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Frank Verheest, Manfred A. Hellberg, and Willy A. Hereman

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Head-on collisions of electrostatic solitons in nonthermal plasmas

Frank Verheest
*Sterrenkundig Observatorium, Universiteit Gent, Krijgslaan 281, B–9000 Gent, Belgium &
School of Chemistry and Physics, University of KwaZulu-Natal, Durban 4000, South Africa*

Manfred A. Hellberg
School of Chemistry and Physics, University of KwaZulu-Natal, Durban 4000, South Africa

Willy A. Hereman
*Department of Applied Mathematics and Statistics,
Colorado School of Mines, Golden CO 80401–1887, U.S.A.*

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In contrast to overtaking interactions, head-on collisions between two electrostatic solitons can only be dealt with by an approximate method, which limits the range of validity but offers valuable insights. Treatments in the plasma physics literature all use assumptions in the stretching of space and time and in the expansion of the dependent variables that are seldom if ever discussed. All models force a separability to lowest order, corresponding to two linear waves with opposite but equally large velocities. A systematic exposition of the underlying hypotheses is illustrated by considering a plasma composed of cold ions and nonthermal electrons. This is general enough to yield critical compositions that lead to modified rather than standard Korteweg-de Vries equations, an aspect not discussed so far. The nonlinear evolution equations for both solitons and their phase shifts due to the collision are established. A Korteweg-de Vries description is the generic conclusion, except when the plasma composition is critical, rendering the nonlinearity in the evolution equations cubic, with concomitant repercussions on the phase shifts. In the latter case, the solitons can have either polarity, so that combinations of negative and positive solitons can occur, contrary to the generic case, where both solitons necessarily have the same polarity.

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I. INTRODUCTION

Standard nonlinear evolution equations, like the Korteweg-de Vries (KdV) equation, are derived in a frame which travels at the linear acoustic speed with respect to an inertial frame and can have N-soliton solutions (for arbitrary N, due to their integrability properties). Hence, in an inertial frame, they are all seen as propagating in the direction underlying the derivation of the original evolution equation, so that only overtaking interactions are covered in this way. A more general description, on a par with the N-soliton methods for treating overtaking solitons, does not at present appear to be available for the interaction between solitons traveling in opposite directions.

Thus, to describe head-on collisions (in an inertial frame) between two solitons, an approximate method is necessary. This limits the range of validity of the description, but offers some valuable insights. The framework is based on an extension of the Poincaré-Lighthill-Kuo formalism of strained coordinates [1], which was used three decades ago to study the head-on collision of nonlinear waves on the surface of an inviscid homogeneous fluid [2]. To lowest nonlinear order, the problem of colliding solitons leads to KdV equations, and also yields the phase shifts that occur in the interaction.

In recent years there has been much interest in the problem of head-on collisions of acoustic solitary waves in various multispecies plasmas, for parallel [3–15] or oblique [16–19] propagation with respect to a static magnetic field. The papers quoted represent a typical selection of recent papers, without any claim at being exhaustive. Although the models vary quite a bit between them, all the papers in the literature follow a similar methodology, apparently inspired by the seminal paper by Su and Mirie [2], and lead to KdV solitons with their phase shifts. We will focus more specifically, in what follows, on the problem of parallel propagation, thus avoiding extraneous analytical complications due to oblique propagation, where the method runs along broadly similar lines as in the parallel case.

The treatments in the plasma physics literature all use assumptions that are seldom, if ever, discussed. For instance, choices made in imposing the stretching of the space and time coordinates, and the expansion of the dependent variables, hide assumptions that are not spelt out. Among the restrictions which are assumed by almost all cited authors, but not explicitly motivated, is that while the stretching for the co-moving coordinates starts in a smallness parameter ε, the perturbations in the dependent variables start at quadratic order [3, 5–8, 10–13, 15], except when an equivalent ordering in $\varepsilon^{1/2}$ and ε is chosen [4]. The only exceptions are where the expansions in the dependent variables are also assumed to start at order ε [9, 14]. Unfortunately, however, these exceptions contain serious errors which invalidate the resulting algebra [20].

Before going on, let us briefly recall that the traditional stretching for the KdV class of equations starts from

$$\xi = \varepsilon(x - t), \qquad \tau = \varepsilon^3 t. \tag{1}$$

Here we have taken ε and ε^3 rather than the traditional $\varepsilon^{1/2}$ and $\varepsilon^{3/2}$, in accordance with present usage in almost all papers dealing with head-on collisions of electrostatic solitons in plasmas. Next, expansions are chosen of the form $\varphi = \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \varepsilon^4\varphi_4 + \ldots$ [our choice for the subscripts] for the electrostatic potential in normalized form. The standard reductive perturbation treatment thus leads to the KdV equation [21, 22],

$$\frac{\partial\varphi_2}{\partial\tau} + A\,\varphi_2\,\frac{\partial\varphi_2}{\partial\xi} + B\,\frac{\partial^3\varphi_2}{\partial\xi^3} = 0. \tag{2}$$

Given the vast literature on single solitons and their KdV description, we have on purpose omitted all references of this kind, barring [21, 22].

This is all fine as long as A only has one definite sign, as is the case for the original description of shallow water waves [2] or of ion-acoustic solitons in simple electron-ion plasmas [4]. In both these physical situations, $A > 0$ and the nonlinear modes are compressive, showing density and/or electrostatic potential humps.

For more complex plasma compositions, this simple picture no longer holds, and one can encounter plasma parameter values which allow A to vanish. For those critical values the quadratic nonlinearity will disappear from (2), and the expansion must be changed to $\varphi = \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \ldots$. This leads to a modified KdV (mKdV) equation,

$$\frac{\partial\varphi_1}{\partial\tau} + C\,\varphi_1^2\,\frac{\partial\varphi_1}{\partial\xi} + D\,\frac{\partial^3\varphi_1}{\partial\xi^3} = 0, \tag{3}$$

having a cubic nonlinearity.

One could, of course, have started in both cases directly with $\varphi = \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \ldots$, but then a careful analysis is needed, leading to a bifurcation in the treatment, where either $A = 0$ or $\varphi_1 = 0$ [23]. We will encounter a similar bifurcation in the present paper.

As will also be seen, a natural consequence of the method is that one necessarily has to work with linear phase velocities which are opposite but of equal magnitude. This is sometimes assumed *ab initio* [3, 4, 6, 8, 10–15], or explicitly proven at lowest order [5]. As to the type of acoustic waves studied, one finds dust-acoustic [3, 10, 12], dust-ion-acoustic [11], ion-acoustic [4–9, 13] and electron-acoustic modes [14, 15].

