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Mathieu function solutions for photoacoustic waves in sinusoidal one-dimensional structures

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The photoacoustic effect for a one-dimensional structure whose sound speed varies sinusoidally in space is shown to be governed by an inhomogeneous Mathieu equation with the forcing term dependent on the spatial and temporal properties of the exciting optical radiation. New orthogonality relations, traveling wave Mathieu functions, and solutions to the inhomogeneous Mathieu equation are found which are used to determine the character of photoacoustic waves in infinite and finite length phononic structures. Floquet solutions to the Mathieu equation give the positions of the band gaps, the damping of the acoustic waves within the band gaps, and the dispersion relation for photoacoustic waves. The solutions to the Mathieu equation give the photoacoustic response of the structure, show the space equivalent of subharmonic generation, and acoustic confinement when waves are excited within band gaps.

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I INTRODUCTION

Since the photoacoustic effect is generated by the absorption of electromagnetic radiation, the character of the sound produced depends on both the temporal characteristics of the exciting radiation and its distribution in space. Waves generated by the photoacoustic effect have been shown to possess information on the geometry and acoustic properties of optically thin bodies[1], to produce acoustic radiation patterns dependent on the positions of objects in turbid media making possible photoacoustic medical imaging[2], and to generate radiation patterns with unique directionalities[3–6]. For a given spatial and temporal profile of the exciting radiation, additional factors that determine properties of photoacoustic waves are the space dependence of the optical absorption and the acoustic properties of the irradiated body, examples including structures designed with periodic absorption[7] in space to produce directional ultrasound, or surfaces with periodically varying acoustic properties resulting in the generation of surface acoustic waves[8–11], the latter showing the existence of band gaps.

For periodic structures, the same kinds of band gaps and dispersion relations[12–15] must obtain for photoacoustic waves as are found in phononic structures[16–18]. Virtually any structure whose properties vary periodically in space can be expected to have several properties in common with those derived for a periodic square well potential, which forms the basis for the Kronig-Penny model for electron wave propagation in crystals. Even simpler than a periodic square well potential is a structure whose properties vary sinusoidally in space, which would appear to represent mathematically the most straightforward case of a structure that gives rise to band phenomena, but which appears not to have been treated in the published literature. Here we investigate the problem of wave propagation in one-dimensional, sinusoidally modulated structures using Mathieu functions and their Flouquet representations to describe all properties of the solution to a wave equation. The simplicity of the solutions is evidenced in that all properties of the waves, their dispersion relation, and character of the band gaps are given in closed form. Furthermore we determine the properties of the photoacoustic effect in such a structure. What is perhaps unique about the photoacoustic effect in periodically modulated structures is that since the launching of photoacoustic waves is determined by the characteristics of the optical source, they can, in principle, be excited at any frequency, even within band gaps, showing interference effects not commonly seen in other forms of wave motion in periodic structures.

In this paper, all properties of the sinusoidally modulated structure, including band gaps, dispersion relations, damping within forbidden regions, and the character of the photoacoustic waves generated by several different kinds of sources are found in terms of properties of Mathieu functions. The space equivalent of sub-harmonic generation characteristic of Mathieu equations solutions is identified. The paper gives a derivation of an inhomogeneous Mathieu equation from the wave equation for pressure in Section II. Methods for solving the inhomogeneous Mathieu equation as well as new identities for Mathieu functions given in the Appendix are used in Sections III and IV to obtain solutions to the photoacoustic effect in an infinite structure and a structure with a finite length. Section V introduces new traveling wave Mathieu functions and gives solutions for an infinite structure where the optical excitation is confined in space. Floquet solutions to the Mathieu equation that describe the traveling wave solutions are given. The Appendix gives expressions for the Mathieu characteristic values lying along a straight line in the stability plot, orthogonality relations for the integer and fractional order Mathieu functions, and a completeness relation, all involving a new factor dependent on the modulation that have not been previously published. As well, a Green's function for a finite length structure whose properties are sinusoidally modulated is derived. The focus of the mathematical derivations is development of methods for solving the inhomogeneous Mathieu equation.

II WAVE EQUATION FOR THE PHOTOACOUSTIC EFFECT

The wave equation[1, 5, 19, 20] for the photoacoustic pressure p for an inviscid fluid were the effects of heat diffusion can be ignored[1, 5, 19] is described by the wave equation

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\mathrm{p} = -\frac{\beta}{C_P}\frac{\partial H}{\partial t}, \tag{1}$$

where p is the pressure, c is the sound speed, β is the thermal expansion coefficient, C_P is the specific heat capacity, H is the energy per unit volume and time added by absorption of light, and t is the time. Consider a 1-dimensional structure where the sound speed is modulated sinusoidally in space according to

$$\frac{1}{c^2} = \frac{1}{c_0^2}[1 - \hat{n}\cos(\frac{2\pi x}{\bar{a}})], \tag{2}$$

where the parameter $\hat{n}$ describes the modulation depth of the sound speed in the x direction, $\bar{a}$ is the lattice spacing, and c_0 is the unmodulated sound speed. Since $1/c_0 = \sqrt{\rho\kappa_s}$, where ρ is the density and κ_s is the compressibility, the structure could be modulated either in its density or in its compressibility to give the space dependence specified by Eq. 2. Substitution of Eq. 2 into Eq. 1 and transformation of the resulting expression into the frequency domain through use of

$$H = \bar{\alpha} I(x) e^{-i\omega t} \text{ and } \mathrm{p}(x,t) = p(x)e^{-i\omega t}, \tag{3}$$

where ω is the modulation frequency, $\bar{\alpha}$ is the optical absorption, and $I(x)$ is the intensity of the optical radiation gives

$$\frac{d^2}{dz^2}p(z) + [a - 2q\cos(2z)]p(z) = f(z), \tag{4}$$

where the following dimensionless quantities have been defined,

$$z = \tfrac{\pi}{\bar{a}}x, \quad a = (\tfrac{\omega\bar{a}}{\pi c_0})^2, \quad \hat{\omega} = (\tfrac{\omega\bar{a}}{\pi c_0}), \quad \gamma = \hat{n}/2$$
$$q = \tfrac{1}{2}(\tfrac{\omega\bar{a}}{\pi c_0})^2\hat{n}, \quad \text{and} \quad f(z) = \tfrac{i\,\hat{\omega}\,\bar{\alpha}\,\beta\,\bar{a}\,c_0}{\pi C_P}I(z).$$

Equation 4 without the forcing term, *i.e.* the homogeneous differential equation, is known as the Mathieu equation[12, 21, 22]. It is known that solutions to the Mathieu equation can be periodic, aperiodic, or unbounded, the last of these, which are found in unstable regions of the a versus q plot gives rise to forbidden bands. Figure 1 shows the stability plot for Mathieu functions from which the character of the Mathieu function for any point (q, a) can be determined.

