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Dynamical predictive power of the generalized Gibbs ensemble revealed in a second quench

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We show that a quenched and relaxed completely integrable system is hardly distinguishable from the corresponding generalized Gibbs ensemble in a dynamical sense. To be specific, the response of the quenched and relaxed system to a second quench can be accurately reproduced by using the generalized Gibbs ensemble as a substitute. Remarkably, as demonstrated with the transverse Ising model and the hard-core bosons in one dimension, not only the steady values but even the transient, relaxation dynamics of the physical variables can be accurately reproduced by using the generalized Gibbs ensemble as a pseudo-initial state. This result is an important complement to the previously established result that a quenched and relaxed system is hardly distinguishable from the generalized Gibbs ensemble in a static sense. The relevance of the generalized Gibbs ensemble in the nonequilibrium dynamics of completely integrable systems is then greatly strengthened.

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I. INTRODUCTION

Recently, non-equilibrium dynamics of many-body systems has attracted a lot of attention [1]. One common concern is whether an initially out-of-equilibrium system can thermalize to behave like a textbook Gibbs ensemble, and how integrability [2, 3] or non-integrability of the system will affect its relaxation dynamics. An important achievement on this issue is identification of the relevance of the generalized Gibbs ensemble (GGE) in the relaxation dynamics of a completely integrable system [4]. The so called generalized Gibbs ensemble is constructed according to the principle of maximum entropy [5] while taking into account all the constants of motion, whose values are determined by the initial state. With the same philosophy behind the construction, it is a natural counterpart of the usual Gibbs ensembles for a non-integrable system. So far, the GGE has been found to predict correctly the asymptotic values of physical variables in a variety of integrable systems [1, 4, 6–16].

The fact that asymptotically, the true, constantly evolving system agrees well with the GGE on the physical quantities is definitely a non-trivial and pleasant one. However, one should not be content with this fact only. Our daily experience in the (mostly non-integrable) macroscopic world is that, if a system relaxes to some steady state, it relaxes in the sense that not only its static properties (i.e., values of the physical quantities) but also its dynamical properties agree with the steady state. To be specific, the system should respond to later perturbations as if it were indeed in the steady state. Therefore, it is necessary to check whether the GGE has this merit. If so, it surely adds to the relevance of the GGE in the non-equilibrium dynamics of a completely integrable system. It would mean that the true system is hardly distinguishable from the GGE neither by static nor dynamical criterions, and it would be fair to say the system has thermalized as much as possible. From a practical point of view, it would mean that the GGE can serve as a *pesudo* initial state, a substitute of the true state of the system, to future perturbations. This is definitely a desirable property since the GGE is much simpler than the true, constantly evolving state. In passing, one should note that here the idea is similar to that of the pesudo-potential, which has found wide use in atomic physics and solid state physics [17].

Motivated by this problem, we have studied the transverse Ising model and the hard-core bosons in one-dimension (which can also be mapped to the XX model) individually. The two models are integrable and both have been shown to admit a GGE account of their asymptotic behaviors after a quantum quench. Here our idea is to give them a second quench when they have reached the steady phase [18]. The concern is whether they will respond as if the systems were in the GGE states. The result turns out to be the case.

Here it is worthwhile to point out that the problem we face and the strategy we take here, are similar to what people face and take in the field of spin glass. There, the typical aging process can be accounted for either in terms of the droplet model or a hierarchical description, and the issue was to find an experimental procedure to discriminate between the two approaches. By subjecting the system to a cycle of different temperature quenches, it is observed that the system shows apparent rejuvenation and memory effect, which is in favor of the hierarchical description [19]. Here, in our case, the situation is that the true, constantly evolving system agrees well with the GGE as far as some physical observables are concerned, and the issue is whether they can be discriminated somehow, say, by quenching the system a second time. The answer, as indicated by our study below, is no, and this adds to the usefulness of the GGE in the non-equilibrium dynamics of integrable systems.