The inertial species are usually cold [3–5, 7–15], polytropic [6], sometimes in the presence of a neutralizing background [11, 14, 15], while the hot species can have Boltzmann [3–6, 9, 10, 12], Cairns nonthermal [10], kappa superthermal [7], Tsallis nonextensive [9, 13, 14] or quantum distributions [8, 11, 15]. Even dust charge fluctuations are sometimes included [3]. Despite being based on such a rich variety of models, all papers arrive at KdV equations as the governing evolution equations [3–7, 9–15], even when the derivation cannot be trusted [9, 14], except when viscosity is added and the resulting equation is of the KdV-Burgers type [8].

Surveying the relevant literature, there is a need to go more deeply into the details of the analytical derivation, to see in a reasoned way where the restrictions come in and what they imply. To do this in an easily tractable way, we will start from a fairly general stretching and expansion scheme for a plasma composed of cold positive ions and nonthermal electrons obeying a Cairns distribution [24]. This is sufficiently general to permit critical parameters leading to mKdV rather than KdV equations. We also include a discussion of why the model forces a separability to lowest order, with opposite velocities of equal magnitudes in the stretching, and clarify the various assumptions.

One can investigate more sophisticated plasma compositions. In the end, however, one always arrives at KdV equations in the generic case or at mKdV equations when some of the compositional parameters are sufficiently special. As far as we have been able to ascertain, the latter aspect is not addressed at all in the plasma literature on head-on collisions.

Having mentioned a fair selection of the theoretical papers, we would like to draw attention to a recent paper [28] with experimental observations of the interaction of two counter-propagating solitons of equal amplitude in a monolayer strongly coupled dusty plasma. This will be discussed later, when our theoretical results can be properly compared with the observations.

The paper is structured as follows. In Section II, the basic formalism is introduced. The exposition is given in the generic case with due attention being paid to the various assumptions needed to make the method work and to the physical and mathematical restrictions that these assumptions imply. This leads to two KdV equations, for the right and the left traveling soliton, respectively, plus equations giving the phase changes due to the head-on collision between the two solitons. Section III is devoted to the special case when the compositional parameters take on values such that the coefficient of the quadratic nonlinearity in the KdV equations vanishes. In this case, one has to adapt the treatment and derive instead an mKdV equation with corresponding changes in the determination of the phases during interaction. Section IV then briefly summarizes our conclusions.

II. BASIC FORMALISM AND GENERIC COMPOSITION

The basic fluid equations for the positive ions are the continuity and momentum equations,

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0, \tag{4}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial\varphi}{\partial x} = 0. \tag{5}$$

The dimensionless variables n and u are the density and fluid velocity of the ion species, respectively, with charge e and mass m_i, normalized in terms of the equilibrium density n_0 and of an ion-acoustic velocity $V_{ia} = \sqrt{T_e/m_i}$. We note that this is not the true sound speed for the model plasma, as the effect of the Cairns nonthermal parameter has been ignored (see below). Nonetheless, it is a useful normalizing speed, and the nonthermal effects will then appear explicitly in the calculations. The electrostatic potential φ is given in units of T_e/e, where T_e is the kinetic temperature the hot electrons would have in the absence of nonthermal effects.

All species are coupled through Poisson's equation,

$$\frac{\partial^2\varphi}{\partial x^2} + n - (1 - \beta\varphi + \beta\varphi^2)\exp(\varphi) = 0, \tag{6}$$

where $(1 - \beta\varphi + \beta\varphi^2)\exp(\varphi)$ represents the hot Cairns electron contribution in terms of a nonthermality parameter β. More details can be found in the original paper introducing the Cairns nonthermal distribution [24].

To model head-on collisions of two electrostatic solitons, and inspired by methods in the literature [1, 2], the stretching is introduced as

$$\begin{aligned}\xi &= \varepsilon(x - ct) + \varepsilon^2 P(\xi,\eta,\tau) + ...,\\ \eta &= \varepsilon(x + ct) + \varepsilon^2 Q(\xi,\eta,\tau) + ...,\\ \tau &= \varepsilon^3 t,\end{aligned} \tag{7}$$

referring in ξ and η to a right- and to a left-propagating soliton, s_ξ and s_η, respectively. For the derivation of a single standard KdV equation to work, the velocity used in the coordinate stretching has to be the appropriate phase speed for the linear acoustic wave type in the plasma considered. In the present problem, the stretching, to the lowest order, treats the colliding waves as separate entities, and both ξ and η involve the unique linear acoustic phase velocity (in absolute value). This argument could have been made in the papers which assume equal velocities without further justification. However, it was not [3, 4, 8, 12] or only in an indirect way [10, 11, 13, 14].

In our model, the linear acoustic phase velocity is $c = 1/\sqrt{1-\beta}$. Although, by definition [24], $0 \leqslant \beta < 4/3$, we will limit ourselves in this paper to $0 \leqslant \beta < 4/7$, so that certainly $\beta < 1$ holds. The reason for this reduced limit is that for $\beta > 4/7$, the phase space Cairns distribution [24] may develop bump-on-tail or beam instabilities, so this range is best avoided. The contribution of the nonthermal effects to c, expressed in β, is made explicit.

The functions P and Q will be seen later to represent phase shifts that arise through the interaction between the two solitons. For now, we prefer to leave them as functions of all three stretched variables, to be determined later, rather than to impose some restrictions at this early stage of the calculations. We shall need

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta},\\ \frac{\partial}{\partial t} &= \frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta} + \frac{\partial\tau}{\partial t}\frac{\partial}{\partial\tau},\end{aligned} \tag{8}$$

which in turn requires

$$\begin{aligned}\frac{\partial\xi}{\partial x} &= \varepsilon + \varepsilon^2\left(\frac{\partial P}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial P}{\partial\eta}\frac{\partial\eta}{\partial x}\right) + ...,\\ \frac{\partial\eta}{\partial x} &= \varepsilon + \varepsilon^2\left(\frac{\partial Q}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial Q}{\partial\eta}\frac{\partial\eta}{\partial x}\right) + ...,\\ \frac{\partial\xi}{\partial t} &= -\varepsilon c + \varepsilon^2\left(\frac{\partial P}{\partial\xi}\frac{\partial\xi}{\partial t} + \frac{\partial P}{\partial\eta}\frac{\partial\eta}{\partial t} + \frac{\partial P}{\partial\tau}\frac{\partial\tau}{\partial t}\right) + ...,\\ \frac{\partial\eta}{\partial t} &= \varepsilon c + \varepsilon^2\left(\frac{\partial Q}{\partial\xi}\frac{\partial\xi}{\partial t} + \frac{\partial Q}{\partial\eta}\frac{\partial\eta}{\partial t} + \frac{\partial Q}{\partial\tau}\frac{\partial\tau}{\partial t}\right) + ... \,.\end{aligned} \tag{9}$$