III PHOTOACOUSTIC EFFECT IN AN INFINITE STRUCTURE

Variation of parameters solution

For periodic excitation along an infinite structure, the solution to Eq. 4 for the photoacoustic pressure can be written as the sum of Eq. A9 for the inhomogeneous differential equation, plus solutions to the homogeneous Mathieu equation so that

$$p(z) = -\frac{ce(a,\gamma a,z)}{W}\int_0^z se(q,\gamma a,z')\,f(z')dz' - \frac{se(a,\gamma a,z)}{W}\int_z^{2\pi} ce(a,\gamma a,z')\,f(z')dz' + A\,ce(a,\gamma a,z) + B\,se(a,\gamma a,z), \tag{5}$$

where $W\{ce,se\} = ce(a,\gamma a,z)se'(a,\gamma q,z) - se(a,\gamma a,z)ce'(a,\gamma a,z)$, and A and B are constants. Since the acoustic properties of the structure are periodic, both the acoustic pressure and its first derivative must be repetitive in space and obey the boundary conditions $p(0) = p(2\pi)$ and $p'(0) = p'(2\pi)$, respectively. Imposition of these conditions on $p(z)$ gives the constants as

$$A = \frac{\Delta s\, se'(0) - s(0)\,\Delta s'}{\Delta c\,\Delta s' - \Delta s\,\Delta c'}\int_0^{2\pi} \frac{ce(z')}{W(z')} f(z')dz' + \frac{ce(2\pi)\,\Delta s' - ce'(2\pi)\,\Delta s}{\Delta c\,\Delta s' - \Delta s\,\Delta c'}\int_0^{2\pi} \frac{se(z')}{W(z')} f(z')dz'$$

$$B = \frac{se(0)\,\Delta c' - se'(0)\,\Delta c}{\Delta c\,\Delta s' - \Delta s\,\Delta c'}\int_0^{2\pi} \frac{ce(z')}{W(z')} f(z')dz' + \frac{ce'(2\pi)\,\Delta c - ce(2\pi)\,\Delta c'}{\Delta c\,\Delta s' - \Delta s\,\Delta c'}\int_0^{2\pi} \frac{se(z')}{W(z')} f(z')dz',$$

where the following have been defined

$$\Delta c = ce(a,q,2\pi) - ce(a,q,0) \qquad \Delta s = se(a,q,2\pi) - se(a,q,0)$$
$$\Delta c' = ce'(a,q,2\pi) - ce'(a,q,0) \quad \Delta s' = se'(a,q,2\pi) - se'(a,q,0),$$

and where the abbreviations $se(z) = se(a,\gamma a,z)$ and $ce(z) = ce(a,\gamma a,z)$ have been used.

Expansion in integer order Mathieu functions

Since the photoacoustic pressure in an infinite periodic structure must be periodic, the solution to Eq. 4 can be written as expansions in $ce_m(z, q_m^{(c)})$ and $se_m(z, q_m^{(s)})$ as

$$p(z) = \sum_{m=0}^{\infty} A_m\, ce_m(z, q_m^{(c)}) + \sum_{m=1}^{\infty} B_m\, se_m(z, q_m^{(s)}), \tag{6}$$

where the A_m and B_m's are constants. Values of $q_m^{(c)}$, $q_m^{(s)}$, $\bar{\pi}_m^{(c)}$, and $\bar{\pi}_m^{(s)}$ are used in the expansion so that the condition $a = q/\gamma$ is automatically satisfied. Values for $q_m^{(c)}$, $q_m^{(s)}$ as well as a straight line along which the expansion is carried out can be seen in Fig. 1 for $\gamma = 0.35$. After substitution of Eq. 6 for p into Eq. 4 and use of the orthogonality relations given in Eqs. A4, the spatial component of the photoacoustic pressure is found to be

$$\begin{aligned} p(z) = &\sum_{m=0}^{\infty} \frac{ce_m(z,q_m^{(c)})}{\bar{\pi}_m^{(c)}[a-a_m(q_m^{(c)})]} \int_0^{2\pi} ce_m(z',q_m^{(c)})\, f(z')dz' \\ &+ \sum_{m=1}^{\infty} \frac{se_m(z,q_m^{(s)})}{\bar{\pi}_m^{(s)}[a-b_m(q_m^{(s)})]} \int_0^{2\pi} se_m(z',q_m^{(s)})\, f(z')dz'. \end{aligned} \tag{7}$$

In practice, Eq. 7 is evaluated by limiting the summation to N terms: first, N solutions to Eqs. A3 and A4 for $q_m^{(c)}$, $q_m^{(s)}$, $\bar{\pi}_m^{(s)}$, and $\bar{\pi}_m^{(s)}$, are found, the indicated integrals carried out, the series summed, p(z,t) determined from Eq. 3, and its real part taken. For the determination of the acoustic pressure at a different frequencies it is necessary to change only a in Eq. 7 as the other quantities in this expression are frequency independent. It follows from Eq. 7 that photoacoustic resonances are found at $\hat{\omega} = [a_m(q_m^{(c)})]^{1/2}$ and $\hat{\omega} = [b_m(q_m^{(s)})]^{1/2}$ for all allowed values of m. Values of the resonance frequencies for $\gamma = 0.35$, as determined by a_m and b_m, can be found from Fig. 1

For delta function heating of the structure periodically at points located distances z_0 from the beginning of each cell, Eq. 7 is evaluated with $f(z)$ of the form $f(z) = \bar{f}_D\, \delta(z - z_0)$, which gives

$$\begin{aligned} p(z) = &\bar{f}_D \sum_{m=0}^{\infty} \frac{ce_m(z,q_m^{(c)})}{\bar{\pi}_m^{(c)}[a-a_m(q_m^{(c)})]} ce_m(z_0,q_m^{(c)}) \\ &+ \bar{f}_D \sum_{m=1}^{\infty} \frac{se_m(z,q_m^{(s)})}{\bar{\pi}_m^{(s)}[a-b_m(q_m^{(s)})]} se_m(z_0,q_m^{(s)}). \end{aligned} \tag{8}$$

A plot of the amplitude of the photoacoustic pressure versus frequency from Eq. 8 is shown in Fig. 2 for delta function excitation.

If the intensity of the radiation in space is uniform along the entire structure so that $f(z)$ is a constant $\bar{f}_C$ then Eq. 7 reduces to

$$p(z) = \bar{f}_C \sum_{m=0}^{\infty} \frac{ce_m(z,q_m^{(c)})}{\bar{\pi}_m^{(c)}[a - a_m(q_m^{(c)})]} \int_0^{2\pi} ce_m(z',q_m^{(c)})dz' \tag{9}$$

In Fig. 3 a plot of the amplitude of the photoacoustic effect versus coordinate is shown for uniform irradiation of the structure. For small values of γ the resonances appear at nearly integer values of the dimensionless frequency. For larger values of γ, the average sound speed in the structure increases so that the peaks are shifted to higher frequencies.

A second approach to determining the photoacoustic pressure is to expand $p(z)$ along a vertical line in the stability plot at a constant value of q as shown in Fig. 1 so that $ce_m(z,q)$ and $se_m(z,q)$

replace $ce_m(z, q_m^{(c)})$ and $se_m(z, q_m^{(s)})$ in Eqs. 6. Using the usual[21] orthogonality relations noted above in Eq. A6, the photoacoustic pressure becomes

$$\begin{aligned} p(z) = \sum_{m=0}^{\infty} \tfrac{ce_m(z,q)}{\pi[a-a_m(q)]} \int_0^{2\pi} ce_m(z', q) f(z') dz' \\ + \sum_{m=1}^{\infty} \tfrac{se_m(z,q)}{\pi[a-b_m(q)]} \int_0^{2\pi} se_m(z', q) f(z') dz'. \end{aligned} \tag{10}$$

For each different frequency, all parameters in this expression must be calculated.

