Two-scale renormalization-group classification of diffusive processes

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Renormalization group operators are used to classify stochastic processes on two time scales. Repeated application of one operator is associated with the long time behavior of the process while the other is associated with the short time behavior of the process. This approach is shown to be robust even in the presence of nonstationary increments and infinite second moments. Fixed points of the operators can be used for further subclassification of processes when appropriate limits exist. Several processes are classified using the renormalization group scheme. The processes to be classified include advection-diffusion in an ergodic velocity field, and a model of diffusion in the human bronchial tree.

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I. INTRODUCTION

A diffusive process, $X(t)$, is often classified according to it's mean square displacement (MSD). This classification scheme assumes that $X(t)$ has power law MSD, $\left\langle [X(t)-X(0)]^2 \right\rangle \propto t^\gamma$ for some γ, where $\langle\cdot\rangle$ denotes an ensemble average. These processes are then put into three broad categories. If $\gamma = 1$, $X(t)$ is called normally or classically diffusive, because normal/classical Brownian motion has this MSD pattern. If $\gamma > 1$, $X(t)$ is called superdiffusive, and if $\gamma < 1$, $X(t)$ is called subdiffusive. This power law MSD approach to classification has limited applicability [1] because many diffusive processes do not exhibit power law MSDs and some do not even have finite MSDs, and many processes with linear MSD are not Brownian.

An alternative classification scheme that builds on the notion of self-similarity and is based on a family of renormalization group operators will be presented. This approach can be considered a generalization of the MSD scheme, because processes that have a known power law MSD can be immediately classified in the renormalization group scheme. It is also a generalization of the renormalization group method in [1] because it accounts for both short and long trajectories. A process that has finite MSD that is not a power law can also be classified, as can processes with infinite MSD. An additional advantage is that this renormalization group approach is a two-scale classification scheme. This is important because diffusive processes often exhibit different behaviors on the short and long time scales.

A number of examples will be discussed to highlight the advantages of the renormalization group scheme. A brief introduction to some stochastic processes will be presented to clarify these examples. The classical and simplest diffusive process is Brownian motion. A process $B(t)$ is said to be Brownian if [2]

1. Every increment $B(t+s)-B(s)$ is normally distributed with mean zero and variance $\sigma^2 t$, *i.e.*, $B(t+s)-B(s) \sim N(0,\sigma^2 t)$.
2. For every pair of disjoint time intervals (t_1,t_2) and (t_3,t_4), the increments $B(t_4)-B(t_3)$ and $B(t_2)-B(t_1)$ are independent random variables with the distribution given above.
3. With probability 1, $B(0)=0$ and $B(t)$ is continuous.

These properties state that Brownian motion is characterized by independent, Gaussian increments.

Two generalizations of Brownian motion are widely studied. The first generalizes the assumption of normal increments, instead assuming that the increments follow an α-stable distribution, $S_\alpha(\mu,\beta,c)$. The parameters of this distribution, α, μ, β, and c are known as the stability, shift, skewness, and spread parameters respectively. This process can be formally defined by saying that $L(t)$ is an α-stable Lévy motion if [3]

1. Every increment $L(t+s)-L(s) \sim S_\alpha(0,\beta,ct^{1/\alpha})$.
2. For every pair of disjoint time intervals (t_1,t_2) and (t_3,t_4), the increments $L(t_4)-L(t_3)$ and $L(t_2)-L(t_1)$ are independent random variables with the distribution given above.
3. L(0)=0 with probability 1.

When $\alpha = 2$, α-stable Lévy motion is the same as Brownian motion. When $\alpha < 2$, the distribution of increments has heavy tails (resulting in infinite MSD), and the motion is characterized by large jumps.

The second commonly employed generalization of Brownian motion is fractional Brownian motion [4]. Instead of modifying the assumption of normality, fractional Brownian motion modifies the assumption of independent increments. A process, $B_H(t)$, is said to be fractional Brownian motion with Hurst exponent H, $(0 < H < 1)$ if [2]

1. For any $t > s \geq 0$ the increment $B_H(t) - B_H(s)$ is normally distributed with mean zero and variance $\sigma^2(t-s)^{2H}$, *i.e.*, $B(t+s) - B(s) \sim N(0, \sigma^2(t-s)^{2H})$.

2. With probability 1, $B_H(t)$ is continuous and $B_H(0) = 0$.

Fractional Brownian motion is a non-Markovian stochastic process with persistent increments when $H > 1/2$ and anti-persistent increments when $H < 1/2$. When $H = 1/2$ fractional Brownian motion is the same as Brownian motion.