Up to third order, the solution of this intricate set of equations yields

$$\begin{aligned}\frac{\partial\xi}{\partial x} &= \varepsilon + \varepsilon^3\left(\frac{\partial P}{\partial\xi} + \frac{\partial P}{\partial\eta}\right) + ...,\\ \frac{\partial\eta}{\partial x} &= \varepsilon + \varepsilon^3\left(\frac{\partial Q}{\partial\xi} + \frac{\partial Q}{\partial\eta}\right) + ...,\\ \frac{\partial\xi}{\partial t} &= -\varepsilon c + \varepsilon^3 c\left(\frac{\partial P}{\partial\eta} - \frac{\partial P}{\partial\xi}\right) + ...,\\ \frac{\partial\eta}{\partial t} &= \varepsilon c + \varepsilon^3 c\left(\frac{\partial Q}{\partial\eta} - \frac{\partial Q}{\partial\xi}\right) + ... \,.\end{aligned} \tag{10}$$

We next introduce the operators

$$\begin{aligned}\widehat{X} &= \frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta},\\ \widehat{T} &= c\left(\frac{\partial}{\partial\eta} - \frac{\partial}{\partial\xi}\right),\\ \widehat{X}' &= \left(\frac{\partial P}{\partial\xi} + \frac{\partial P}{\partial\eta}\right)\frac{\partial}{\partial\xi} + \left(\frac{\partial Q}{\partial\xi} + \frac{\partial Q}{\partial\eta}\right)\frac{\partial}{\partial\eta},\\ \widehat{T}' &= c\left(\frac{\partial P}{\partial\eta} - \frac{\partial P}{\partial\xi}\right)\frac{\partial}{\partial\xi} + c\left(\frac{\partial Q}{\partial\eta} - \frac{\partial Q}{\partial\xi}\right)\frac{\partial}{\partial\eta},\end{aligned} \tag{11}$$

to write the operators in (8) in a compact way, viz.,

$$\begin{aligned}\frac{\partial}{\partial x} &= \varepsilon\widehat{X} + \varepsilon^3\widehat{X}' + ...,\\ \frac{\partial}{\partial t} &= \varepsilon\widehat{T} + \varepsilon^3\widehat{T}' + \varepsilon^3\frac{\partial}{\partial\tau} + ... \,.\end{aligned} \tag{12}$$

The series expansions of the dependent variables are

$$\begin{aligned}n &= 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + \varepsilon^4 n_4 + ...,\\ u &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4 + ...,\\ \varphi &= \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \varepsilon^4\varphi_4 + ... \,.\end{aligned} \tag{13}$$

Although it might be anticipated that the first order contributions will vanish in the generic case, it is prudent not to assume that from the outset, but to let the algebra decide this, should that be the case. Hence, to lowest nonzero order (4), (5) and (6) give

$$\begin{aligned}&\widehat{T}n_1 + \widehat{X}u_1 = 0,\\ &\widehat{T}u_1 + \widehat{X}\varphi_1 = 0,\\ &n_1 - (1-\beta)\varphi_1 = 0.\end{aligned} \tag{14}$$

By operating on the first equation in (14) by $-\widehat{T}$, on the second by $\widehat{X}$ and on the third by $\widehat{T}^2$ and adding the results, we eliminate n_1 and u_1, to find that

$$4\,\frac{\partial^2\varphi_1}{\partial\xi\partial\eta}=0. \tag{15}$$

Hence, the first-order variables will consist of a term depending on ξ and τ, but not η, and another term depending on η and τ, but not ξ. This leads to

$$\begin{aligned} n_1 &= (1-\beta)\left(\varphi_{1\xi}+\varphi_{1\eta}\right), \\ u_1 &= \sqrt{1-\beta}\left(\varphi_{1\xi}-\varphi_{1\eta}\right), \\ \varphi_1 &= \varphi_{1\xi}+\varphi_{1\eta}, \end{aligned} \tag{16}$$

with obvious compact notations to denote the dependence on the space arguments ξ or η.

To the next higher order (4), (5) and (6) give

$$\begin{aligned} &\widehat{T}n_2+\widehat{X}u_2+\widehat{X}(n_1u_1)=0, \\ &\widehat{T}u_2+u_1\widehat{X}u_1+\widehat{X}\varphi_2=0, \\ &n_2-(1-\beta)\varphi_2-\tfrac{1}{2}\,\varphi_1^2=0. \end{aligned} \tag{17}$$

For those terms that depend on τ, and on either ξ or η, but not both, we find that

$$\begin{aligned} n_{2\xi} &= (1-\beta)\varphi_{2\xi}+\tfrac{3}{2}\,(1-\beta)^2\varphi_{1\xi}^2, \\ n_{2\eta} &= (1-\beta)\varphi_{2\eta}+\tfrac{3}{2}\,(1-\beta)^2\varphi_{1\eta}^2, \\ u_{2\xi} &= \sqrt{1-\beta}\,\varphi_{2\xi}+\tfrac{1}{2}\,(1-\beta)^{3/2}\varphi_{1\xi}^2, \\ u_{2\eta} &= -\sqrt{1-\beta}\,\varphi_{2\eta}-\tfrac{1}{2}\,(1-\beta)^{3/2}\varphi_{1\eta}^2, \end{aligned} \tag{18}$$

coupled to

$$\begin{aligned} &(2-6\beta+3\beta^2)\varphi_{1\xi}^2=0, \\ &(2-6\beta+3\beta^2)\varphi_{1\eta}^2=0. \end{aligned} \tag{19}$$

These results include an integration with respect to ξ or η, with zero boundary conditions at infinity, in the undisturbed medium.