IV FINITE LENGTH STRUCTURES

Expansion in fractional order Mathieu functions

Consider a one-dimensional structure that extends along the x axis from 0 to L, where L is an integer number of $\bar{a}$. Since the fractional Mathieu functions $se_{m+p/s}(z, q)$ have the property that for integer values of s and p they are zero for $z = 0$ and $s\pi$, it is possible to expand the photoacoustic pressure in the form

$$p(z) = \sum_{m=0,\ p=0}^{\infty, s-1} K_{m+p/s}\, se_{m+p/s}(z, q_{m+p/s}^{(s)}), \tag{11}$$

where the $K_{m+p/s}$ are constants, $s = L/\bar{a}$ and it is understood here and in the following that the summation is restricted to where m and p cannot be zero at the same time. The fractional order Mathieu functions $se_{m+p/s}$ along the line $a = q/\gamma$ obey

$$\frac{d^2}{dz^2} se_{m+p/s}(z, q_{m+p/s}^{(s)}) = -b_{m+p/s}(q_{m+p/s}^{(s)})[1 - 2\gamma\cos(2z)] se_{m+p/s}(z, q_{m+p/s}^{(s)}),$$

where the $b_{m+p/s}(q_{m+p/s}^{(s)})$ are characteristic values that satisfy $q_{m+p/s}^{(s)} = \gamma b_{m+p/s}(q_{m+p/s}^{(s)})$. The fractional Mathieu functions $se_{m+p/s}$ obey (in addition to the orthogonality relations noted following Eq. A4) the orthogonality relation

$$\int_0^{s\pi} [1 - 2\gamma\cos(2z)] se_\mu(z, q_\mu^{(s)})\, se_\nu(z, q_\nu^{(s)}) dz = \frac{s}{2} \bar{\pi}_\mu^{(s)} \delta_{\mu,\nu}, \tag{12}$$

where μ and ν are of the form $m+p/s$. Substitution of Eq. 11 into Eq. 4 followed by multiplication by $se_{m'+p'/s}$ and integration over the range from 0 to $s\pi$ yields

$$p(z) = 2 \sum_{m=0,\ p=0}^{\infty,\ s-1} \frac{se_{m+p/s}(z, q_{m+p/s}^{(s)})}{s\bar{\pi}_{m+p/s}^{(s)}[a - b_{m+p/s}(q_{m+p/s}^{(s)})]} \int_0^{s\pi} se_{m+p/s}(q_{m+p/s}^{(s)}, z')\, f(z') dz' \tag{13}$$

A stability plot for Mathieu functions showing the expansion of the solution along the line $a = q/\gamma$ indicated in Eq. 13 is shown in Fig. 4. Plots of the solution at a fixed time for two values of $\hat{\omega}$ are shown in Fig. 5. Note that the properties of fractional Mathieu functions guarantee that solutions at the origin and at $z = s\pi$ are zero. A plot of the photoacoustic amplitude versus frequency for a structure with $s = 2$ is given in Fig. 5. It is of note that in the limit when γ approaches zero the sine elliptical function becomes a sine function, and the double summation in Eq. 13 reduces to a single summation, giving

$$p(z) = \frac{2\pi^2 L}{\bar{a}^2} \sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi x}{L})}{(\omega L/c_0)^2 - n^2\pi^2} \int_0^L \sin(\frac{n\pi x'}{L}) f(x')dx', \tag{14}$$

which is the same result obtained as an eigenfunction expansion solution to the wave equation for pressure for a structure of length L without sound speed variations.

Green's function solution

Using the Green's function for a finite length structure given by Eq. A10 it is straightforward to obtain $p(z)$ as

$$p(z) = -\frac{se(a, \gamma a, z - s\pi)}{W_G} \int_0^z se(a,\, \gamma a,\, z')f(z')dz' - \frac{se(a, \gamma a, z)}{W_G} \int_z^{s\pi} se(a,\, qa,\, z' - s\pi)f(z')dz', \tag{15}$$

The Wronskian W_G, although dependent on frequency, is independent of the coordinate, and has been removed from the integral.

V CONFINED EXCITATION

As a result of either manipulation of the absorbance of the structure or by appropriate direction of the optical beam, it is possible to deposit heat in a confined region of an infinite structure as opposed to its entirety. Consider the case where photoacoustic waves in an infinite periodic structure are excited in a region of space extending from $-L$ to L along the x axis. The acoustic waves within the region of excitation for amplitude modulated continuous heat deposition will be standing waves, but those outside of the excitation region must be traveling waves. For the waves outside the excitation region, two linearly independent traveling waves $he^{(1)}(a, q, z)$ and $he^{(2)}(a, q, z)$ can be defined according to

$$\begin{aligned} he^{(1)}(a, q, z) &= ce(a, q, z) + i\, se(a, q, z) \\ he^{(2)}(a, q, z) &= ce(a, q, z) - i\, se(a, q, z), \end{aligned} \tag{16}$$

which satisfy the Mathieu equation and which have the appropriate form for large z. These two solutions, as they are linear combinations of ce and se can describe standing waves in the region with optical absorption, as well. Solution to the inhomogeneous wave equation, Eq. 4, can be found from the method of variation of parameters solution given above by Eq. A9 to yield

$$\begin{aligned} p(z) = &-\tfrac{he^{(1)}(z)}{\bar{W}} \int_{-\hat{L}}^{z} he^{(2)}(z')\, f(z')dz' \\ &- \tfrac{he^{(2)}(z)}{\bar{W}} \int_{z}^{\hat{L}} he^{(1)}(z')\, f(z')dz' \end{aligned} \tag{17}$$

where $\hat{L} = L/\bar{a}$ and the Wronskian $\bar{W}$ is related to the Wronskian W for Mathieu functions through $\bar{W}\{he^{(1)}, he^{(2)}\} = -2iW\{ce, se\}$. Depending on whether the solution to the right or the left of the origin is bounded, the sign of $\hat{L}$ is reversed in Eq. 17 to give bounded solutions.

Solutions in Floquet form

In general, the solutions to the homogeneous Mathieu equation, $se(a, q, z)$ and $ce(a, q, z)$, can be expressed in Floquet form[21, 22] as

$$p(z) = Ae^{\mu z}\phi(z) + Be^{-\mu z}\phi(-z), \tag{18}$$

where ϕ is a periodic function, A and B are arbitrary constants, and μ is the Mathieu characteristic exponent. The characteristic exponent is a function of q and a, and is written, in general, as $\mu = \alpha + i\beta$ where α and β are real numbers. Whenever $\alpha = 0$, the dispersion relation for photoacoustic waves[23] can be found for a given value of γ as the imaginary component of the characteristic exponent β as a function of $\hat{\omega}$. The band gaps are determined by those values of $\hat{\omega}$ that give μ with a nonzero real component.

In the case of confined excitation, solutions to the Mathieu equation can be written in the form of either Eq. 18 or 16. The general form for $he^{(1)}$ must be $he^{(1)}(a, q, z) = Ae^{\mu z}\phi(z) + Be^{-\mu z}\phi(-z)$; as well, ce and se, which are even and odd functions respectively, can be written as

$$\begin{aligned} ce(a, q, z) &= e^{\mu z}\phi_c(z) + e^{-\mu z}\phi_c(-z) \\ se(a, q, z) &= e^{\mu z}\phi_s(z) - e^{-\mu z}\phi_s(-z). \end{aligned} \tag{19}$$

If these two expressions are substituted into the first of Eqs. 16 then the pair of equations

$$\begin{aligned} A\phi(z) &= \phi_c(z) + i\phi_s(z) \\ B\phi(z) &= \phi_c(z) - i\phi_s(z), \end{aligned} \tag{20}$$

is found, from which it follows that

$$\frac{\phi_c(z)}{\phi_s(z)} = i\frac{A+B}{A-B} = C, \tag{21}$$

where C must be a purely imaginary number. Since $\int_{period} ce\, ce^* dz = \int_{period} se\, se^* dz$, it can be shown that $\phi_c(z) = C\phi_s(z)$ and $\phi_c^*(z) = -C\phi_s^*(z)$, from which it follows that $|C|^2 = 1$. From examination of Eq. 21 either A or B must equal 0. The same arguments are valid for $he^{(2)}$ as well. The Floquet form of the functions he are thus given by

$$\begin{aligned} he^{(1)}(a,q,z) &= Ae^{\mu z}\phi(z) \\ he^{(2)}(a,q,z) &= Ae^{-\mu z}\phi(-z), \end{aligned} \tag{22}$$

where ϕ is a periodic function, and the signs of α are chosen according to whether the solution is taken along the positive or negative z axis.