In addition to employing these fundamental stochastic processes (all of which have stationary increments), modified forms with nonstationary increments will be employed. These modified forms are obtained by starting with a stochastic process, $X(t)$, and defining a new stochastic process, $Y(t) = X(F(t))$. These types of processes have been previously studied [1, 5–10], and $F(t)$ is referred to as the clock. By choosing $F(t)$ to be nonlinear, a process with stationary increments, $X(t)$, can be transformed into a process with nonstationary increments, $Y(t)$.

II. RENORMALIZATION GROUP CLASSIFICATION

To begin, a set of renormalization group operators will be defined that act on stochastic processes. Let $X(t)$ be a stochastic process, a and b be positive real numbers, and define

$$R_{a,b}X(t) = \frac{X(at)}{b}. \tag{1}$$

These operators form a group: $R_{1,1}$ is the identity, closure and associativity follow from the fact that

$$R_{a_1,b_1}R_{a_2,b_2}X(t) = R_{a_1a_2,b_1b_2}X(t), \tag{2}$$

for the positive real numbers a_1, a_2, b_1, b_2. The inverse of $R_{a,b}$ is given by $R_{1/a,1/b}$.

A subset of this group will be the operators employed to classify diffusion with a shorthand notation to describe the elements of this subset. Define

$$S_{p,r} = R_{1/r^{1/p},1/r}, \tag{3}$$

$$L_{p,r} = R_{r,r^p}. \tag{4}$$

Note that for fixed p, the sets $\{S_{p,r} : r > 0\}$ and $\{L_{p,r} : r > 0\}$ each form a semigroup (further, a monoid) under composition, because

$$S_{p,r_1}S_{p,r_2} = S_{p,r_1r_2} \tag{5}$$

$$L_{p,r_1}S_{p,r_2} = L_{p,r_1r_2}. \tag{6}$$

This notation also serves as a mnemonic device, because $S_{p,r}$ will be used to study the behavior of diffusive processes on the short time scale, and $L_{p,r}$ will be used to study the behavior on the long time scale. Applying these maps to a stochastic process, and observing the behavior as $r \to \infty$ for different values of p will supply the necessary information for the classification.

The classification scheme that we are employing is based on the behavior on both short and long time scales. The approach for the long time scale is similar to the approach employed in [1], except that the renormalization group operators act directly on a stochastic process rather than acting on the process' increments. This approach is also able to capture the short time behavior, extending the classification scheme to cover two-scales instead of one.

A. Relation to self-similarity

The operators $S_{p,r}$ and $L_{p,r}$ are closely linked to self-similar behavior. A stochastic process, $X(t)$, is called p-self-similar when $X(t)$ and $r^{-p}X(rt)$ have the same finite dimensional distributions for all positive values of r. This proposition is identical to the proposition that $X(t)$ and $L_{p,r}X(t)$ have the same finite dimensional distributions for all positive values of r.

It can be shown that p-self-similarity is also equivalent to $X(t)$ and $S_{p,r}X(t)$ having the same finite dimensional distributions for all positive values of r. To see this, suppose that $X(t)$ is p-self-similar, *i.e.*, $X(t)$ and $r_0^{-p}X(r_0 t)$ have the same finite dimensional distributions for all positive values of r_0. Choose $r_0 = r^{-1/p}$, so that

$$X(t) \sim \frac{X(r_0 t)}{r_0^p} \sim \frac{X(r^{-1/p}t)}{\left(r^{-1/p}\right)^p} \sim \frac{X(t/r^{1/p})}{1/r} \sim S_{p,r}X(t), \tag{7}$$

where $\sim$ is used to denote that two processes have the same finite dimensional distributions. Therefore we see that $X(t)$ being p-self-similar implies that $X(t)$ and $S_{p,r}X(t)$ have the same finite dimensional distributions. The proof of the converse is very similar.

These arguments reveal the relationship between these operators and self-similar processes. The set of p-self-similar processes is the same as the set of fixed points of the operator, $L_{p,r}$ (or, equivalently, the fixed points of the operator, $S_{p,r}$). This implies that the classification scheme based on these operators will be closely tied to the notion of self-similarity.

B. Long time scale

Consider applying the operator $L_{p,r}$ to some stochastic process, $X(t)$,

$$L_{p,r}X(t) = \frac{X(rt)}{r^p}. \tag{8}$$

If p is too large, then letting r approach ∞ will in general cause the denominator to be much larger than the numerator on the right hand side of Eq. 8. The effect is that $L_{p,r}X(t)$ will go to zero in a sense that will be made clear.