The remainder of the information to this order comes from the elimination of u_2 between the first two equations in (17) and using the third equation to replace n_2, but only for the terms which depend on ξ and η together, which leads to

$$\begin{aligned} &\frac{\partial^2\widetilde{\varphi}_2}{\partial\xi\partial\eta}+\frac{\beta(2-\beta)}{2(1-\beta)}\,\frac{\partial\varphi_{1\xi}}{\partial\xi}\,\frac{\partial\varphi_{1\eta}}{\partial\eta} \\ &-\frac{2-2\beta+\beta^2}{4(1-\beta)}\left(\varphi_{1\eta}\,\frac{\partial^2\varphi_{1\xi}}{\partial\xi^2}+\varphi_{1\xi}\,\frac{\partial^2\varphi_{1\eta}}{\partial\eta^2}\right)=0. \end{aligned} \tag{20}$$

Here $\widetilde{\varphi}_2=\varphi_2-\varphi_{2\xi}-\varphi_{2\eta}$ has been defined as the part of φ_2 which depends on ξ and η together in a way which cannot be disentangled, with analogous definitions for other variables and higher orders.

The structure of (19) points to two possibilities: either the nonthermality parameter is very special, in that $\beta_c=(3-\sqrt{3})/3\simeq 0.423$, or else $\varphi_{1\xi}=\varphi_{1\eta}=0$. Not surprisingly, there is a close correspondence to standard KdV theory, where critical parameters annul the coefficient of the quadratic nonlinearity and one has to go to cubic nonlinearities in an mKdV equation. This will be investigated in detail in Section III. There is also a correlation with the large amplitude analysis of nonlinear modes by pseudopotential theory, where the same value of β_c is found for the reversal of polarities of the KdV-like and nonKdV-like modes in a number of physically different plasma systems [25–27].

Apart from the papers by Demiray [4] and Chatterjee *et al.* [7] whose plasma models do not give rise to critical parameters, the other models [3, 5, 6, 8, 10–13, 15] are sufficiently sophisticated to yield critical compositions. However, none of the authors discusses the possibility of having such special cases, because they have immediately started the expansions of the dependent variables, outside equilibrium, at order ε^2. Since in the generic case all first-order quantities vanish, it has been *implicitly assumed* in the papers quoted [3, 5, 6, 8, 10–13, 15] that this is the only case worth considering. Eslami *et al.* [9, 14] start the expansions at order ε, but then proceed as though this were the generic case, and get thoroughly confused, so that their results are wrong [20].

Assuming now that we are in the generic case, for which we have to take $\varphi_{1\xi}=\varphi_{1\eta}=0$, one is, from (18) and (20), once again, led to separability:

$$\begin{aligned} n_2 &= (1-\beta)\left(\varphi_{2\xi}+\varphi_{2\eta}\right), \\ u_2 &= \sqrt{1-\beta}\left(\varphi_{2\xi}-\varphi_{2\eta}\right), \\ \varphi_2 &= \varphi_{2\xi}+\varphi_{2\eta}. \end{aligned} \tag{21}$$

Continuing up the ladder, we find for the third-order variables that

$$\begin{aligned} &\widehat{T}n_3+\widehat{X}u_3=0, \\ &\widehat{T}u_3+\widehat{X}\varphi_3=0, \\ &n_3-(1-\beta)\varphi_3=0. \end{aligned} \tag{22}$$

Since these variables will not appear at the next order, we simply put $n_3=u_3=\varphi_3=0$, and thus find that the series expansions (13) go in even orders of ε, much as is the case for the usual derivation of the KdV equation.

Finally, we arrive at the order where interesting new contributions appear,

$$\begin{aligned} &\widehat{T}n_4+\widehat{T}'n_2+\frac{\partial n_2}{\partial\tau}+\widehat{X}u_4+\widehat{X}(n_2u_2)+\widehat{X}'u_2=0, \\ &\widehat{T}u_4+\widehat{T}'u_2+\frac{\partial u_2}{\partial\tau}+u_2\widehat{X}u_2+\widehat{X}\varphi_4+\widehat{X}'\varphi_2=0, \\ &\widehat{X}^2\varphi_2+n_4-(1-\beta)\varphi_4-\tfrac{1}{2}\,\varphi_2^2=0. \end{aligned} \tag{23}$$

Combining again the parts of these equations which contain terms that only depend on ξ or on η (besides τ), yields the typical KdV equations

$$\begin{aligned} &\frac{\partial\varphi_{2\xi}}{\partial\tau}+A\,\varphi_{2\xi}\,\frac{\partial\varphi_{2\xi}}{\partial\xi}+B\,\frac{\partial^3\varphi_{2\xi}}{\partial\xi^3}=0, \\ &\frac{\partial\varphi_{2\eta}}{\partial\tau}-A\,\varphi_{2\eta}\,\frac{\partial\varphi_{2\eta}}{\partial\eta}-B\,\frac{\partial^3\varphi_{2\eta}}{\partial\eta^3}=0, \end{aligned} \tag{24}$$

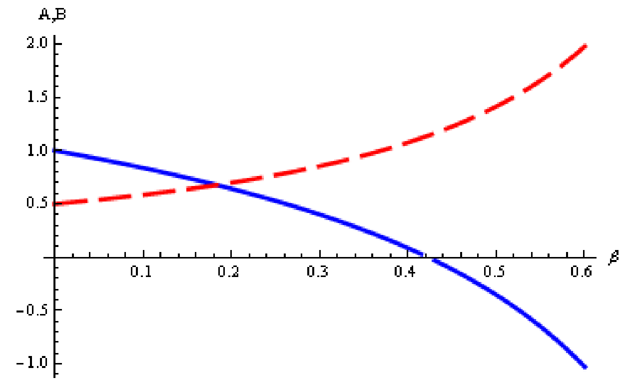

FIG. 1. (Color online) Changes of A and B (full and dashed curves, respectively), as β is increased, showing how A goes from positive to negative values at β_c and how B remains positive and increases monotonically from 1/2.

for the right- and left-going solitary waves, respectively, with

$$A = \frac{2-6\beta+3\beta^2}{2(1-\beta)^{3/2}}, \qquad B = \frac{1}{2(1-\beta)^{3/2}}. \tag{25}$$

The coefficient of the quadratic nonlinearity, A, has already been encountered in a similar role in (19), except for factors $(1-\beta)$ which have been omitted there. It will be recalled that this factor is related to the normalized linear phase speeds, which are given by $c = 1/\sqrt{1-\beta}$.

As seen already in the discussion about (19), A can change sign at β_c. For good measure, we include in Figure 1 a graph of how A and B vary as β is increased from 0 to 0.6.