Photoacoustic waves can be excited within or outside of any band gap, since, according to Eq. 4, the frequency of the acoustic wave is determined by forcing term. Waves excited by a source limited to a fixed region of the structure at frequencies within the band gaps are damped in space depending on the magnitude of α. For a structure that does not possess acoustic damping, as considered here, when $\hat{\omega}$ approaches the edge of the band gap the photoacoustic amplitude increases without limit and the wave extends farther in space. Waves within the band gaps thus exhibit a confinement effect where the stored acoustic energy increases as the modulation frequency approaches the edges of the band gaps.

Outside of the band gaps, the acoustic waves fill the entire space: they are neither damped in space nor do they diverge. The Mathieu characteristic exponent as indicated by Eq. 23 does not have a real component and the Floquet form of the solutions to the Mathieu equation can be written

$$\begin{aligned} he^{(1)}(a,q,z) &= e^{i\beta z}\phi(z) \\ he^{(2)}(a,q,z) &= e^{-i\beta z}\phi(-z). \end{aligned} \tag{23}$$

It is known from the theory of Mathieu functions[21], that if β can be expressed as a quotient of two integers, then the solutions of the Mathieu equation are periodic. The functions $he^{(1)}$ and $he^{(2)}$ will thus be combinations of either integer or fractional order Mathieu functions. If, on the other hand, β is an irrational number, the solutions ce and se are not periodic with the result that $he^{(1)}$ and $he^{(2)}$ also will not be periodic. As can be seen from Eqs. 23, when β is a rational number, $he^{(1)}$ and $he^{(2)}$are products of two periodic functions, $\exp(\pm i\beta z)$ and $\phi(\pm z)$, so that the photoacoustic pressure outside the band gaps appears as amplitude modulated carrier waves that fill the entire space.

VI DISCUSSION

Although the series expansion in integer Mathieu functions solution Eq. 7 requires programming to evaluate the normalization constants and characteristic values, its numerical evaluation is faster than that for the variation of parameters solution given by Eq. 5. As well, the the series expansion in fractional Mathieu functions Eq. 13 is more easily evaluated than the Green's function solution, Eq. 15. In both cases, the relative ease of computation depends on the number of integrations that must be carried out. Of further note is that the solution for the pressure for an expansion in integer Mathieu functions along the line $a = q/\gamma$, given by Eq. 7, requires a smaller number of terms than that along the line of constant q from Eq. 10 to obtain the same accuracy.

The Floquet solution given by Eq. 18 has a useful feature in that the important properties of wave propagation can be found from the characteristic exponent which is available from contemporary computer programs such as Mathematica. Thus, the dispersion relation, the positions of band gaps, and the damping within a band gap can be found without resort to numerical integration. The same computer programs can give the frequencies of the resonances indicated in Eqs. 7 or 13 through use of Eqs. A3 as well the solutions for the acoustic pressure both inside and outside the band gaps.

As noted in Ref. [23] for the periodically excited infinite structure, the frequencies at which the resonances are found change dependent on the magnitude of γ. For small γ, the positions of the resonances, such as those shown in Fig. 2, are found lie at nearly integer values of $\hat{\omega}$. If the resonances for small γ are considered to lie roughly at $n\hat{\omega}$ for nonzero integer values of n, then the acoustic wavelength for the lowest frequency resonance is not at the expected value of $\bar{a}$ but rather at $2\bar{a}$, which is a characteristic of solutions to the Mathieu equation, and which can be considered to the space domain equivalent of what is referred to in the time domain as subharmonic resonance[21].

An effect not reported previously for photoacoustic effect is that acoustic waves can be excited by uniform illumination of a structure, which follows from evaluation of Eq. 7 with constant f, as shown by Eq. 9. Since the integral over $ce_m(z, q_m^{(c)})$ does not vanish, sound whose wavelength is dependent on $\bar{a}$ is generated, showing that variations in acoustic impedance make possible the production of sound when optical heat deposition is uniform. A property of the photoacoustic effect at low frequencies, as can be seen in the contrasting behavior of the pressure in the plots given in Figs. 2 and 6, is that in the fixed length structure, the pressure amplitude approaches zero whereas in the infinite structure the pressure tends to infinity. The contrasting behavior comes as a result of the boundary conditions, where, in the former, the structure expands with optical energy deposition, whereas in the latter, the structure cannot undergo a net expansion along its entirety.

In so far as application of the results given here are concerned, it is of note that photoacoustic radiators similar to those discussed in Section IV constructed to have optically thin, periodic absorptions have been shown to exhibit strong photoacoustic interference effects[7], with both the strength and frequency selectivity of the resonances increasing with the number of periods. The directional properties of this source were found to increase, as well, with the length of the device. Given the availability of high repetition rate pulsed lasers, such as mode locked lasers, it appears straightforward to construct directional photoacoustic sources that operate at frequencies up to one hundred MHz by irradiation of simple, modulated structures. Even higher frequencies are possible through use of fs lasers, in which case, the upper frequency limit would be contingent more on the capability of fabrication of the structure rather than on the characteristics of the laser.

Solutions to other one-dimensional problems can be found in Brillouin's review, *Wave Propagation in Periodic Structures*[12]. It is of note that the results given here for the dispersion relation for a sinusoidally modulated structure are qualitatively similar to those derived for Cauchy, Baden-Powell, and Kelvin models for point masses attached with Hooke's Law forces[12]. The dispersion relation for the acoustical branch of a one-dimensional lattice also shows behavior strongly similar to that derived for the infinite structure treated here. The solution for the photoacoustic effect in a finite length structure can be shown to be related to the problem of an oscillating string with a periodic density along its length, as treated in Ref. [21] In addition, Brillouin shows that electrical LC circuits can be constructed with stop bands and dispersion relations similar to those derived from the Floquet solution to the Mathieu equation given here. Further work on sound propagation in waveguides with sinusoidal variations in their geometries that bear some similarity to the present one-dimensional problem can be found in Refs. [24] and [25]. Such structures exhibit band gaps that arises from the character of the interfaces rather than from impedance variations, and do not admit to simple Floquet solutions.

Perhaps the unique characteristic of photoacoustic excitation of a periodic structure is that waves can be generated within band gaps. For structures where excitation is restricted in space, plots of acoustic amplitude versus frequency obtained from Eq. 17 (see Ref. [23]) show that at the centers of the band gaps the damping is greatest, and that as the modulation frequency of the radiation approaches the edges of the band gaps, the amplitude of the photoacoustic pressure becomes arbitrarily large. At the interior edges of the gaps, the acoustic waves show a confinement with the acoustic amplitude decaying in space away from the region of excitation, whereas immediately outside of the band gaps, in the allowed regions, the acoustic amplitude, which becomes large as well, is distributed throughout the structure. Such effects are general, and must be present for waves other than photoacoustic waves in sinusoidally modulated structures–the clarity with which these effects are seen in the case of the photoacoustic effect comes from its description by an inhomogeneous wave equation, the forcing term in which allows for wide variation both the temporal and spatial character of wave excitation.