A simple example will help illuminate this. Suppose that $B(t)$ is a Brownian motion. The arguments in the previous section and the fact that Brownian motion is 1/2-self-similar suggest that an "appropriate" value of p for Brownian motion is 1/2. The denominator in the classical central limit theorem also suggests that dividing by $r^{1/2}$ in Eq. 8 will produce meaningful results. If we think of the graph of a Brownian motion under these transformations, then when $p > 1/2$ the compression of the image in the x direction will be too large relative to the compression of the image in the t-direction. The result will be that as r becomes larger, the graph of $L_{p,r}X(t)$ will begin to converge to the line $x = 0$ (see Fig. 1). For a fixed t_0, this picture can be written formally as

$$L_{p,r}B(t_0) = \frac{B(rt_0)}{r^p} \sim \frac{N(0, rt_0)}{r^p} \sim N\left(0, r^{1-2p}t_0\right) \to 0 \tag{9}$$

as $r \to \infty$ when $p > 1/2$.

Returning to the general case, suppose that

$$p = \inf\{q \geq 0 : L_{q,r}X(t) \to 0 \;\forall t\} \tag{10}$$

then we say that $X(t)$ is p-diffusive on the long time scale. Convergence in this context means that $L_{p,r}X(t)$ converge in distribution to 0 (*i.e.*, the cumulative density function for $L_{p,r}X(t)$ converges pointwise to the Heaviside step function). If the set in the infimum of Eq. 10 is the empty set, then we say that $X(t)$ is ∞-diffusive on the long time scale. Examples of processes that are ∞-diffusive on the long time scale will be presented in section II D This forms the foundation of the classification scheme on the long time scale.

The reason that the operator $L_{p,r}$ is associated with the long time scale is that for $r > 1$, it compresses the time axis. The effect of this is that the long time behavior of a p-self-similar process, $X(t)$, is exhibited on a much shorter time scale by $L_{p,r}X(t)$ as r becomes large.

C. Short time scale

The short time scale classification proceeds along the same lines as on the long time scale, except that $S_{p,r}$ is used in place of $L_{p,r}$ and a supremum takes the place of an infimum in Eq. 10. Consider applying the operator $S_{p,r}$ to some stochastic process, $X(t)$,

$$S_{p,r}X(t) = rX\left(\frac{t}{r^{1/p}}\right). \tag{11}$$

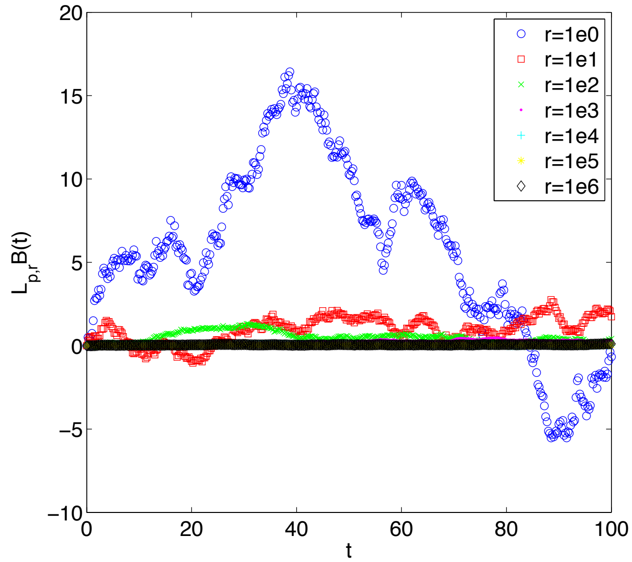

FIG. 1: (Color online) The image of a Brownian motion after application of $L_{p,r}$ for increasing values of r and $p = 1$ is shown. The image converges to zero for all values of t as $r \to \infty$.

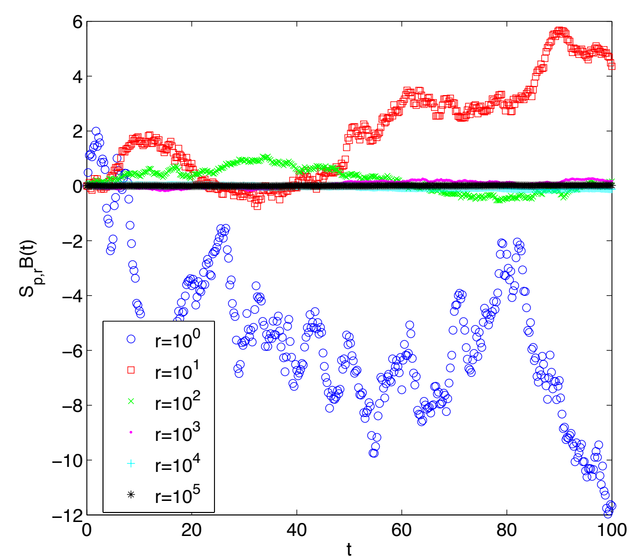

FIG. 2: (Color online) The image of a Brownian motion after application of $S_{p,r}$ for increasing values of r and $p = 1/3$ is shown. The image converges to zero for all values of t as $r \to \infty$.