There is more information still, in the terms which contain both ξ and η, besides τ, giving

$$\begin{aligned}&\frac{\partial^2}{\partial\xi\partial\eta}\left[\widetilde{\varphi}_4 + \frac{\beta(2-\beta)}{2(1-\beta)}\,\varphi_{2\xi}\,\varphi_{2\eta}\right]\\&+\frac{\partial}{\partial\xi}\left\{\left[\frac{\partial P}{\partial\eta} - S\,\varphi_{2\eta}\right]\frac{\partial\varphi_{2\xi}}{\partial\xi}\right\}\\&+\frac{\partial}{\partial\eta}\left\{\left[\frac{\partial Q}{\partial\xi} - S\,\varphi_{2\xi}\right]\frac{\partial\varphi_{2\eta}}{\partial\eta}\right\} = 0,\end{aligned} \tag{26}$$

with

$$S = \frac{2-2\beta+\beta^2}{4(1-\beta)} \geqslant \frac{1}{2}. \tag{27}$$

The second and third terms in (26) will generate secular contributions at the next higher order, so these are to be annulled, leading to

$$\begin{aligned}\frac{\partial P}{\partial\eta} &= S\,\varphi_{2\eta},\\ \frac{\partial Q}{\partial\xi} &= S\,\varphi_{2\xi}.\end{aligned} \tag{28}$$

Thus the phase shifts can be determined also. The structure of these equations is such that $\partial P/\partial\eta$ cannot depend on ξ, since the right hand side does not contain ξ, and thus P itself might contain an *additive part* which would depend on ξ and τ, but not on η. Such a part plays no role here and is not interesting, because it would refer to changes in the phase of the right-traveling soliton due to its own propagation. It can therefore be omitted altogether, as was assumed at the outset, without discussion, in all the papers mentioned [3–15]. Analogous arguments hold for the absence of η in Q.

Finally, the other remaining term in (26) will give rise to a contribution

$$\widetilde{\varphi}_4 = -\frac{\beta(2-\beta)}{2(1-\beta)}\,\varphi_{2\xi}\,\varphi_{2\eta} \tag{29}$$

to φ_4, besides the parts $\varphi_{4\xi}$ and $\varphi_{4\eta}$, which will have to be determined from higher-order contributions.

Turning now to the one-soliton solutions of (24), these are the well-known "sech squared" solitons of KdV theory, here

$$\begin{aligned}\varphi_{2\xi} &= \frac{3v_\xi}{A}\,\mathrm{sech}^2[\kappa_\xi(\xi - v_\xi\tau)],\\ \varphi_{2\eta} &= \frac{3v_\eta}{A}\,\mathrm{sech}^2[\kappa_\eta(\eta + v_\eta\tau)],\end{aligned} \tag{30}$$

with the amplitudes and inverse width (related to κ_ξ or κ_η) expressed in terms of the velocities v_ξ and v_η, respectively, for the right- and left-propagating solitons. Here

$$\kappa_\xi = (1-\beta)^{3/4}\sqrt{\frac{v_\xi}{2}}, \qquad \kappa_\eta = (1-\beta)^{3/4}\sqrt{\frac{v_\eta}{2}}. \tag{31}$$

This requires that $v_\xi > 0$ and $v_\eta > 0$ and also indicates that the two interacting solitons must have the same polarity, given by the sign of A. Thus, for $0 \leqslant \beta < \beta_c$, hence $A > 0$, both solitons must have positive polarity, and for $\beta_c < \beta < 1$ (where $A < 0$), negative polarity. These ranges for the soliton polarities agree with those found in recent pseudopotential studies for KdV-like solitons in plasmas with a Cairns nonthermal component [26, 27].

It is thus clear that the present approximate way of dealing with head-on collisions is unable to handle interactions between two modes of opposite polarities in the same physical system. Furthermore, even though at the level of the stretching (7) we have opposing velocities c of equal magnitude, this does not hold for the one-soliton solutions, which are superacoustic in their direction of propagation but can have quite different amplitudes.

To obtain the phase shifts after the head-on collision between the two solitons, we assume that the right- and left-propagating solitons s_ξ and s_η are, asymptotically, far from each other at the initial time ($t = -\infty$), i.e., s_ξ is at $\xi = 0$, $\eta = -\infty$ and s_η is at $\eta = 0$, $\xi = +\infty$, respectively. After the collision ($t = +\infty$), s_ξ is far to the right of s_η, i.e., s_ξ is at $\xi = 0$, $\eta = +\infty$ and s_η is at $\eta = 0$, $\xi = -\infty$. These or similar boundary conditions have been used in almost all papers [3–7, 9–15]. Hence, the phase shifts expressed by P and Q are found from the substitution of (30) into (28) and integration, yielding

$$P = \frac{3S\sqrt{2v_\eta}}{A(1-\beta)^{3/4}}\,\{\tanh[\kappa_\eta(\eta + v_\eta\tau)] + 1\},$$
$$Q = \frac{3S\sqrt{2v_\xi}}{A(1-\beta)^{3/4}}\,\{\tanh[\kappa_\xi(\xi - v_\xi\tau)] - 1\}. \qquad (32)$$

As in all proper KdV problems, there is an intimate link in the co-moving frame between the soliton amplitude, width and excess velocity, and two of the characteristics can be expressed in terms of a third. Here the choice has been made to start from the excess velocities and use that to give the amplitudes and widths. Increases in v_ξ entail increases in the amplitudes of $\varphi_{2\xi}$ as well as in the phase shift of $\varphi_{2\eta}$. Thus, the larger of the two solitons travels faster than the smaller one, but is less affected by the phase shift when emerging from the collision region.

The special case of a plasma with Boltzmann electrons, as treated in the literature [4], is obtained at $\beta = 0$, so that $A = 1$, $B = 1/2$ and $S = 1/2$. Thus (30) and (32) are simplified to

$$\varphi_{2\xi} = 3v_\xi \operatorname{sech}^2\left[\sqrt{\frac{v_\xi}{2}}(\xi - v_\xi\tau)\right],$$
$$\varphi_{2\eta} = 3v_\eta \operatorname{sech}^2\left[\sqrt{\frac{v_\eta}{2}}(\eta + v_\eta\tau)\right], \qquad (33)$$

and

$$P = 3\sqrt{\frac{v_\eta}{2}}\left\{\tanh\left[\sqrt{\frac{v_\eta}{2}}(\eta + v_\eta\tau)\right] + 1\right\},$$
$$Q = 3\sqrt{\frac{v_\xi}{2}}\left\{\tanh\left[\sqrt{\frac{v_\xi}{2}}(\xi - v_\xi\tau)\right] - 1\right\}. \qquad (34)$$

Moreover, the contribution to $\widetilde{\varphi}_4$ of the form $\varphi_{2\xi}\varphi_{2\eta}$ now disappears, so that φ_4 is separable, as were the lower orders, as is u_4, but not n_4. Indeed, one can show that $\widetilde{\varphi}_4 = 0$ induces $\widetilde{u}_4 = 0$ but $\widetilde{n}_4 = \varphi_{2\xi}\varphi_{2\eta}$, provided the simplified versions of (24) and (28) hold.