Appendix: Methods of solution

Expansions in Mathieu functions with q=γa

The Sturm-Liouville equation [26] can be written as

$$\frac{d^2y(z)}{dz^2} - \bar{q}(z)y(z) + \lambda\bar{\rho}(z)y(z) = 0. \tag{A1}$$

If $\bar{q}(z)$ is taken to be zero and the weighting function $\bar{\rho}(z)$ to be $[1-2\gamma\cos(2z)]$, then the linear operator L in the Sturm-Liouville equation is given by $L = d^2/dz^2$. Thus, with the imposition of the requirement $a = q/\gamma$, the integer order cosine elliptic $ce_m(z,q)$ and sine elliptic $se_m(z,q)$ Mathieu functions obey

$$\begin{aligned} \frac{d^2}{dz^2}ce_m(z,q_m^{(c)}) &= -a_m(q_m^{(c)})[1-2\gamma\cos(2z)]ce_m(z,q_m^{(c)}), \\ \frac{d^2}{dz^2}se_m(z,q_m^{(s)}) &= -b_m(q_m^{(s)})[1-2\gamma\cos(2z)]se_m(z,q_m^{(s)}), \end{aligned} \tag{A2}$$

where m is a positive integer that ranges from 0 to ∞ for ce_m and 1 to ∞ for se_m. As the Mathieu characteristic values a_m and b_m are found along the line $a = q/\gamma$ they must obey the implicit equations

$$q_m^{(c)} = \gamma a_m(q_m^{(c)}) \text{ or } q_m^{(s)} = \gamma b_m(q_m^{(s)}), \tag{A3}$$

where the superscripts (c) and (s) refer to the cosine and sine elliptic functions respectively. The following orthogonality relations also follow from the Sturm-Liouville equation

$$\begin{aligned} \int_0^{2\pi}[1-2\gamma\cos(2z)]ce_m(z,q_m^{(c)})\,ce_n(z,q_n^{(c)})dz &= \bar{\pi}_m^{(c)}\delta_{m,n} \\ \int_0^{2\pi}[1-2\gamma\cos(2z)]se_m(z,q_m^{(s)})\,se_n(z,q_n^{(s)})dz &= \bar{\pi}_m^{(s)}\delta_{m,n} \\ \int_0^{2\pi}[1-2\gamma\cos(2z)]ce_m(z,q_m^{(c)})\,se_n(z,q_n^{(s)})dz &= 0, \end{aligned} \tag{A4}$$

where $\bar{\pi}_m^{(c)}$ and $\bar{\pi}_m^{(c)}$ are constants. As well, the completeness relations for ce_m and se_m are of the form

$$\delta(z - z') = [1 - 2\gamma \cos(2z)] \sum_m ce_m(z, q_m^{(c)})\, ce_m(z', q_m^{(c)}). \tag{A5}$$

The Sturm-Liouville equation also applies to fractional Mathieu functions; thus, in Eq. A4 and A5 the integer order Mathieu functions ce_m and se_m can be replaced by the corresponding fractional order Mathieu functions by substituting indices $m + p/s$ for m and $m' + p'/s$ for n, changing the range of integration from 0 to $2\pi s$, where s is an integer, and replacing $\bar{\pi}_m^{(c)}$ and $\bar{\pi}_m^{(s)}$ by $s\bar{\pi}_{m+p/s}^{(c)}$ and $s\bar{\pi}_{m+p/s}^{(s)}$. For the present problem, repetitive solutions are required, thus p and s must be integers so that p/s is a rational number.

Expansions in Mathieu functions with q constant

There is a second choice of $\bar{\rho}$ and $\bar{q}$ in the Sturm-Liouville equation, that is, where $\bar{q}(z) = 2q \cos 2z$, and $\bar{\rho}(z) = 1$ are taken. The Sturm-Liouville operator then becomes $L = d^2/dz^2 - 2q \cos 2z$. This identification leads to the usual orthogonality relations found in the literature for Mathieu functions[21] for ce_m and se_m of the form

$$\int_0^{2\pi} ce_m(z, q)\, ce_n(z, q) dz = \pi \delta_{m,n}, \tag{A6}$$

and orthogonality between se_m and ce_m as well. Two completeness relations follow from the choice of $\bar{q}$ and $\bar{\rho}$, the first being $\delta(z - z') = \sum_m ce_m(z, q)\, ce_m(z', q)$, and the second the same identity but with se_m substituted for ce_m. In these relations, q is a frequency dependent quantity determined by the choice of $\hat{\omega}$ and $\hat{n}$ and is not governed by Eqs. A3. The orthogonality relations of the form of Eq. A4 also are valid for fractional Mathieu functions[21] with m and n replaced as in the preceding section, but with the range of integration extending from 0 to $2\pi s$ and the constants $\bar{\pi}_m^{(c)}$ and $\bar{\pi}_m^{(c)}$ replaced by $s\pi$.

Variation of parameters solution

The inhomogeneous Mathieu equation can be solved by using the variation of parameters method[27] where two independent solutions to the homogeneous Mathieu equation, y_1 and y_2

are combined with two arbitrary functions u_1 and u_2 to form a solution p according to

$$y = u_1(z)y_1(z) + u_2(z)y_2(z). \tag{A7}$$

The procedure is to substitute the solution given by Eq. A7 into Eq. 4. Since the function u_1 and u_2 are as yet unspecified, it is possible to impose the condition $u_1'(z)y_1(z) + u_2'(z)y_2(z) = 0$, which then yields expressions for the derivatives,

$$u_1' = -\frac{y_2(z)\, f(z)}{W(z)} \text{ and } u_2' = \frac{y_1(z)\, f(z)}{W(z)}, \tag{A8}$$

where $W(z) = y_1y_2' - y_1'y_2$ is a Wronskian. Upon integration of the expressions in Eq. A8, the particular integral of the Mathieu equation becomes,

$$y(z) = -y_1(z)\int_{\alpha}^{z} \frac{y_2(z')}{W(z')} f(z')dz' + y_2(z)\int_{\beta}^{z} \frac{y_1(z')}{W(z')} f(z')dz', \tag{A9}$$

where α and β are constants to be determined by the boundary conditions.

Green's function

Consider a structure that extends along the x axis from 0 to L, which corresponds to z ranging from 0 to $s\pi$, where $s = L/\bar{a}$. Two independent solutions for y_1 and y_2 are $se(a, \gamma a, z)$ and $se(a, \gamma a, z - s\pi)$, which, since se is zero at $z = 0$, satisfy both the boundary condition of zero pressure at the two ends of the structure and obey the relation $a = q/\gamma$. A Green's function $G(z, z')$ can be constructed for Eq. 4 of the form

$$G(z,z') = \begin{cases} A\, se(a,q,z) & z < z' \\ B\, se(a,q,z-s\pi) & z > z' \end{cases},$$

where A and B are constants. Using standard methods[26], $G(z, z')$ is found to be

$$G(z,z') = \begin{cases} \frac{se(a,q,z)\, se(a,q,z'-s\pi)}{W_G} & z < z' \\ \frac{se(a,q,z-s\pi)\, se(a,q,z')}{W_G} & z > z' \end{cases}, \tag{A10}$$

where $W_G\{se(z - s\pi, q), se(z, q)\} = se(a, q, z - s\pi)se'(a, q, z) - se'(a, q, z - s\pi)se(a, q, z)$

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FIGURE CAPTIONS

Fig. 1 Stability plot, a versus q, for Mathieu functions. Solutions in the regions in white give either aperiodic, or periodic fractional Mathieu functions. In the gray regions the solutions are unbounded. Integer value Mathieu functions are found at the borders between the shaded and white regions. The straight line $a = q/\gamma$ is plotted for a value of $\gamma = 0.35$. Several Mathieu characteristic values a_m and b_m are shown along this line that are solutions to Eq. A3 used for evaluation of Eq. 7. The intersections of the vertical line $q = 9$ at the boundaries between the stable and unstable regions give values of a_m and b_m used in the solution given by Eq. 10.