This time, if p is too small, then letting r approach ∞ will cause the time axis to be stretched (by dividing by $r^{1/p}$) much faster than the position axis will be stretched (by multiplying by r) in 11. The effect will be that $S_{p,r}X(t)$ will go to zero.

Again it is helpful to consider the familiar example of Brownian motion. Due to Brownian motion's 1/2-self-similarity, the "appropriate" value of p is expected to be 1/2. The graph $S_{p,r}B(t)$ with $p < 1/2$ will approach the graph of $x = 0$ as r becomes large, as it did for $L_{p,r}$ when $p > 1/2$ (see Fig. 2).

We say that a diffusive process, $X(t)$ is p-diffusive on the short time scale if

$$p = \sup \{p \geq 0 : S_{p,r}X(t) \to 0 \ \forall t\} . \tag{12}$$

If the set in the supremum in Eq. 12 does not have a supremum (*i.e.*, it is unbounded), then we say that $X(t)$ is ∞-diffusive on the short time scale.

The reason that the operator $S_{p,r}$ is associated with the short time scale is that for $r > 1$, it stretches the time axis. The effect of this is that the short time behavior of a p-self-similar process, $X(t)$, is exhibited on a much longer time scale by $S_{p,r}X(t)$ as r becomes large.

D. Processes with no self-similarity

We have shown that the fixed points of $L_{p,r}$ and $S_{p,r}$ are p-self-similar processes. It can also be shown that a p-self-similar process is p-diffusive on both the short and long time scales. The argument for this is very similar to the argument presented in [1] to show this for a slight reformulation of p-diffusivity for the long time scale.

All of this might give some indication that the infimum and supremum in Eqs. 10 and 12 are unnecessary, and that we can just look for the value of p that results in a fixed point. This approach would work if all processes exhibited self-similarity, however many processes do not. Processes without self-similarity can be split into two groups. Those that have asymptotic self-similarity, and those that have no self-similarity (*i.e.*, not even asymptotic self-similarity).

We say that a process, $X(t)$ is asymptotically p-self-similar on the long time scale if $L_{p,r}X(t)$ approaches a non-zero fixed point as r approaches ∞. This is a generalization of the definition of asymptotical self-similarity given in [11]. Likewise, we say that a process, $X(t)$ is asymptotically p-self-similar on the short time scale if $S_{p,r}X(t)$ approaches a non-zero fixed point as r approaches ∞. Processes that are not self-similar, but are asymptotically self-similar, abound. Mathematical processes can be found in, *e.g.*, discrete random walks that are Brownian or Lévy motions in the long time limit. A physical example can be found in the diffusion of a monomer in a large polymer which is Brownian in the long time limit, but more like fractional Brownian with $H = 1/4$ on a shorter time scale [9, 12, 13]. If a process is asymptotically p-self-similar on the long or short time scale, it will also be p-diffusive on the long or short time scale, respectively.

The more complicated case involves processes that display neither self-similarity nor asymptotic self-similarity. These are the processes that make the study of fixed points inadequate for classifying diffusive processes. Such processes are easy to construct mathemtically, but, perhaps, more difficult to identify physically. Consider a mathematical example

$$X(t) = \sum_{n=2}^{\infty} \frac{B_{1-1/n}(t)}{2^n} \tag{13}$$

where $B_{1-1/n}(t)$ are independent fractional Brownian motions with Hurst exponent $1 - 1/n$. Subsequently, we show $X(t)$ is 1/2-diffusive and asymptotically 1/2-self-similar on the short time scale, and on the long time scale, it is 1-diffusive, but not asymptotically 1-self-similar.

Observe that when $p < 1$, $L_{p,r}X(t)$ does not approach 0 as r becomes large, because

$$\left\langle [L_{p,r}X(t)]^2 \right\rangle = \sum_{k=2}^{\infty} \left\langle \left[\frac{B_{1-1/k}(rt)}{r^p 2^k}\right]^2 \right\rangle > \frac{\left\langle \left[B_{1-1/n}(rt)\right]^2 \right\rangle}{r^{2p}4^n} = \frac{(rt)^{2-2/n}}{r^{2p}4^n} \tag{14}$$

for all n. The right hand side of Eq. 14 does not approach zero when $n > \frac{1}{1-p}$, so the left hand side cannot possibly approach a delta distribution about zero. This shows that the infimum in Eq. 10 must be greater than or equal to 1. It remains to show that the infimumin Eq. 10 is no more than 1. In order to show that $L_{p,r}X(t) \to 0$ in distribution, it is sufficient to show that $\left\langle [L_{p,r}X(t)]^2 \right\rangle \to 0$ as $r \to \infty$. This can be verified analytically,

$$\left\langle [L_{p,r}X(t)]^2 \right\rangle = \sum_{n=2}^{\infty} \frac{\left\langle \left[B_{1-1/n}(rt)\right]^2 \right\rangle}{r^{2p}4^n} = t^2 r^{2-2p} \sum_{n=2}^{\infty} \frac{1}{(rt)^{2/n}4^n}. \tag{15}$$

The right hand side of 15 goes to zero as $r \to \infty$ when $p > 1$. Therefore the infimum in Eq. 10 is no more than 1. Combining this with the previous result, we see that the infimum is exactly 1, and $X(t)$ is 1-diffusive on the long time scale.

