where \( S_i \) is node \( i \)'s strength, \( k_i \) is its degree, \( \mathbf{W} \) is the weighted adjacency matrix, and \( \mathbf{A} \) is the underlying binary adjacency matrix. (ii) The weighted clustering coefficient \( C_w(i) \) measures the strength of local connectivity. (iii) It is constrained by the underlying contact network structure, so we expect it to have a high positive correlation with the binary clustering coefficient \( C(i) \). Indeed, the Pearson correlation coefficient between the binary and weighted clustering coefficients over the experimental runs is \( r \approx 0.94 \) (with a \( p \)-value of \( p \approx 2 \times 10^{-9} \)). Both diagnostics tend to attain their highest values on the edges of the packing, where nodes' immediate neighbors are most likely to also be connected to one another.

5. Geodesic node betweenness \( B_w \) [88]: (i) Geodesic betweenness is defined for the \( t^{th} \) node in a network \( \mathcal{G} \) as

\[
B_w(i) = \sum_{j,m \in \mathcal{G}} \tilde{\psi}_{j,m}(i),
\]

where all three nodes \( j, m, \) and \( i \) must be different from each other, \( \tilde{\psi}_{j,m} \) denotes the number
of geodesic weighted paths between nodes \( j \) and \( m \), and \( \psi_{j,m}(i) \) denotes the number of geodesic weighted paths between \( j \) and \( m \) that pass through node \( i \). (As with weighted global efficiency, the weighted shortest path between nodes \( i \) and \( j \) is \( d_{ij}^w \).) The weighted geodesic betweenness of an entire network \( B_w \) is defined as the mean of \( B_w(i) \) over all nodes \( i \). (ii) One can interpret weighted geodesic betweenness as a measure of traffic flow on a network. (iii) We expect betweenness in weighted networks to correlate positively with strength, just as betweenness in binary networks correlates positively with degree [59]. In a 2D granular packing, we expect particles in the center of the system to have high values of weighted betweenness because more paths must pass through them to connect the edges of the system. We indeed find this to be the case.

6. Geodesic edge betweenness \( B_{ew} \) [68, 88]: (i) We define geodesic edge betweenness on weighted networks using the number of shortest weighted paths between pairs of nodes that run along it. (We again determine the path distance between nodes \( i \) and \( j \) using \( d_{ij}^w \).) For the edge connecting nodes \( j \) and \( m \), the weighted geodesic edge betweenness is therefore

\[
B_{ew}(j, m) = \sum_{i,k} \tilde{\psi}_{i,k}(j, m),
\]

(26)

where \( \tilde{\psi}_{i,k}(j, m) \) is the number of shortest paths between nodes \( i \) and \( k \) that pass through the edge connecting nodes \( j \) and \( m \). (ii) Weighted edge betweenness indicates the influence of an edge on traffic flow through a network. (iii) In a 2D granular packing, the edge betweenness should give an indication of the influence of a contact on a hypothetical flow through the network. We find that edge betweenness is largest in the center of the system because more paths must pass through these edges to connect all pairs of particles.

7. Modularity index \( Q_w \) [36–38]: The weighted modularity of a network partition is

\[
Q_w = \frac{1}{2W} \sum_{ij} \left[ W_{ij} - \frac{S_i S_j}{2W} \right] \delta_{g_i, g_j},
\]

(27)

where \( S_i \) is node \( i \)'s strength, \( W \) is the total strength of the edges in a network, \( W_{ij} \) is an element of the weighted adjacency matrix, \( \delta_{ij} \) is the Kronecker delta, and \( g_i \) is the label of the community to which node \( i \) has been assigned. As with unweighted networks, one uses some computational heuristic to find a partition that maximizes \( Q \). As with the binary networks, we have used the Louvain locally greedy optimization method [39]. (ii) The maximum value of \( Q \) is a measure of how well a network can be partitioned into cohesive communities. (iii) In a 2D granular packing, in which forces between particles are represented as edge weights, we expect communities to be localized in space because the forces between nodes in potential communities are constrained geographically. Indeed, as shown in Fig. 2, this is indeed the case.

8. Transitivity \( T \) [87]: (i) The weighted transitivity \( T(i) \) of node \( i \) is

\[
T(i) = \frac{2}{k_i(k_i+1)} \sum_{j,k} \left( \tilde{W}_{ij} \tilde{W}_{jk} \tilde{W}_{ik} \right)^{1/3},
\]

(28)

where we have normalized weights by the maximum edge weight in the matrix:

\[
\tilde{W}_{ij} = \frac{W_{ij}}{\max(W_{ij})}.
\]

(29)

The transitivity \( T \) of the entire network is the mean of \( T(i) \) over all nodes \( i \). (ii) Weighted transitivity is a generalization of the local clustering coefficient in unweighted networks in which one computes the sum of the weights of edges in a network’s triangles instead of computing simply the number of triangles. (iii) We expect weighted transitivity to be similar to the weighted clustering coefficient \( C_w \). Indeed, we find that the two variables are highly correlated over experimental runs (with a Pearson correlation coefficient of \( r \approx 0.99 \) and a p-value of \( p \approx 1 \times 10^{-16} \)).