The rest of this paper is organized as follows. In the first two subsections of Sec. II, the transverse Ising model and hard-core bosons are studied separately. The results obtained are then compared in Sec. II C with that associated with the Bose-Hubbard model. We summarize in Sec. III.

II. TWO CASE STUDIES

A. Transverse Ising model

The Hamiltonian of the transverse Ising model is

$$H(g) = -\sum_{l=1}^{N} \left(\sigma_l^x \sigma_{l+1}^x - g\sigma_l^z\right), \tag{1}$$

where $\sigma_l^{x,z}$ are Pauli matrices acting on a 1/2-spin at site l. Here periodic boundary condition is assumed and N is an even integer large enough. Below, quenches of the system correspond to changing the value of g (strength of the transverse magnetic field) suddenly. We will consider a double quench scenario. Initially the value of g is g_0 and the system is in its ground state $|G_0\rangle$. Then the value of g is changed successively to g_1 and g_2.

Under the Jordan-Wigner transform

$$\frac{1}{2}(\sigma_l^x + i\sigma_l^y, \sigma_l^x - i\sigma_l^y) = (a_l^\dagger, a_l)\exp(-i\pi\sum_{r=1}^{l-1} a_r^\dagger a_r), \tag{2a}$$

$$\sigma_l^z = 2a_l^\dagger a_l - 1, \tag{2b}$$

where $a_l^\dagger$ and a_l are fermionic operators, the Hamiltonian is rewritten as

$$H(g) = -\sum_{l=1}^{N}[(a_l^+ a_{l+1}^+ + a_l^+ a_{l+1} + h.c.) - 2g a_l^\dagger a_l], \tag{3}$$

with a constant term dropped [20]. Note that here the boundary condition is anti-periodic [21]. Taking the Fourier transform $b_k = \frac{1}{\sqrt{N}}\sum_l e^{i2\pi kl/N} a_l$, with $k = -N/2+1/2, \cdots, -1/2, 1/2, \cdots, N/2-1/2$ so as to comply with the anti-periodic boundary condition, we can rewrite the Hamiltonian as ($\phi_k = 2\pi k/N$)

$$H(g) = \sum_k \left[2(g-\cos\phi_k) b_k^\dagger b_k - i\sin\phi_k (b_{-k}b_k + b_{-k}^\dagger b_k^\dagger)\right].$$

It is ready to verify that b_k and $b_{-k}^\dagger$ are coupled in their equations of motion and this suggests the Bogoliubov transformation $\eta_k = u_k b_k + i v_k b_{-k}^\dagger$. With $\varepsilon_k = 2\sqrt{1+g^2-2g\cos\phi_k} \geq 0$, $(\cos\theta_k, \sin\theta_k) = 2(g - \cos\phi_k, \sin\phi_k)/\varepsilon_k$, and $(u_k, v_k) = (\cos\frac{\theta_k}{2}, \sin\frac{\theta_k}{2})$, the Hamiltonian is finally diagonalized as $H = \sum_k \varepsilon_k \eta_k^\dagger \eta_k$. Here again the constant term is dropped. Note that u_k, v_k, θ_k, and ε_k all depend on g. The dependence will be displayed explicitly when necessary.

We are interested in the correlation functions $\langle\sigma_i^x\sigma_j^x\rangle$, $\langle\sigma_i^z\sigma_j^z\rangle$, and the transverse magnetization $\langle M_z\rangle \equiv \langle\sum_l \sigma_l^z\rangle = \langle\sum_k(2b_k^\dagger b_k - 1)\rangle$. Here the expectation values may be taken with respect to various states as shown below. Introducing $A_l = a_l^\dagger + a_l$ and $B_l = a_l^\dagger - a_l$, we can rewrite them as $\langle\sigma_i^x\sigma_j^x\rangle = \langle B_i A_{i+1} B_{i+1} \cdots A_{j-1} B_{j-1} A_j\rangle$ and $\langle\sigma_i^z\sigma_j^z\rangle = \langle B_i A_i B_j A_j\rangle$ [20]. These forms allow us to use Wick's theorem to do the calculation. The correlation functions will be decomposed into sums of products of the basic correlators $\langle A_l A_m\rangle$, $\langle B_l B_m\rangle$, and $\langle B_l A_m\rangle$.