We can now illustrate the above discussion with some graphs, starting with the Boltzmann case ($\beta = 0$) in Fig. 2. Both head-on colliding modes have positive polarities and are therefore compressive in the electron and ion densities. Plots with β intermediate between 0 and β_c will show a qualitatively similar behavior, as the nonthermal effects are not strong enough to reverse the polarity.

In order to show the phase shift more clearly, we plot in Fig. 3 for $\beta = 0.25$ a smaller (slower) right-going soliton (with a larger phase shift) as viewed from above. We have omitted the larger left-going soliton (with a smaller phase shift) as it would obscure the right-going soliton.

If one looks at (7), P and Q are corrections to both the space and time coordinates, in a way which is not immediately obvious. However, Fig. 3 clearly shows that it is not so much the soliton velocity which is affected, if one compares the propagation characteristics long before and after the interaction region, but the important change is in a kind of phase shift. One should also keep in mind that the results are restricted to order ε^3, as befits the expansion scheme leading to the KdV equations.

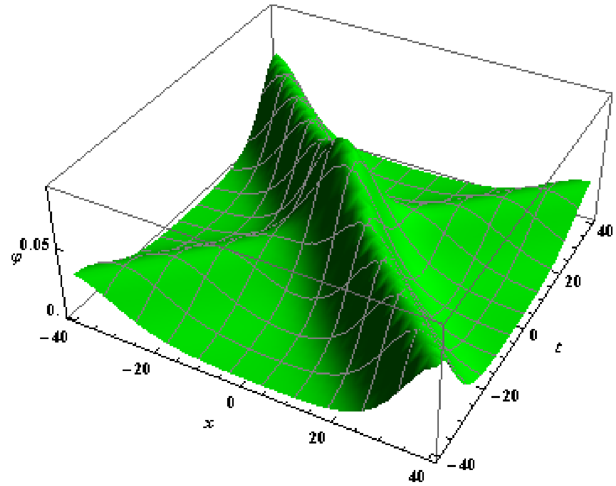

FIG. 2. (Color online) Head-on collision for plasma composition with Maxwellian electrons ($\beta = 0$), for $v_\xi = 0.01$ and $v_\eta = 0.02$. The solitons have positive polarity and are compressive in the densities.

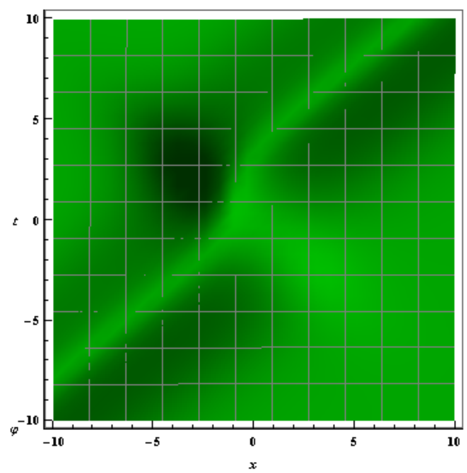

FIG. 3. (Color online) Slower soliton in the interaction zone, viewed from above, for the head-on collision for plasma composition with $\beta = 0.25$, for $v_\xi = 0.1$ and $v_\eta = 0.2$. The soliton is compressive and has a positive polarity. The soliton speeds have been chosen as rather large, for graphical clarity, but one cannot normally take the amplitudes too large, when using reductive perturbation theory.

Once $\beta > \beta_c$, the polarities become negative and the solitons are rarefactive in the densities. This is shown in Fig. 4 for $\beta = 0.5$, a typical value used in literature on nonthermal plasmas [24, 26, 27]. Plots for other values of $\beta > \beta_c$ will yield graphs which are qualitatively similar to Fig. 4.

As is clear from the literature surveyed [3–15], the inclusion of other superthermal electron effects, such as kappa distributions, or more cold ion species, can easily be done by modifying the derivation at the appropriate

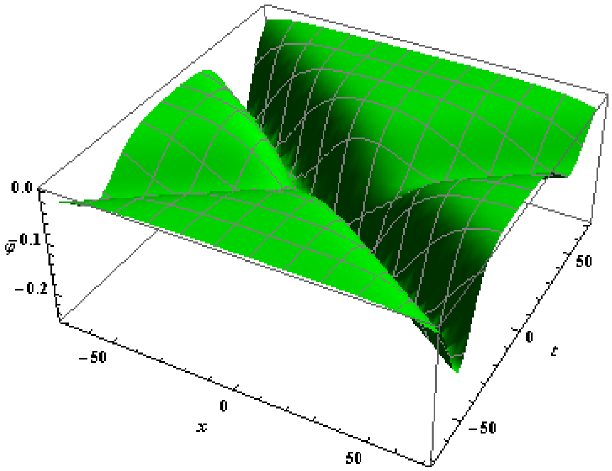

FIG. 4. (Color online) Head-on collision for plasma composition with $\beta = 0.5$, beyond $\beta_c = 0.423$, for $v_\xi = 0.01$ and $v_\eta = 0.02$. Here the solitons have negative polarity and are rarefactive in the densities.

places. This will result in changes of the coefficients in various expressions, but not in the intrinsic structure of the nonlinear evolution equations or in the phase shifts.

None of the papers in the literature [3–7, 9–14] presents graphs of this kind, except for the very recent paper by El-Labany *et al.* [15] which shows interactions of two rarefactive or two compressive solitons, respectively. This is a clear indication that in their model $A = 0$ is possible, but no discussion has been given of what is needed at critical composition. The figures in the paper by Han *et al.* [8] cannot be directly compared with our graphs, because they studied the interaction between two colliding shocks obeying KdV-Burgers equations.