We implemented all computational and simple statistical operations (such as t-tests and correlations) using MATLAB® (2009a, The MathWorks Inc., Natick, MA). We estimated network diagnostics using a combination of in-house software, the Brain Connectivity Toolbox [89], the MATLAB Boost Graph Library, and the fast unfolding community detection code for the Louvain optimization of modularity [39] from Peter Mucha [90].

V. APPENDIX B: RELIABILITY OF NETWORK STRUCTURE

We quantify the reliability of each diagnostic by calculating the coefficient of variation (a normalized measure of dispersion) over the 17 experimental runs: \( CV = \sigma/|\mu| \), where \( \sigma \) is the standard deviation and \( \mu \) is the mean. Values of \( CV \lesssim 0.2 \) are commonly considered to be acceptable, as they indicate that a variable is reliable [77, 91, 92]. See Table I for CV values for all binary network diagnostics and Fig. 9B for a corresponding bar graph. Interestingly, reliable diagnostics are dispersed among the quantities we considered rather than focused on sets of related diagnostics.

For the force-weighted granular networks, we find lower reliability (i.e., higher values of CV) for the diagnostics than for the (binary) contact networks. Compare Fig. 8B
TABLE I. Reliability of network diagnostics for the binary contact networks. We measure reliability using coefficient of variance (CV).

<table>
<thead>
<tr>
<th>Binary Contact Network Diagnostic</th>
<th>Variable</th>
<th>CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Connection Distance</td>
<td>mcd</td>
<td>0.0046</td>
</tr>
<tr>
<td>Geodesic Edge Betweenness</td>
<td>B_e</td>
<td>0.0090</td>
</tr>
<tr>
<td>Global Efficiency</td>
<td>E_g</td>
<td>0.0114</td>
</tr>
<tr>
<td>Rent Exponent</td>
<td>p</td>
<td>0.0171</td>
</tr>
<tr>
<td>Random-Walk Node Betweenness</td>
<td>B_w</td>
<td>0.0176</td>
</tr>
<tr>
<td>Number of Nodes</td>
<td>N</td>
<td>0.0179</td>
</tr>
<tr>
<td>Geodesic Node Betweenness</td>
<td>B</td>
<td>0.0213</td>
</tr>
<tr>
<td>Modularity: Louvain Optimization</td>
<td>Q_L</td>
<td>0.0071</td>
</tr>
<tr>
<td>Modularity: Spectral Optimization</td>
<td>Q_s</td>
<td>0.0252</td>
</tr>
<tr>
<td>Closeness Centrality</td>
<td>C_c</td>
<td>0.0267</td>
</tr>
<tr>
<td>Number of Edges</td>
<td>D</td>
<td>0.0360</td>
</tr>
<tr>
<td>Synchronizability</td>
<td>s</td>
<td>0.0408</td>
</tr>
<tr>
<td>Robustness, Random</td>
<td>R_r</td>
<td>0.0456</td>
</tr>
<tr>
<td>Clustering Coefficient</td>
<td>C</td>
<td>0.0496</td>
</tr>
<tr>
<td>Subgraph Centrality</td>
<td>C_s</td>
<td>0.0528</td>
</tr>
<tr>
<td>Local Efficiency</td>
<td>E_l</td>
<td>0.0573</td>
</tr>
<tr>
<td>Robustness, Targeted</td>
<td>R_t</td>
<td>0.0727</td>
</tr>
<tr>
<td>Communicability</td>
<td>C_o</td>
<td>0.0829</td>
</tr>
<tr>
<td>Hierarchy</td>
<td>h</td>
<td>0.1080</td>
</tr>
<tr>
<td>Eigenvector Centrality</td>
<td>C_e</td>
<td>0.1150</td>
</tr>
<tr>
<td>Degree Assortativity</td>
<td>a</td>
<td>0.3219</td>
</tr>
</tbody>
</table>

TABLE II. Reliability of network diagnostics for the force-weighted contact networks. We measure reliability using coefficient of variance (CV).