The initial state $|G_0\rangle$ is defined as $\eta_k(g_0)|G_0\rangle = 0$ for all k, or explicitly, $|G_0\rangle \propto \prod_k \eta_k(g_0)|\psi\rangle$ where $|\psi\rangle$ can be an arbitrary state as long as $\eta_k(g_0)|\psi\rangle \neq 0$. After the first quench of changing g from g_0 to g_1 at $t=0$, we have $\langle G_0|A_l(t)A_m(t)|G_0\rangle \to \delta_{lm}$ for t large enough [22, 23], and similarly $\langle G_0|B_l(t)B_m(t)|G_0\rangle \to -\delta_{lm}$ for t large enough. But $\langle G_0|B_l(t)A_m(t)|G_0\rangle \to G_{l,m}^{(1)}$, which has the value of

$$G_{l,m}^{(1)} = -\frac{1}{N}\sum_k e^{i\phi_k(m-l)+i\theta_k(g_1)}\cos(\Delta\theta_k^{10}). \tag{4}$$

Here and hereafter $\Delta\theta_k^{ij} \equiv \theta_k(g_i) - \theta_k(g_j)$. Thus for t large enough, $\langle G_0|\sigma_i^x(t)\sigma_j^x(t)|G_0\rangle \to C_{ij}^x$:

$$C_{ij}^x = \mathrm{Det}\begin{pmatrix} G_{i,i+1}^{(1)} & G_{i,i+2}^{(1)} & \cdots & G_{i,j}^{(1)} \\ G_{i+1,i+1}^{(1)} & G_{i+1,i+2}^{(1)} & \cdots & G_{i+1,j}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ G_{j-1,i+1}^{(1)} & G_{j-1,i+2}^{(1)} & \cdots & G_{j-1,j}^{(1)} \end{pmatrix}, \tag{5}$$

and

$$\langle G_0|\sigma_i^z(t)\sigma_j^z(t)|G_0\rangle \to C_{ij}^z = G_{i,i}^{(1)}G_{j,j}^{(1)} - G_{i,j}^{(1)}G_{j,i}^{(1)}. \tag{6}$$

As for the transverse magnetization, $\langle\Psi_0|M_z(t)|\Psi_0\rangle$ has the asymptotic value of

$$M_z^{(1)} = -\sum_k \cos\theta_k(g_1)\cos(\Delta\theta_k^{10}). \tag{7}$$

On the other hand, from g_0 to g_1, the (first) GGE density matrix is defined as

$$\rho_{gge1} = \frac{1}{Z_1}\prod_k \exp\left(-\lambda_k^{(1)}\eta_k^\dagger(g_1)\eta_k(g_1)\right), \tag{8}$$

with the Lagrange multiplier $\lambda_k^{(1)}$ determined by the condition $\langle G_0|\eta_k^\dagger(g_1)\eta_k(g_1)|G_0\rangle = tr(\eta_k^\dagger(g_1)\eta_k(g_1)\rho_{gge1})$, and $Z_1 = \prod_k(1+e^{-\lambda_k^{(1)}})$. It can be verified that $\langle A_l A_m\rangle_{gge1} = -\langle B_l B_m\rangle_{gge1} = \delta_{lm}$, and $\langle B_l A_m\rangle_{gge1} = G_{l,m}^{(1)}$. Here the subscript means averaging over ρ_{gge1}. Thus the basic correlators are of the same values with respect to the GGE density matrix ρ_{gge1} and the evolving state $e^{-iH(g_1)t}|G_0\rangle$ for t large enough. This fact then indicates that the asymptotic values of the correlation functions (5) and (6) can be recovered with the GGE. Likewise, the asymptotic value of the transverse magnetization (7) is exactly predicted by the GGE, i.e., $M_z^{(1)} = tr(M_z\rho_{gge1})$ [1, 21].