It will now be shown that $X(t)$ is not asymptotically self-similar on the long time scale. This demonstrates that classifying processes based on their self-similarity or asymptotic self-similarity is not sufficient. Some processes, such as $X(t)$, cannot be classified based on self-similarity, because they do not exhibit self-similarity. The arguments showing that $X(t)$ is 1-diffusive show that $X(t)$ is not aysmptotically p-self-similar for all values of p other than 1. It remains to show that $X(t)$ is not asymptotically 1-self-similar. Substituting $p = 1$ into Eq. 15 combined with some algebra gives

$$\left\langle [L_{1,r}X(t)]^2 \right\rangle = t^2 \sum_{n=2}^{\infty} \frac{1}{2^{2n}(rt)^{2/n}} \to 0. \tag{16}$$

as $r \to \infty$. The proof that the sum converges to zero is straightforward and is omitted here. It can therefore be concluded that, $L_{1,r}X(t) \to \mathbf{0}$ for all t, and $X(t)$ is not asymptotically 1-self-similar.

Any process that is ∞-diffusive on a given time scale also lacks any self-similarity on that same time scale. In this case it is essentially because no value of p is big enough. A simple example that displays this behavior is a Brownian motion with nonlinear clock in the form

$$X(t) = B(e^t - 1). \tag{17}$$

$X(t)$ is ∞-diffusive on the long time scale. This process diffuses so rapidly that no matter how large p is, $L_{p,r}$ for large values of r will not be able to transform $X(t)$ into something like a delta distribution. This process will subsequently be studied in more detail.

E. Subclassification of p-diffusive processes

In addition to classifying processes as p-diffusive on the short and/or long time scales, an approach based on these renormalization group operators enables a more refined distinction to be made. If a process is asymptotically p-self-similar on the short or long time scales, then it is also p-diffusive on the short or long time scale, respectively. In addition to associating the p-diffusive classification with this process, we can further associate it's behavior on the short or long time scale with a fixed point of the operators $S_{p,r}$ or $L_{p,r}$ respectively. This association is the basis for the more refined classification mentioned above. The cases where no asymptotic self-similarity is present (see section II D for some examples) must all be grouped together in this subclassification.

F. Classification of experimental or numerical data

This classification approach was primarily developed to classify mathematical models of anomalous diffusion processes. However, it is often desirable to classify processes based on experimental or numerical data sets, and such classifications are possible with this two-scale classification scheme. We will briefly describe two approaches to classification of data sets here. It is likely that other approaches are possible, and there is ample room for further research in this direction.

One approach is to use existing techniques to develop a model for the process that underlies a given data set. The model can then be fit into the two-scale renormalization group scheme. Given that the classification scheme was originally intended to classify models, not data, this approach is natural. However, it is also somewhat unsatisfying, because it fails to provide any new tools for data analysis.

Some work has been carried out on an automated approach to classifying diffusive processes using the long time scale classification [1]. This approach involves applying the renormalization group, $L_{p,r}$ operator to the data, $X^i(j)$, for increasing values of r, say $r = 1, 2, \ldots, 2^k$, and for various values of p. For values of p that are too small, the distribution of $L_{p,r}X(t)$ will move away from a delta distribution as r increases, and for values of p that are too big, the distribution of $L_{p,r}X(t)$ will converge to a delta distribution as r increases. All of this can be quantified, as was done in [1], and the result is a numerical approach to classifying the process underlying a data set. The short time scale classification can be carried out analogously, except that the operator $S_{p,r}$ is used in place of the operator $L_{p,r}$.

These two approaches provide a way to classify data sets from experiments or simulations. We again emphasize the likelihood that alternative avenues for classification of data with this scheme exist and should be explored in future work.

III. CLASSIFICATION OF SOME PROCESSES

Some processes will subsequently be classified using the p-diffusive renormalization group approach. A broad swath of processes will be classified at once by observing proving that p-self-similar processes are also p-diffusive on both the short and long time scales. Advection diffusion in an ergodic velocity field will also be classified. This type of process could be expected, for example, in a layered velocity field in a porous medium. Finally, diffusion in a model of the human bronchi will be considered.