<table>
<thead>
<tr>
<th>Force-Weighted Contact Network Diagnostic</th>
<th>Variable</th>
<th>CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transitivity</td>
<td>T</td>
<td>0.0339</td>
</tr>
<tr>
<td>Clustering Coefficient</td>
<td>C_w</td>
<td>0.0549</td>
</tr>
<tr>
<td>Geodesic Node Betweenness</td>
<td>B_w</td>
<td>0.0282</td>
</tr>
<tr>
<td>Geodesic Edge Betweenness</td>
<td>B_e</td>
<td>0.0199</td>
</tr>
<tr>
<td>Normalized Strength</td>
<td>s'</td>
<td>0.0000</td>
</tr>
<tr>
<td>Modularity: Louvain Optimization</td>
<td>Q_w</td>
<td>0.0135</td>
</tr>
<tr>
<td>Diversity</td>
<td>V</td>
<td>0.0412</td>
</tr>
<tr>
<td>Global Efficiency</td>
<td>E_w</td>
<td>0.1094</td>
</tr>
</tbody>
</table>

VI. APPENDIX C: COMPARISON OF CONTACT NETWORKS TO RANDOM GEOMETRIC GRAPHS

Many of the diagnostics that we compute for the granular networks are highly correlated with one another (see Fig. 9A). In the (binary) contact networks, they form roughly two families, where the correlations among the diagnostics in a given family are high. The diagnostics that we used for the weighted networks also exhibit some correlations (see Fig. 8A), and (unsurprisingly) this is particularly evident for diagnostics that are known to be closely related mathematically. For example, the weighted clustering coefficient is highly correlated with transitivity, and geodesic node betweenness is highly correlated with geodesic edge betweenness.

It is important to think about the possible origins...
of correlations between the various network diagnostics. Some of them might be specific features of the precise granular system under consideration, but others might arise because our granular packings are confined in 2D rather than in 3D or because of our particular preparation protocol. Still others might be general properties of spatially-embedded systems (in any dimension), of granular materials, or of networks in general.

To examine such issues, it is desirable to compare network diagnostics computed for the networks obtained from experimental data with those obtained from appropriate ensembles of null-model networks. It is common to compare the structures of networks under study to those that would be expected in Erdős-Rényi random graphs or from some other random graph ensemble [1]. The networks that we study in the present paper — namely, contact networks in granular packings — are spatially embedded (in the plane) because of physical constraints. The development of null-models is a wide open area of research for spatially-embedded networks [8], but we can make some progress for the binary contact networks by comparing the network diagnostics in those networks to computations of the same diagnostics using an ensemble of random geometric graphs (RGGs). As we will now discuss, we find that all diagnostics (except for the ones that we fix when defining the RGG ensemble to match their counterparts in the real networks) are significantly different in the real versus random networks.

The simplest RGG [8, 32, 94] contains \( N \) nodes that are randomly and distributed according to some probability distribution throughout an ambient space, which in our case is \( \mathbb{R}^2 \). One then places an edge between any pair of nodes \( i \) and \( j \) that are separated by a distance of at most \( 2r \), where one should think of the parameter \( r \) as the radius of a ball (using some metric) in the confining space. In planar Euclidean space, one considers a disk in \( \mathbb{R}^2 \) and uses ordinary (Euclidean) distance.

To compare the networks that we study to RGGs, we generated RGGs in which we placed nodes randomly and uniformly within the 2D space of the granular packing. For each experimental run, we created an ensemble of 100 RGGs in which the number of nodes was identical to that in the experimental system. We likewise fix the number of edges in each RGG to be identical to that in the real system \( (D) \) by choosing the threshold \( 2r \) so that the number of inter-node distances less than \( 2r \) is equal to \( D \). We calculate the other 19 binary diagnostics (i.e., except for the number of nodes and the number of edges, as those have been fixed to be equal in the two sets of networks) and computed their mean over the 100 RGGs in each ensemble. By performing these computations for each experimental run, we created 1 estimate of each of the 19 diagnostic values for each of the 17 runs. We report in Table III the mean values for both the real networks and the networks generated from the RGG ensembles. We also report t-values and p-values for two-sample t-tests between the network-diagnostic values of the two families of networks (EGs and RGGs).