Now consider giving the system a second quench, i.e., changing the value of g from g_1 to g_2 at some time $t = t_1$. It is tedious but straightforward to show that at the time of $t = t_1 + t_2$, for large t_2 [21], $\langle G_0|A_l(t)A_m(t)|G_0\rangle \simeq \delta_{lm}$+oscillating terms depending on t_1, and similarly $\langle G_0|B_l(t)B_m(t)|G_0\rangle \simeq -\delta_{lm}$+oscillating terms depending on t_1. However, $\langle G_0|B_l(t)A_m(t)|G_0\rangle \simeq G^{(2)}_{l,m}$+oscillating terms depending on t_1, where

$$G^{(2)}_{l,m} = -\frac{1}{N}\sum_k e^{i\phi_k(m-l)+i\theta_k(g_2)}\cos(\Delta\theta_k^{10})\cos(\Delta\theta_k^{21}). \quad (9)$$

As for the transverse magnetization, $\langle G_0|M_z(t)|G_0\rangle \to M_z^{(2)}$+oscillating terms depending on t_1, with

$$M_z^{(2)} = -\sum_k \cos\theta_k(g_2)\cos(\Delta\theta_k^{10})\cos(\Delta\theta_k^{21}). \quad (10)$$

The oscillating terms depending on t_1 consist of $O(N)$ components of different non-zero frequencies and thus they virtually vanish for t_1 large enough. Therefore, for t_1 and t_2 large enough, the correlation functions $\langle G_0|\sigma_i^x(t)\sigma_j^x(t)|G_0\rangle$ and $\langle G_0|\sigma_i^z(t)\sigma_j^z(t)|G_0\rangle$ have the same form as Eqs. (5) and (6) but with $G^{(1)}_{m,l}$ replaced by $G^{(2)}_{m,l}$, and $\langle G_0|M_z(t)|G_0\rangle$ has the value of $M_z^{(2)}$.

On the other hand, if the second quench is imposed on the first GGE density matrix ρ_{gge1}, we have the same asymptotic behaviors of the basic correlators and the transverse magnetization for large t_2. That is, $\langle A_l(t_2)A_m(t_2)\rangle = -\langle B_l(t_2)B_m(t_2)\rangle \simeq \delta_{lm}$, $\langle B_m(t_2)A_l(t_2)\rangle \simeq G^{(2)}_{m,l}$, and $\langle M_z(t_2)\rangle \simeq M_z^{(2)}$ [21]. Here $(A_l(t_2), B_l(t_2), M_z(t_2)) = e^{iH(g_2)t_2}(A_l, B_l, M_z)e^{-iH(g_2)t_2}$ and the average is taken over ρ_{gge1}. We see that the transverse magnetization as well as the basic correlators possess the same asymptotic values regardless of the initial state being $e^{-iH(g_1)t_1}|G_0\rangle$ or ρ_{gge1}. The latter fact implies that the correlation functions have the same property. However, it is not only the asymptotic values that can be accurately reproduced by using ρ_{gge1} as a substitute for $e^{-iH(g_1)t_1}|G_0\rangle$. In Fig. 1, the transient dynamics of M_z after the second quench is shown. There we see that as long as t_1 is large enough, the relaxation dynamics of M_z (the correlation functions have the same property; see the supplementary material) is independent of t_1 and can be reproduced by ρ_{gge1} even to minute details. Therefore, as long as t_1 is large enough, or as long as the second quench comes when the system has equilibrated to agree with the first GGE ρ_{gge1} after the first quench, the model reacts as if it were indeed in the GGE state ρ_{gge1}. That is, the GGE density matrix ρ_{gge1} can serve as a pseudo initial state to the second quench.