Fig. 2 Magnitude of the photoacoustic pressure amplitude p in arbitrary units versus dimensionless frequency $\hat{\omega}$ for an infinite structure with $\gamma = 0.3$ for delta function heat deposition along the entire structure for $z_0 = \pi/14$.

Fig. 3 Photoacoustic pressure p in arbitrary units versus dimensionless coordinate z for (top plot) $\hat{\omega} = 1$, and (bottom plot) $\hat{\omega} = 7$ for an infinite structure with uniform irradiation and $\gamma = 0.35$.

Fig. 4 Stability plot of a versus q for the homogeneous Mathieu equation. The straight line is a plot of $a = q/\gamma$ for a value of $\gamma = 0.35$. The points marked with subscripts are characteristic values for fractional Mathieu functions from solution of Eq. A3 with $s = 2$. The unmarked points show solutions to Eq. A3 for integer order Mathieu functions.

Fig. 5 Photoacoustic pressure p in arbitrary units versus dimensionless coordinate z for (top plot) $\hat{\omega} = 1$, and (bottom plot) $\hat{\omega} = 7$ for a finite structure with $\gamma = 0.35$ and $s = 2$.

Fig. 6 Magnitude of the photoacoustic pressure amplitude p in arbitrary units versus dimensionless frequency $\hat{\omega}$ for a finite structure with $\gamma = 0.35$ and $s = 2$ for delta function heat deposition at $z_0 = \pi/14$.