The recent paper by Harvey *et al.* [28] includes experimental observations that are qualitatively similar to our results in Figs. 2 and 4. It was found that the solitons are delayed after the collision, with solitons of higher amplitude experiencing longer delays. The amplitude of the overlapping solitons during the collision was less than the sum of the initial soliton amplitudes. We do not claim that our model is the only one which produces graphs which are qualitatively similar to those obtained by Harvey *et al.* [28], only that none of the papers in the literature dealing with head-on collisions even refers to the experimental results of Harvey *et al.*, and that El-Labany *et al.* [15] might have reached an analogous conclusion had they chosen to discuss the experimental results.

Harvey *et al.* [28] discuss a comparison with the theoretical analysis of Su and Mirie [2], which, however, like all papers using the same methodology, predicts a superposition of the amplitudes during the interaction. Presumably, this discrepancy between the theory and the observations of Harvey *et al.* [28] highlights a weak point of the present approximate approach, which works well for the description of the solitons outside the interaction region and gives the right phase shifts after the collision. However, the decomposition $\varphi_2 = \varphi_{2\xi} + \varphi_{2\eta}$ amounts to a *linear* superposition of two KdV solitons, which, particularly during the collision, is quite different from a two-soliton solution, of a single KdV equation, to describe *overtaking* collisions.

III. SOLITONS AND PHASE SHIFTS AT CRITICAL COMPOSITIONS

Now we suppose that the electron nonthermality is critical, in that β_c annuls $2 - 6\beta + 3\beta^2$. Note that the other root of this quadratic, $\beta = (3 + \sqrt{3})/3$, is outside the definition range $0 \leqslant \beta < 4/3$. Ignoring this root and taking $\beta = \beta_c$, the quadratic nonlinearity in the KdV equations (24) disappears and in (18) we must keep the contributions in φ_1. Moreover, (20) simplifies to

$$\frac{\partial^2 \widetilde{\varphi}_2}{\partial\xi\partial\eta} + \frac{\sqrt{3}}{3}\left(\frac{\partial^2}{\partial\xi\partial\eta} - \frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2}\right)\varphi_{1\xi}\varphi_{1\eta} = 0. \quad (35)$$

While it is clear that $\widetilde{\varphi}_2 \neq 0$, we can cancel the solutions of the linear operator without loss of generality, hence $\varphi_{2\xi} = \varphi_{2\eta} = 0$, leaving us with

$$\begin{aligned} n_{2\xi} &= \frac{1}{2}\,\varphi_{1\xi}^2, \\ n_{2\eta} &= \frac{1}{2}\,\varphi_{1\eta}^2, \\ u_{2\xi} &= \frac{\sqrt[4]{3}}{6}\,\varphi_{1\xi}^2, \\ u_{2\eta} &= -\frac{\sqrt[4]{3}}{6}\,\varphi_{1\eta}^2, \end{aligned} \quad (36)$$

besides contributions $\widetilde{n}_2$ and $\widetilde{u}_2$, involving $\widetilde{\varphi}_2$ and combinations of $\varphi_{1\xi}\varphi_{1\eta}$.

At critical composition, the interesting new contributions appear in the equations for the third-order variables,

$$\begin{aligned} &\widehat{T} n_3 + \widehat{T}' n_1 + \frac{\partial n_1}{\partial\tau} + \widehat{X} u_3 + \widehat{X}(n_1 u_2 + n_2 u_1) + \widehat{X}' u_1 = 0, \\ &\widehat{T} u_3 + \widehat{T}' u_1 + \frac{\partial u_1}{\partial\tau} + \widehat{X}\,(u_1 u_2) + \widehat{X}\varphi_3 + \widehat{X}'\varphi_1 = 0, \\ &\widehat{X}^2\varphi_1 + n_3 - \frac{\sqrt{3}}{3}\,\varphi_3 - \varphi_1\varphi_2 - \frac{4-\sqrt{3}}{6}\,\varphi_1^3 = 0. \end{aligned} \quad (37)$$

Combining again the parts of these equations which contain terms that only depend on ξ or on η (besides τ) yields the typical mKdV equations

$$\begin{aligned} &\frac{\partial\varphi_{1\xi}}{\partial\tau} + (2-\sqrt{3})\sqrt[4]{3}\,\varphi_{1\xi}^2\,\frac{\partial\varphi_{1\xi}}{\partial\xi} + \frac{3^{3/4}}{2}\,\frac{\partial^3\varphi_{1\xi}}{\partial\xi^3} = 0, \\ &\frac{\partial\varphi_{1\eta}}{\partial\tau} - (2-\sqrt{3})\sqrt[4]{3}\,\varphi_{1\eta}^2\,\frac{\partial\varphi_{1\eta}}{\partial\eta} - \frac{3^{3/4}}{2}\,\frac{\partial^3\varphi_{1\eta}}{\partial\eta^3} = 0, \end{aligned} \quad (38)$$

for the right- and left-propagating solitary waves, respectively. Now the nonlinearity is cubic.

Additional information can be found in the terms which contain both ξ and η (in addition to τ), yielding

$$\frac{\partial^2 \widetilde{\varphi}_3}{\partial\xi\partial\eta} + \frac{\partial}{\partial\xi}\left\{\left[\frac{\partial P}{\partial\eta} - \frac{6\sqrt{3}-5}{12}\,\varphi_{1\eta}^2\right]\frac{\partial\varphi_{1\xi}}{\partial\xi}\right\} + \frac{\partial}{\partial\eta}\left\{\left[\frac{\partial Q}{\partial\xi} - \frac{6\sqrt{3}-5}{12}\,\varphi_{1\xi}^2\right]\frac{\partial\varphi_{1\eta}}{\partial\eta}\right\} + R = 0. \quad (39)$$

Here R represents all the terms in which the variables ξ and η are fully mixed, in a way which cannot contribute to the determination of P and Q. Most of these rather complicated terms depend on finding expressions for $\widetilde{n}_2$, $\widetilde{u}_2$ and $\widetilde{\varphi}_2$. This can only be done once (35) has been solved for $\widetilde{\varphi}_2$, and this in turn relies upon the solutions to (38).

The second and third terms in (39) will generate secular contributions at the next higher order. These are to be annulled, leading to

$$\frac{\partial P}{\partial\eta} = \frac{6\sqrt{3}-5}{12}\,\varphi_{1\eta}^2, \qquad \frac{\partial Q}{\partial\xi} = \frac{6\sqrt{3}-5}{12}\,\varphi_{1\xi}^2. \quad (40)$$

Thus, with (40), the phase shifts can also be determined.