Finally, for the quench of ρ_{gge1}, we can define a second GGE density matrix as

$$\rho_{gge2} = \frac{1}{Z_2}\prod_k \exp\left(-\lambda_k^{(2)}\eta_k^\dagger(g_2)\eta_k(g_2)\right), \quad (11)$$

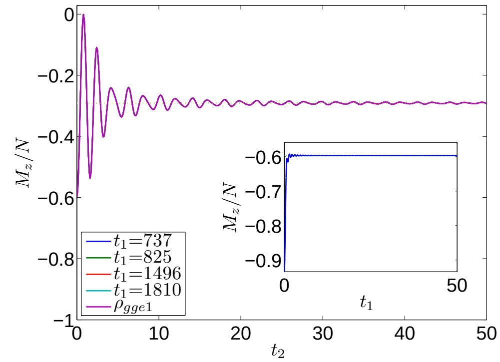

FIG. 1: (Color online) Evolution of the transverse magnetization M_z after the second quench. The parameters are $(N, g_0, g_1, g_2) = (10000, 2, 1, 0.2)$. All the lines, with the "initial" state being the (first) generalized Gibbs ensemble (GGE) density matrix ρ_{gge1} or $e^{-iH(g_1)t_1}|G_0\rangle$, collapse into one. Here the values of t_1 are chosen randomly from $[500, 2500]$. The horizontal dotted line indicates the predicted asymptotic value (10). The insert shows the time evolution of M_z after the first quench.

with the parameter $\lambda_k^{(2)}$ determined by the condition $tr(\eta_k^\dagger(g_2)\eta_k(g_2)\rho_{gge2}) = tr(\eta_k^\dagger(g_2)\eta_k(g_2)\rho_{gge1})$, and $Z_2 = \prod_k(1 + e^{-\lambda_k^{(2)}})$. The point is that the basic correlator $G^{(2)}_{l,m}$ in (9) and the transverse magnetization in (10) can be exactly reproduced by ρ_{gge2}. This is one more support of the argument that ρ_{gge1} can serve as a pseudo initial state to the second quench.

B. Expansion of hard-core bosons in a one dimensional lattice

To make contact with previous works, the scenario studied below is an extension of that in Ref. [4]. There are N hard-core bosons and there is a lattice of M_2 sites, which are numbered from 1 to M_2. Initially the N bosons are confined to the M_0 middle sites by hard-walls on the two sides and the system is in the ground state, which is denoted as ψ_0. At $t = 0$, the hard-walls are suddenly moved outward symmetrically so that now M_1 sites are contained. The system then evolves and as found by Rigol *et al.* [4], the GGE plays an important role in the ensuing dynamics—the momentum distribution of the bosons in its steady value is accurately captured by the GGE density matrix Ξ_{gge1} (see below). Our idea is then at some time t_1, when the momentum distribution has settled down to its steady value, to increase the volume to M_2 sites and let the bosons expand once again. The aim is to see whether the subsequent dynamics can be accurately reproduced with the initial state (to the second expansion) $\psi(t_1)$ replaced by Ξ_{gge1}. Note that since the latter is time independent, this necessarily requires that the subsequent dynamics be insensitive to the specific value of t_1 as long as it is large enough to belong to

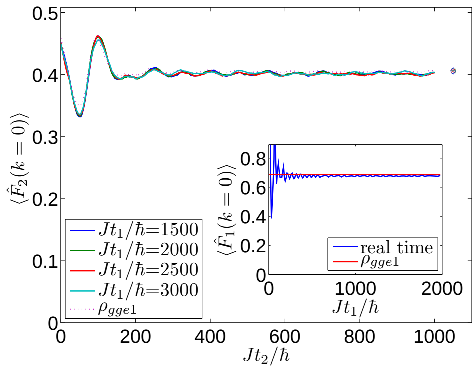

FIG. 2: (Color online) Evolution of the population on the $k = 0$ quasi-momentum state $\langle \hat{F}_2(k = 0)\rangle$ after the second expansion. The parameters are $(N, M_0, M_1, M_2) = (50, 100, 200, 300)$. The dotted line indicates the result with the "initial" state being the (first) generalized Gibbs ensemble (GGE) density matrix Ξ_{gge1}. Other lines correspond to results with the "initial" states being $\psi(t_1)$, with the value of t_1 varied. The markers on the right ends of the lines indicate the predicted values of the second GGEs. The insert shows the time evolution of the population on the $k = 0$ quasi-momentum state $\langle \hat{F}_1(k = 0)\rangle$ after the first expansion.

the steady regime.