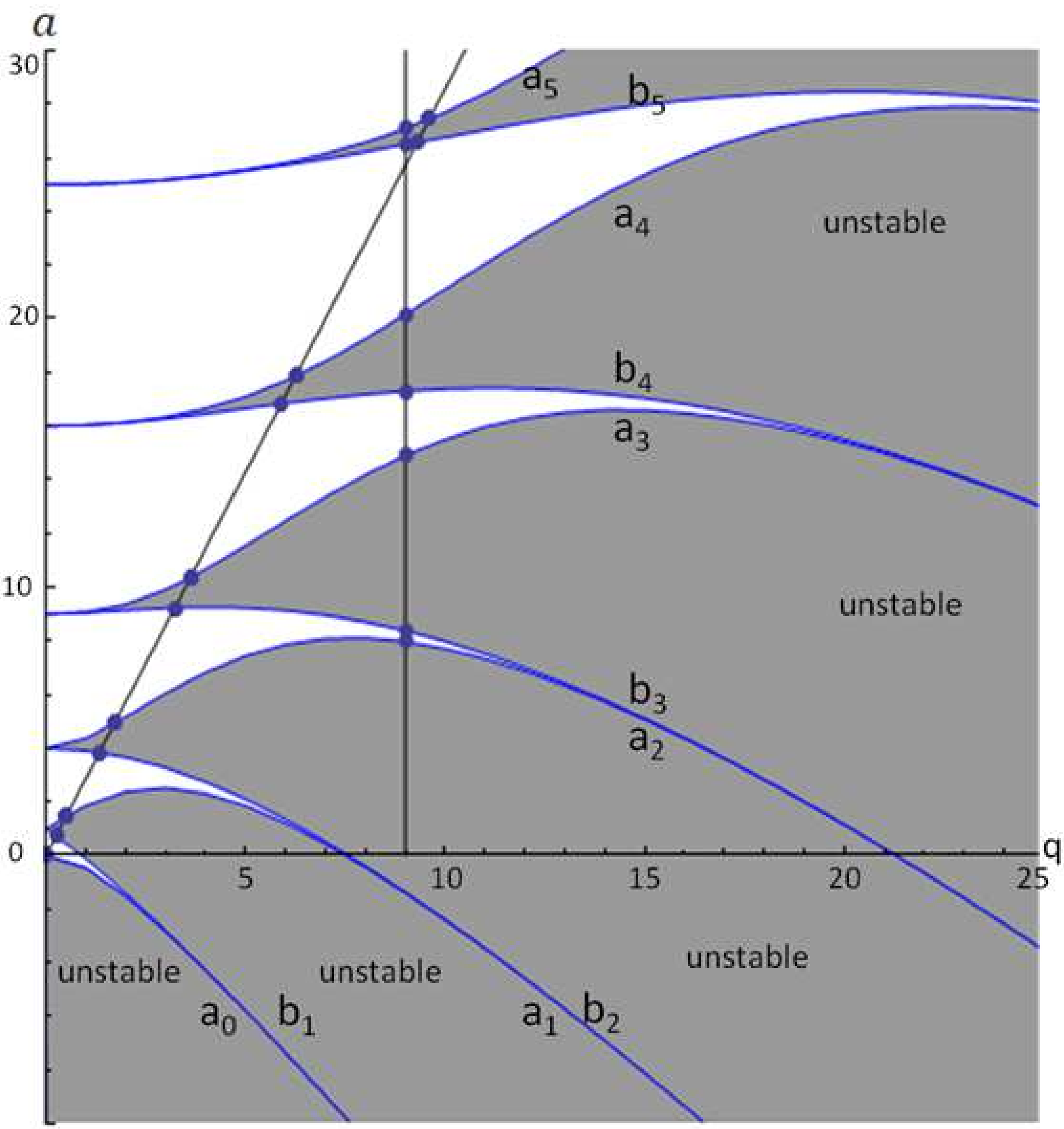


Figure 1 EK10905 22Jun2012

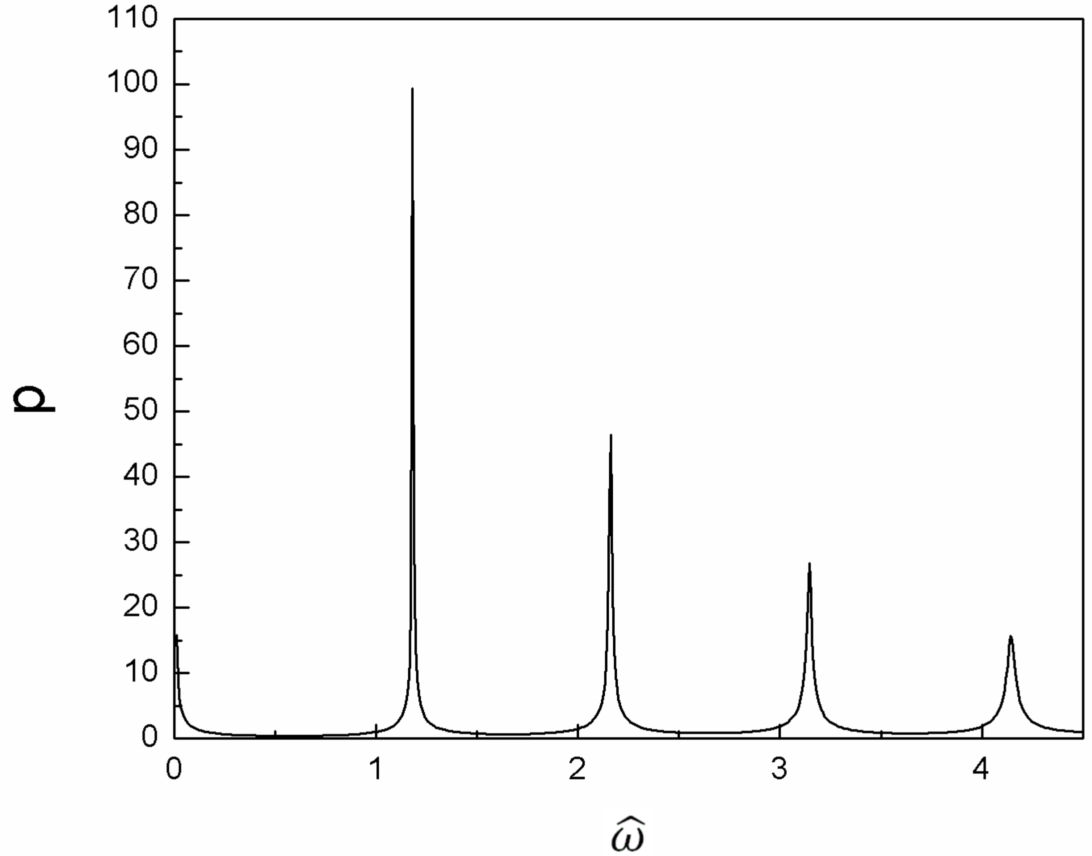
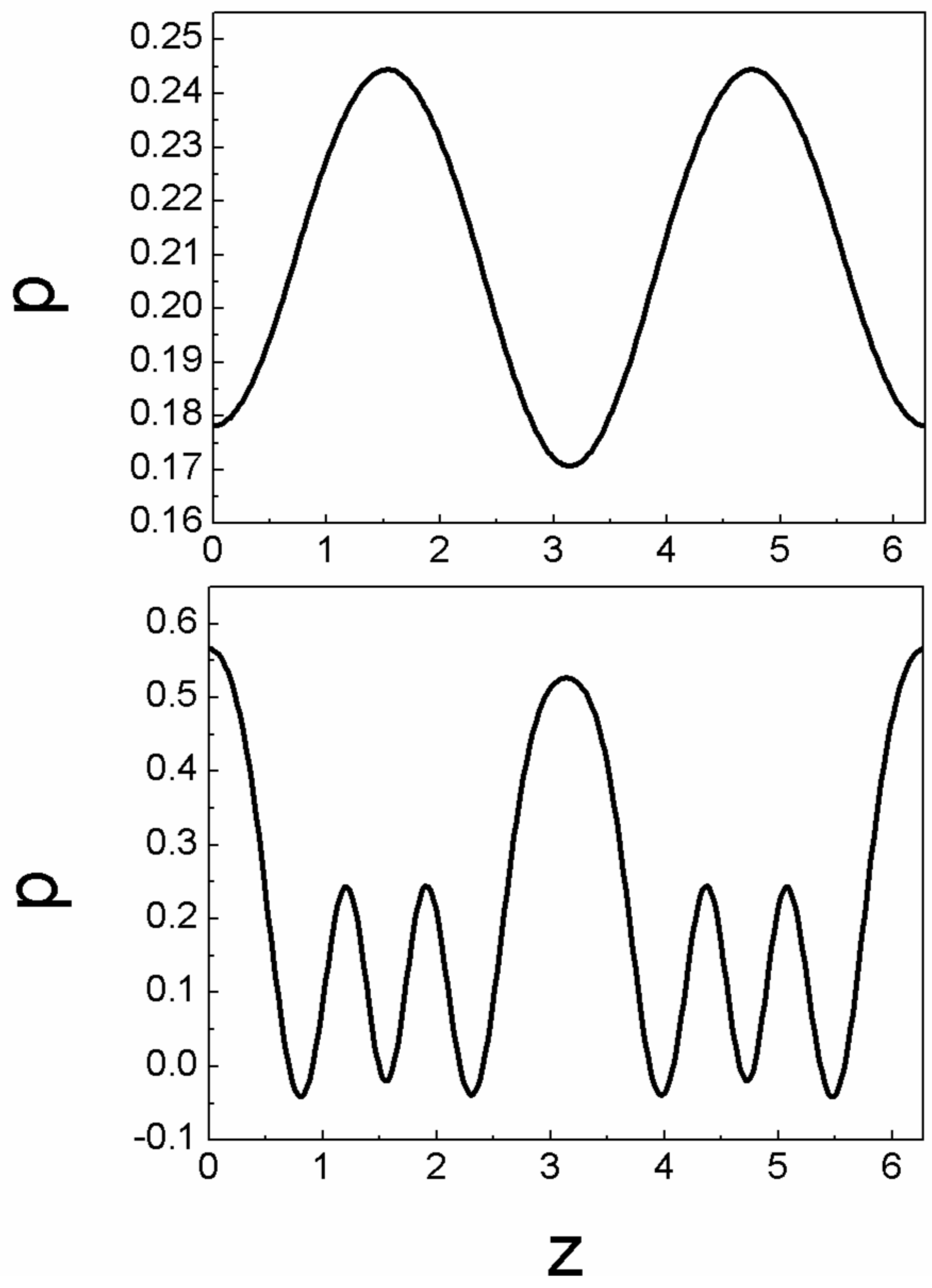


Figure 2 EK10905 22Jun2012



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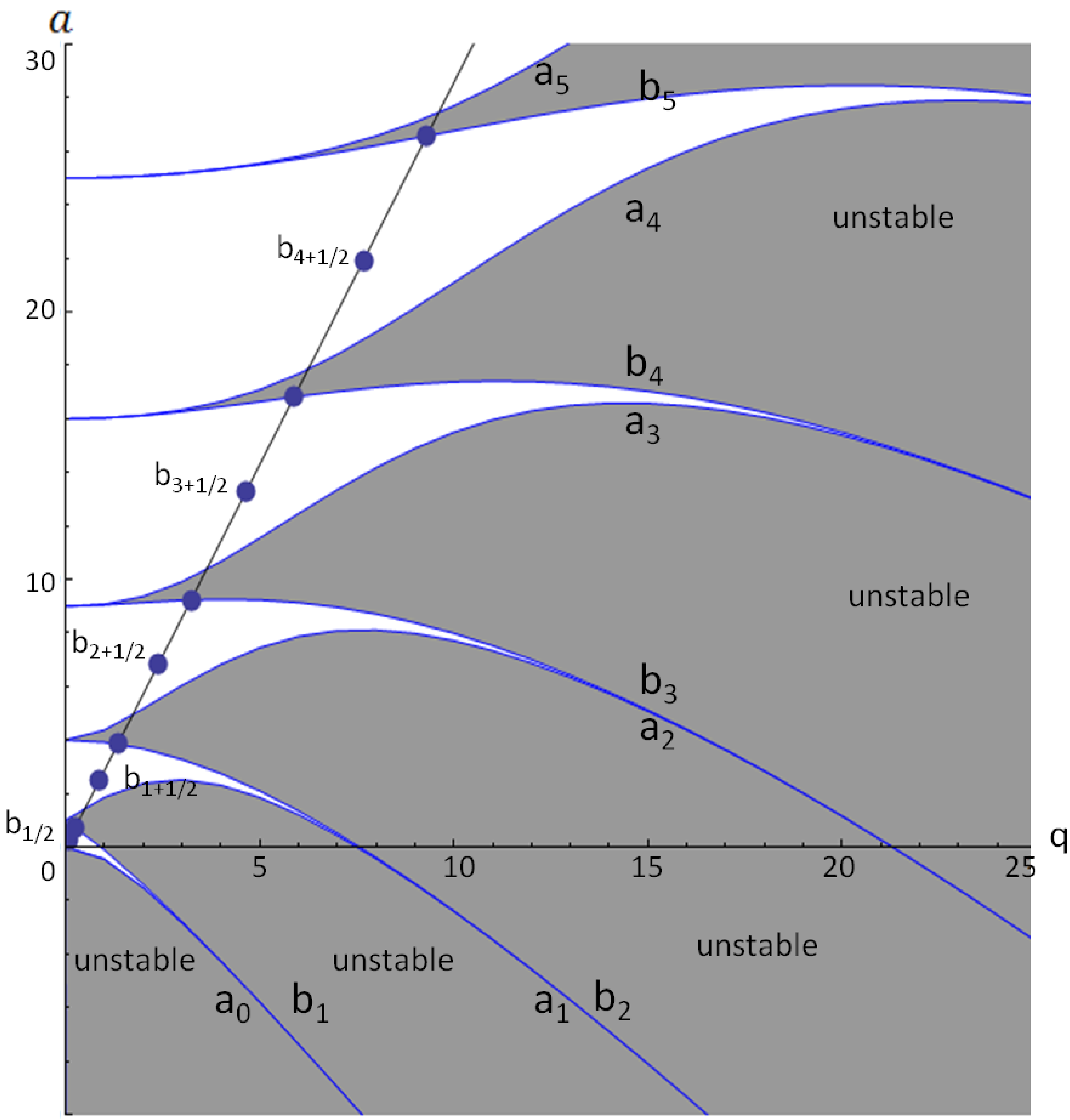