A. p-self-similar processes

As was mentioned previously, a stochastic process, $X(t)$ is p-self-similar if and only if $X(t)$ is a fixed point of both $L_{p,r}$ and $S_{p,r}$. Suppose that $X(t)$ is p-self-similar, and consider the application of $L_{q,r}$ to $X(t)$,

$$L_{q,r}X(t) \sim \frac{X(rt)}{r^q} \sim r^{p-q}\frac{X(rt)}{r^p} \sim r^{p-q}X(t) \tag{18}$$

The distribution on the right hand side of Eq. 18 approaches a delta distribution as $r \to \infty$ exactly when $q > p$. Therefore, the infimum in Eq. 10 is p, and it has been shown that $X(t)$ is p-diffusive on the long time scale. A very similar argument shows that $X(t)$ is p-diffusive on the short time scale. Therefore, any p-self-similar process is p-diffusive on both the short and long time scales.

This result allows for the immediate classification of many commonly used processes. Brownian motion is 1/2-self-similar [3], and therefore 1/2-diffusive on both the long and short time scales. α-stable Lévy motion is $1/\alpha$-self-similar [3], and hence $1/\alpha$-diffusive on both the short and long time scales. Fractional Brownian motion with Hurst exponent H is H-self-similar [3], and thus H-diffusive on both the long and short time scales. Many other processes including some continuous time random walks [14] and fractional Brownian motion with a power law clock [9] are also classified as a consequence of this result.

Even though two different stochastic processes may both be classified as p-diffusive for the same value of p on both the long and short time scales, there is still room to make a distinction in the renormalization group classification scheme. For example, fractional Brownian motion with Hurst exponent 4/5 and 5/4-stable Lévy motion are both 4/5 diffusive on the short and long time scales. However, these two processes are distinct fixed points of the operators $L_{p,r}$ and $S_{p,r}$. This distinction allows for the processes to be differentiated using the subclassification scheme discussed previously.

B. Advection-diffusion in an ergodic velocity field

Suppose that a stochastic process $X(t)$ is determined by the Ito stochastic differential equation [15]

$$dX(t) = a[X(t)]dt + \sqrt{D}dB(t) \tag{19}$$

where $a(x)$ is the velocity field and dB(t) is a standard Brownian noise. Assuming that $a(x)$ and $X(t)$ are continuous, consider the behavior of $X(t)$ upon application of $S_{p,r}$

$$\begin{aligned} S_{p,r}X(t) &\sim r\int_0^{t/r^{1/p}} a[X(s)]ds + r\int_0^{t/r^{1/p}} \sqrt{D}dB(s) \\ &\to N(tr^{1-1/p}a[X(0)], Dtr^{2-1/p}) \end{aligned} \tag{20}$$

The normal distribution in Eq. 20 approaches a delta distribution if $p < 1/2$, $N(0, Dt)$ if $p = 1/2$ and diverges if $p > 1/2$. Therefore, $X(t)$ is 1/2-diffusive on the short time scale. This is indicative of the fact that the erratic behavior of the noise term in Eq. 19 dominates the behavior of $X(t)$ on the short time scale.

On the long time scale, we do not require that $a(x)$ be continuous, but rather that a long time average of $a[X(t)]$ converge in distribution to some nonzero value, $\langle a\rangle$. The notation for this nonzero value is suggestive of an ensemble average. This notation was chosen because, if the velocity field is ergodic then the long time average converges to the ensemble average. Consider the behavior of $X(t)$ upon application of $L_{p,r}$

$$L_{p,r}X(t) \sim r^{-p}\int_0^{rt} a[X(s)]ds + r^{-p}\int_0^{rt} \sqrt{D}dB(s). \tag{21}$$

If $p < 1$ then the advective term in Eq. 21 blows up (by the assumption that the time average of $a[X(t)]$ converges to $\langle a\rangle$), and it is seen that the left hand side of Eq. 21 does not approach a delta distribution about zero. When $p > 1$ the right hand side converges to a delta distribution (the advection term does so because of the assumption on the time average, and the diffusion term will converge to a delta distribution for any $p > 1/2$). Therefore, $X(t)$ is 1-diffusive on the long time scale. This is indicative of the fact that the long term effect of the advection term in 19 dominates the behavior of $X(t)$ on the long time scale.

It has thus been shown that $X(t)$ is 1/2-diffusive on the short time scale and 1-diffusive on the long time scale. The different classifications on different scales indicate a deficiency of any one-scale scheme.