In the intervals of $t \le 0$, $0 < t < t_1$, and $t \ge t_1$, the volume (number of sites) of the system is M_0, M_1, and M_2, and thus the corresponding Hamiltonians will be denoted as H_0, H_1, and H_2, respectively. They are of the form $H_i = -J \sum_{j=L_i}^{R_i-1} (b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j)$, $0 \le i \le 2$. Here J is the hopping strength, and $L_i = (M_2 - M_i)/2 + 1$ and $R_i = (M_2 + M_i)/2$ denote the left- and right-most sites accessible to the bosons, respectively. The creation and annihilation operators satisfy the usual bosonic commutation relations plus the hard-core constraint $b_j^2 = b_j^{\dagger 2} = 0$, so that each site can be occupied by at most one boson. By using the Jordan-Wigner transformation $(b_j^\dagger, b_j) = (c_j^\dagger, c_j) \prod_{j'=1}^{j-1} e^{-i\pi c_{j'}^\dagger c_{j'}}$, where c_j ($c_j^\dagger$) is the fermionic annihilation (creation) operator, H_i is mapped to a free fermion one, $H_i = -J \sum_{j=L_i}^{R_i-1} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$. This Hamiltonian can be readily diagonalized as $H_i = \sum_{q=1}^{M_i} \varepsilon_q^{(i)} c_q^{(i)\dagger} c_q^{(i)}$, with $\varepsilon_q^{(i)} = -2J \cos(\pi q/(M_i + 1))$ and $c_q^{(i)} = \sqrt{\frac{2}{M_i+1}} \sum_{j=L_i}^{R_i} c_j \sin(q\pi(j - L_i + 1)/(M_i + 1))$. The initial state is then simply a Fermi-sea state $\psi_0 = \prod_{q=1}^{N} c_q^{(0)\dagger} |0\rangle$.

From H_0 to H_1, the wave function evolves as $\psi(t_1) = e^{-iH_1 t_1/\hbar}\psi_0$, and the (first) GGE density matrix is defined as $\Xi_{gge1} = \frac{1}{\Theta_1} \exp[-\sum_{q=1}^{M_1} \lambda_q^{(1)} c_q^{(1)\dagger} c_q^{(1)}]$. Here the parameter $\lambda_q^{(1)}$ is determined by the initial state, $tr(c_q^{(1)\dagger} c_q^{(1)} \Xi_{gge1}) = \langle \psi_0 | c_q^{(1)\dagger} c_q^{(1)} | \psi_0 \rangle$, and Θ_1 is a normalization factor (or partition function). It is found in [4], argued in [14], and verified in Fig. 2 below that for t_1 large enough, the momentum distribution (or populations on the quasi-momentum states, here $k = -M_1/2, -M_1/2 + 1, \cdots, M_1/2 - 1$)

$$\hat{F}_1(k) = \frac{1}{M_1} \sum_{j,j'=L_1}^{R_1} e^{-i2\pi k(j-j')/M_1} b_{j'}^\dagger b_j \qquad (12)$$

with respect to $\psi(t_1)$ can be accurately reproduced by using Ξ_{gge1}, i.e., $\langle \psi(t_1)|\hat{F}_1(k)|\psi(t_1)\rangle \simeq tr(\hat{F}_1(k)\Xi_{gge1})$. Note that the quantity $\hat{F}_1(k)$, unlike $c_q^{(1)\dagger} c_q^{(1)}$, is not conserved by the quenched Hamiltonian H_1. Therefore, it is a highly non-trivial fact that its asymptotic value can be accurately accounted for by Ξ_{gge1}, which takes care of only the conserved quantities $c_q^{(1)\dagger} c_q^{(1)}$.