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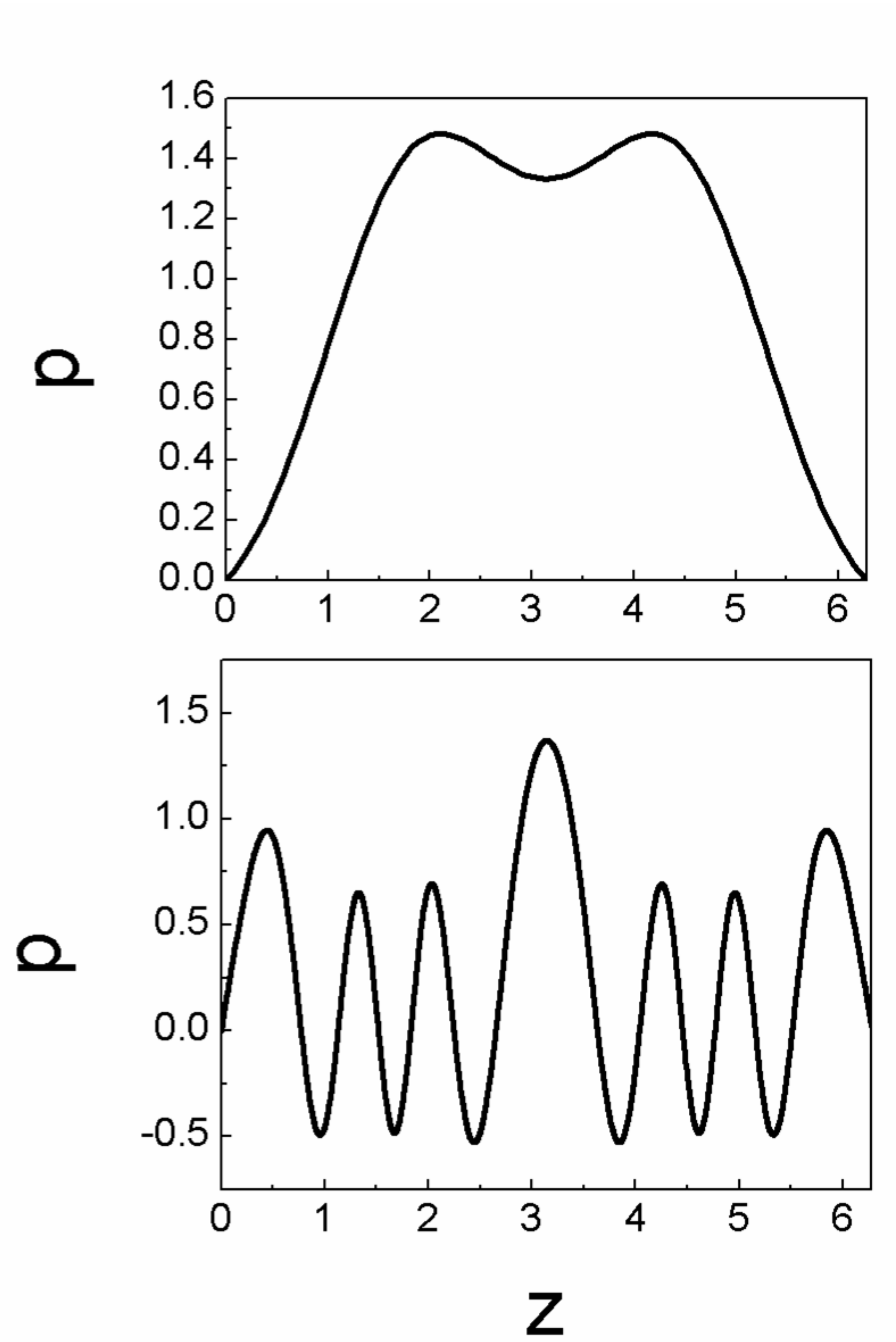

Figure 5 EK10905 22Jun2012

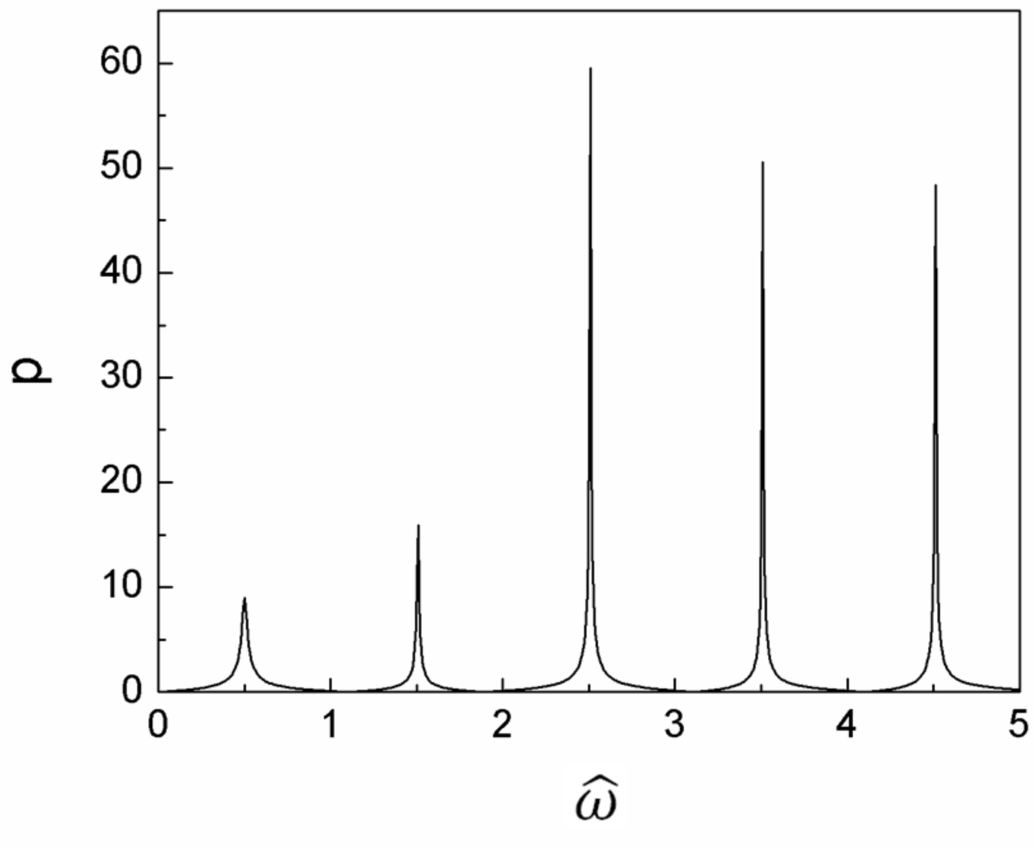

Figure 6 EK10905 22Jun2012