<table>
<thead>
<tr>
<th>Diagnostic Name</th>
<th>EG</th>
<th>RGG</th>
<th>( t )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>EG Less Than RGG</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Local Efficiency</td>
<td>0.81</td>
<td>0.82</td>
<td>59.57</td>
<td>2.6 \times 10^{-20}</td>
</tr>
<tr>
<td>Modularity: Louvain Optim.</td>
<td>0.81</td>
<td>0.82</td>
<td>59.57</td>
<td>2.6 \times 10^{-20}</td>
</tr>
<tr>
<td>Clustering Coefficient</td>
<td>0.26</td>
<td>0.26</td>
<td>85.28</td>
<td>2.5 \times 10^{-30}</td>
</tr>
<tr>
<td>Random-Walk Betweeness</td>
<td>0.00</td>
<td>0.00</td>
<td>39.73</td>
<td>8.2 \times 10^{-29}</td>
</tr>
<tr>
<td>Degree Assortativity</td>
<td>0.10</td>
<td>0.10</td>
<td>38.52</td>
<td>2.1 \times 10^{-28}</td>
</tr>
<tr>
<td>Subgraph Centrality</td>
<td>7.19</td>
<td>7.19</td>
<td>22.41</td>
<td>3.9 \times 10^{-21}</td>
</tr>
<tr>
<td>Communicability</td>
<td>0.15</td>
<td>0.15</td>
<td>17.79</td>
<td>3.6 \times 10^{-18}</td>
</tr>
<tr>
<td>Rent Exponent</td>
<td>0.47</td>
<td>0.47</td>
<td>15.41</td>
<td>2.2 \times 10^{-16}</td>
</tr>
<tr>
<td><strong>EG Greater Than RGG</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Global Efficiency</td>
<td>0.10</td>
<td>0.10</td>
<td>94.41</td>
<td>1.0 \times 10^{-46}</td>
</tr>
<tr>
<td>Mean Connection Distance</td>
<td>39.60</td>
<td>39.60</td>
<td>79.38</td>
<td>2.5 \times 10^{-38}</td>
</tr>
<tr>
<td>Robustness, Random</td>
<td>8.38 \times 10^4</td>
<td>8.38 \times 10^4</td>
<td>54.89</td>
<td>3.0 \times 10^{-33}</td>
</tr>
<tr>
<td>Closeness Centrality</td>
<td>7.84</td>
<td>7.84</td>
<td>50.29</td>
<td>4.9 \times 10^{-32}</td>
</tr>
<tr>
<td>Synchronizability</td>
<td>0.0014</td>
<td>0.0014</td>
<td>44.71</td>
<td>2.2 \times 10^{-30}</td>
</tr>
<tr>
<td>Robustness, Targeted</td>
<td>7.08 \times 10^4</td>
<td>7.08 \times 10^4</td>
<td>36.96</td>
<td>7.9 \times 10^{-28}</td>
</tr>
<tr>
<td>Edge Betweenness</td>
<td>13.33</td>
<td>13.33</td>
<td>34.71</td>
<td>5.6 \times 10^{-27}</td>
</tr>
<tr>
<td>Geodesic Node Betweenness</td>
<td>6.24 \times 10^4</td>
<td>6.24 \times 10^4</td>
<td>30.95</td>
<td>2.0 \times 10^{-25}</td>
</tr>
<tr>
<td>Hierarchy</td>
<td>0.76</td>
<td>0.76</td>
<td>24.42</td>
<td>2.9 \times 10^{-22}</td>
</tr>
<tr>
<td>Eigenvector Centrality</td>
<td>0.0172</td>
<td>0.0172</td>
<td>19.15</td>
<td>4.2 \times 10^{-19}</td>
</tr>
<tr>
<td><strong>Modularity: Spectral Optimization</strong></td>
<td>0.78</td>
<td>0.78</td>
<td>5.06</td>
<td>1.63 \times 10^{-5}</td>
</tr>
</tbody>
</table>

TABLE III. Comparison of binary network diagnostics in the (real) experimental graphs (EGs) and the random geometric graphs (RGGs). We show the mean values of the EGs (column 1), the mean values of the RGGs (column 2), the t-values (column 3), and p-values (column 4) for a two-sample t-test between the network-diagnostic values of the two families of networks (EGs and RGGs).
FIG. 9. [Color online] (A) Relationships between 8 diagnostics applied to the weighted networks: geodesic node betweenness ($B_w$), clustering coefficient ($C_w$), diversity ($V$), geodesic edge betweenness ($B_{ew}$), global efficiency ($E_w$), modularity ($Q_w$) optimized using the Louvain method, and normalized strength ($S'$). We ordered the diagnostics to maximize the correlation along the diagonal for better visualization of highly correlated groups. Color indicates the correlation between global network diagnostics over the 17 experimental runs. (B) Reliability, as measured by the coefficient of variation (CV), over the 17 runs for the 8 weighted graph diagnostics reported in (A).

local connectivity (e.g., clustering coefficient) are higher in the RGG, whereas measures of global connectivity and physical distance are lower. These results illustrate that the networks in the RGG ensemble have more locally connectivity structures than those in the 2D granular system under study.

[22] E. T. Owens and K. E. Daniels, Europhysics Letters 94,
[96] Note that a ‘random geometric graph’ in 2D is a different mathematical object from what is known as a ‘random planar graph’ [95].