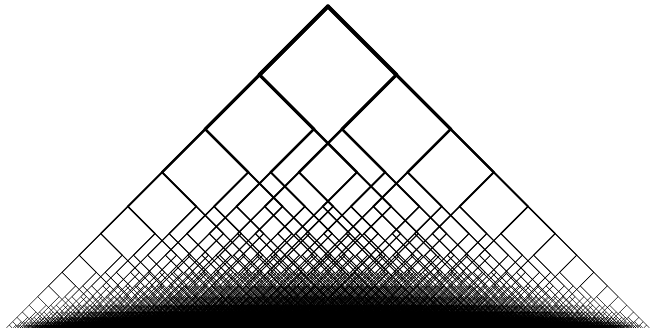

FIG. 3: A two dimensional representation of the human bronchial tree is shown. The diameter of the branches decreases as the tree is descended, but the total cross-sectional area increases. The result is that a particle descending the tree slows as it descends.

C. Diffusion in the human bronchial tree

A model of diffusion in the human bronchial tree based on the description given in [16] will be classified. The model captures all the topological properties, and the geometric properties that are essential to accurately classify the diffusion. The human bronchial tree consists of a binary tree where the diameter of the bifurcations is governed by the recurrence relation

$$d_n^3 = 2d_{n+1}^3 \tag{22}$$

where d_n denotes the diameter of the branch at the n^{th} level. This implies that $d_n = 2^{-n/3}d_0$. From this it is seen that the branches become narrower, but the total cross-sectional area at each level increases. Conservation of mass results in a velocity that decreases as the air descends the bronchial tree, $v_n = 2^{-n/3}v_0$. The length of each branch is assumed to be proportional to the diameter, so $l_n = 2^{-n/3}l_0$. See Fig. 3 for a pictorial representation.

Suppose that $X(t)$ is the horizontal position of a particle descending this tree with velocity v_n at level n, and the choice of a branch at each bifurcation is independent of the other branch choices. Let $\epsilon > 0$, and consider first the short time scale classification,

$$\begin{aligned} \lim_{r\to\infty} P\left(|S_{p,r}X(t)| > \epsilon\right) &= \lim_{r\to\infty} P\left(|rX(t/r^{1/p})| > \epsilon\right) \\ &= \lim_{r\to\infty} P\left(Cr^{1-1/p}tv_0 > \epsilon\right) \end{aligned} \tag{23}$$

where C is a constant depending on the angle between two bifurcations. The last limit converges to zero if $p < 1$ and to 1 if $p > 1$. Since ϵ was arbitrary, the supremum in Eq. 12 is 1, and $X(t)$ is 1-diffusive on the short time scale. This is a natural result given that on the short time scale, the process is almost purely advective (*i.e.*, it is purely advective, except for the choice of one branch).

The long time scale classification remains to be determined. It will be shown that $X(t)$ is 0-diffusive on the long time scale by observing that as $r \to \infty$, $X(rt)$ approaches a stationary distribution. Therefore, renormalizing by a factor of r^p for any $p > 0$ will result in convergence to a delta distribution. Note that the time it takes for the particle to traverse one branch of the tree is given by

$$\frac{v_n}{l_n} = \frac{2^{-n/3}v_0}{2^{-n/3}l_0} = \frac{v_0}{l_0} \tag{24}$$

and is independent of the level, n. Therefore, the horizontal displacement from traversing the branch at the n^{th} level is $X[(n+1)v_0/l_0] - X[nv_0/l_0]$, and the total horizontal distance traveled by the particle from the 0^{th} level to any level is bounded by

$$\sum_{n=0}^{\infty} |X[(n+1)v_0/l_0] - X[nv_0/l_0]| = C\sum_{n=0}^{\infty} l_n = Cl_0 \sum_{n=0}^{\infty} 2^{-n/3} = \frac{Cl_0}{1-2^{-1/3}} \tag{25}$$

Therefore, $X(t) \in \left[-\frac{Cl_0}{1-2^{-1/3}}, \frac{Cl_0}{1-2^{-1/3}}\right]$ for all t, and it is seen that

$$P\left(|X(rt)/r^p| > \frac{Cl_0}{r^p\left(1-2^{-1/3}\right)}\right) = 0. \tag{26}$$

when $p > 0$. Hence $L_{p,r}X(t) \to 0$ for any value of $p > 0$, and the infimum in Eq. 10 is 0. It can be concluded that $X(t)$ is 0-diffusive on the long time scale.

The horizontal position of a particle descending into this tree presents an interesting case. On the short time scale, it is essentially advective and 1-diffusive. On the long time scale, it is 0-diffusive which indicates that it is diffusing very slowly.