Because mKdV equations like (38) are invariant for a sign inversion of $\varphi_{1\xi}$ or $\varphi_{1\eta}$, the one-soliton solutions of (38) are

$$\varphi_{1\xi} = \pm\sqrt{\frac{2\times 3^{3/4}}{2-\sqrt{3}}}\,\sqrt{v_\xi}\,\mathrm{sech}[\kappa_\xi(\xi - v_\xi\tau)], \qquad \varphi_{1\eta} = \pm\sqrt{\frac{2\times 3^{3/4}}{2-\sqrt{3}}}\,\sqrt{v_\eta}\,\mathrm{sech}[\kappa_\eta(\eta + v_\eta\tau)], \quad (41)$$

where the amplitudes are now $4.125\sqrt{v_\xi}$ and $4.125\sqrt{v_\eta}$, respectively, and

$$\kappa_\xi = \sqrt{\frac{2\,v_\xi}{3^{3/4}}} \simeq 0.937\sqrt{v_\xi}, \qquad \kappa_\eta = \sqrt{\frac{2\,v_\eta}{3^{3/4}}} \simeq 0.937\sqrt{v_\eta}. \quad (42)$$

Note that in (41) the respective $\pm$ signs are not coupled.

The phase shifts expressed by P and Q are found from the substitution of (41) into (40) and integration, yielding

$$\begin{aligned} P &= \frac{3^{1/8}(8+7\sqrt{3})}{2\sqrt{2}}\,\sqrt{v_\eta}\,\{\tanh[\kappa_\eta(\eta + v_\eta\tau)] + 1\} \\ &\simeq 8.162\,\sqrt{v_\eta}\,\{\tanh[\kappa_\eta(\eta + v_\eta\tau)] + 1\}, \\ Q &= \frac{3^{1/8}(8+7\sqrt{3})}{2\sqrt{2}}\,\sqrt{v_\xi}\,\{\tanh[\kappa_\xi(\xi - v_\xi\tau)] - 1\} \\ &\simeq 8.162\,\sqrt{v_\xi}\,\{\tanh[\kappa_\xi(\xi - v_\xi\tau)] - 1\}. \end{aligned} \quad (43)$$

Because P and Q are defined in terms of $\varphi_{1\eta}^2$ and $\varphi_{1\xi}^2$, respectively, the polarity of the modes does not play a role in this aspect of the problem.

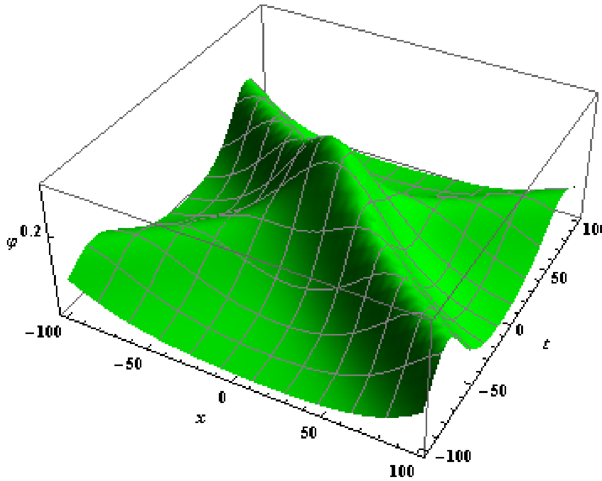

FIG. 5. (Color online) Head-on collision for critical plasma composition, at $\beta = \beta_c$, so that mKdV equations are needed here, for $v_\xi = 0.001$ and $v_\eta = 0.002$ and positive potential solitons.

As shown in Fig. 5 for two modes of positive polarity, the behavior is qualitatively reminiscent of what happens for other $\beta < \beta_c$, namely, there are compressive solitons. Compared to Fig. 2, there are less steep characteristics and larger widths, as one is now plotting "sech" rather than "sech squared" solutions. One could also have chosen two modes with negative polarities, and then the rarefactive solitons would qualitatively look like those in Fig. 4.

The most interesting difference between the generic and the critical case is that, in the critical case, the two mKdV equations (38) admit a combination of *positive and negative* modes. This is shown in Fig. 6 for a weaker right-propagating negative soliton and a stronger left-propagating positive soliton.

IV. CONCLUSIONS

In this paper we have treated the head-on collision between two solitons in a nonthermal plasma, taking great care to analyze in a systematic way the different assumptions needed to derive the corresponding nonlinear evolution equations and phase shifts. It is shown that the typical reductive perturbation expansion restricts one to the case of equal linear phase speeds used in the coordinate stretching.

For the generic case, the solitons are of KdV type, as is the case for simple electron-ion plasmas or plasmas having a more complicated multispecies composition but without critical parameter values. Both left- and right-propagating solitons must have the same electrostatic polarity, namely that given by the sign of the coefficient, A, of the nonlinear term in the KdV equation.

If the plasma parameters take on critical values, the quadratic nonlinearity in the KdV equation disappears

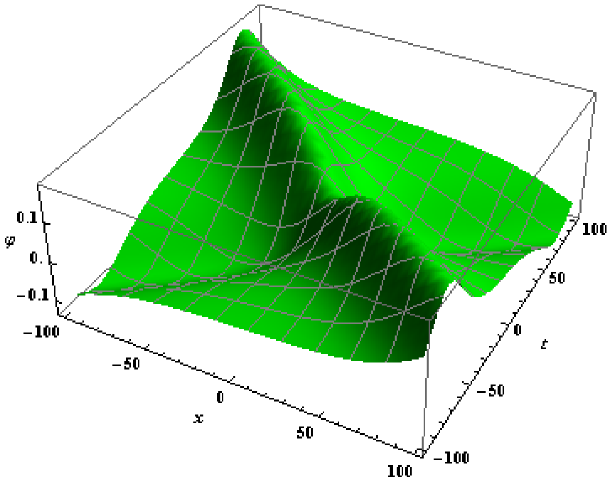

FIG. 6. (Color online) Head-on collision between solitons of opposite polarities, for critical plasma composition, at $\beta = \beta_c$, $v_\xi = 0.001$ (negative polarity) and $v_\eta = 0.002$ (positive polarity).

and the scaling works out in a different way. This leads to an mKdV equation with cubic nonlinearity. This part of the problem has not been treated before in the plasma literature, as far as could be ascertained.

Moreover, since the mKdV equation is invariant for inversion of the electrostatic polarity, it follows that combinations of solitons of different polarities now become possible, e.g., a negative right- and a positive left-propagating soliton. This result has not been pointed out before in a plasma physics context, and can not occur for surface solitons on shallow water [2], which are always compressive.

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