IV. COMPARISON TO OTHER SCHEMES

The two-scale p-diffusive classification scheme has significant advantages over the MSD classification scheme. One of these advantages is that the MSD scheme cannot be applied to processes with infinite MSD such as Lévy motions. Further, it cannot be applied to processes with MSD that does not obey a power law. The two-scale p-diffusive classification can be applied to any process. In any event, when a process does have a finite, power law MSD (say t^γ) and zero mean the process will be classified as $\gamma/2$-diffusive on both the long and short time scale. This follows from the Chebyshev's inequality, $P(|X - E[X]| \geq k) \leq \text{Var}(X)/k^2$. This means that for a process with power-law MSD, the two-scale p-diffusive scheme is just as easy to apply as the MSD scheme. When the MSD is infinite or simply not a power law the MSD scheme does not apply, but the two-scale p-diffusive approach still has something to say.

An alternative would be to simply classify processes based on their self-similarity index. However, in this scheme, processes without self-similarity would be impossible to classify. The two-scale p-diffusive scheme generalizes a self-similarity scheme because, as discussed previously, if a process is p-self-similar it is also p-diffusive on both the long and short time scales. Of course, the renormalization group p-diffusive scheme is also able to classify processes that do not exhibit self-similarity. This fact combined with the discussion in the previous paragraph demonstrates that the renormalization group classification scheme advocated here can be regarded as a generalization of both the MSD and self-similarity schemes.

The fact that the classification scheme proposed here is two-scale gives it a leg up on any one-scale scheme such as the power law MSD scheme or a self-similarity scheme. Consideration of multiple scales is important because many diffusive processes display scale dependent behavior. For example, many processes have a preasymptotic regime before converging to an asymptotic regime [12, 17]

A. Infinite MSD

An important class of processes for which the MSD scheme fails is processes with infinite MSD. Processes with infinite MSD, such as α-stable Lévy motion, have been used to study nonlocal transport [18, 19], movement patterns of marine predators [20], and microbial motion [21] to name a few examples. It can be shown that α-stable Lévy motion is $1/\alpha$-self-similar [3], and therefore $1/\alpha$-diffusive on both the long and short time scales. Since $\alpha \in (0, 2]$, these processes would all be considered superdiffusive on both the short and long time scales in the renormalization group classification except when $\alpha = 2$. This is an appropriate classification because α-stable Lévy motion is widely considered superdiffusive when $\alpha < 2$, and 2-stable Lévy motion is simply another name for Brownian motion. The simplicity with which the renormalization group scheme classifies α-stable Lévy motion demonstrates one of it's primary advantages over the MSD scheme. Namely, that it is able to simply classify processes with infinite MSD.

B. Nonstationary increments

We now consider an example that has nonstationary increments. Let $X(t) = B(e^t - 1)$ where $B(t)$ is a Brownian motion. This process does not have power-law MSD, and therefore cannot be classified using the MSD scheme. Further, it is not self-similar. However, it will be shown that $X(t)$ is ∞-diffusive on the long time scale and 1/2-diffusive on the short time scale.

First consider the long time scale,

$$L_{p,r}X(t) \sim \frac{B(e^{rt} - 1)}{r^p} \sim N(0, r^{-2p}[e^{rt} - 1]) \tag{27}$$

The normal distribution on the right hand side of 27 does not approach a delta distribution about zero as $r \to \infty$ for any value of p. Therefore the set in the infimum of Eq. 10 is empty, and $X(t)$ is by definition ∞-diffusive on the long time scale.

Next consider the short time scale,

$$S_{p,r}X(t) \sim rB(\exp[t/r^{1/p}] - 1) \sim N(0, r^2[\exp(t/r^{1/p}) - 1]) \tag{28}$$

Since this process has finite variance, $S_{p,r}X(t) \to 0$ in distrubtion exactly when the variance on the right hand side of Eq. 28 approaches 0 as $r \to \infty$. A power series representation of this variance is given by

$$tr^{2-1/p} + \frac{t^2 r^{2-2/p}}{2} + \frac{t^3 r^{2-3/p}}{6} + \cdots \tag{29}$$

This series approaches zero as $r \to \infty$ when $2 - 1/p < 0$, *i.e.*, $p < 1/2$. Therefore the supremum in Eq. 12 is 1/2, and it can be concluded that $X(t)$ is 1/2-diffusive on the short time scale.

This multiscale process cannot be classified using power law MSDs or self-similarity, but the brief calculations shown here demonstrate that it fits easily into the p-diffusive renormalization group classification scheme.

V. SUMMARY

A classification scheme for diffusive processes based on renormalization group operators has been presented. This scheme is a two-scale classification scheme that builds upon the idea of self-similarity. It was shown to have a number of advantages over the traditional MSD classification scheme. In particular that it can be applied to processes with infinite MSDs and processes that do not have power-law MSDs. A number of classical stochastic processes were classified, as was advection-diffusion in an ergodic velocity field and diffusion in the human bronchial tree.

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