Now from H_1 to H_2, the H_2-evolved wave function at $t = t_1 + t_2$ is given by $\psi(t_1 + t_2) = e^{-iH_2 t_2/\hbar}\psi(t_1)$. For our purpose, we replace the "initial" state $\psi(t_1)$ by Ξ_{gge1} and define the H_2-evolved GGE density matrix $\Xi_{gge1}(t_2) = e^{-iH_2 t_2/\hbar} \Xi_{gge1} e^{iH_2 t_2/\hbar}$. We then study the momentum distribution ($k = -M_2/2, -M_2/2 + 1, \cdots, M_2/2 - 1$)

$$\hat{F}_2(k) = \frac{1}{M_2} \sum_{j,j'=L_2}^{R_2} e^{-i2\pi k(j-j')/M_2} b_{j'}^\dagger b_j \qquad (13)$$

with respect to $\psi(t_1 + t_2)$ and $\Xi_{gge1}(t_2)$. The results are shown in Fig. 2. Here, we emphasize that $\hat{F}_1(k)$ and $\hat{F}_2(k)$ are defined similar but are different.

In the insert of Fig. 2, we see that after the first expansion, the population on the $k = 0$ quasi-momentum state $\langle \psi(t_1)|\hat{F}_1(k = 0)|\psi(t_1)\rangle$ relaxes to the steady value predicted by the GGE density matrix Ξ_{gge1} eventually. This proves the predictive power of the GGE after the first expansion. What Fig. 2 highlights is that, if the time of the second expansion t_1 is chosen to belong to the steady regime, the later evolution of the population on the $k = 0$ quasi-momentum state $\langle \psi(t_1 + t_2)|\hat{F}_2(k = 0)|\psi(t_1 + t_2)\rangle$ can be accurately reproduced by $tr(\hat{F}_2(k = 0)\Xi_{gge1}(t_2))$. Their lines coincide with each other not only in the asymptotic limit but even on details during the transitory period. Note that since the latter is independent of t_1, this necessarily implies that the former is insensitive to the value of t_1, as is indeed the case. Overall, Fig. 2 is a remarkable demonstration of the fact that the GGE density matrix Ξ_{gge1} shares with the relaxed state $\psi(t_1)$ not only the value of the momentum distribution, but also the response to a second quench. Or in the perspective of the state $\psi(t_1)$, it has relaxed to be virtually indistinguishable from the GGE state Ξ_{gge1}, neither by static nor dynamical criterions.

In Fig. 2, we have also studied whether the steady value of $\hat{F}_2(k = 0)$ after the second quench can be described by a second GGE density matrix Ξ_{gge2}, which is defined as $\Xi_{gge2} = \frac{1}{\Theta_2} \exp[-\sum_{q=1}^{M_2} \lambda_q^{(2)} c_q^{(2)\dagger} c_q^{(2)}]$, with the parameter $\lambda_q^{(2)}$ determined by the condition $tr(c_q^{(2)\dagger} c_q^{(2)} \Xi_{gge2}) = tr(c_q^{(2)\dagger} c_q^{(2)} \Xi_{gge1})$ or $tr(c_q^{(2)\dagger} c_q^{(2)} \Xi_{gge2}) = \langle \psi(t_1)| c_q^{(2)\dagger} c_q^{(2)} |\psi(t_1)\rangle$ depending on whether the "initial" state is Ξ_{gge1} or $\psi(t_1)$. The result is that the second GGEs do predict the steady values correctly; moreover, they agree with each other very well. This is one more evidence that the relaxed wave function $\psi(t_1)$ is virtually indistinguishable from the GGE Ξ_{gge1}.

Finally, some comments on the waiting time t_1 are in order. In both models above, it is observed that for t_1 large enough, $\psi(t_1)$ becomes indistinguishable from the (first) GGE. A natural question is then, how long exactly t_1 should be. The observation is that, as long as t_1 belongs to the steady regime, that is, as long as the second quench comes when the system has settled down after the transient period, the system responds as if it were in the GGE state. Of course, the span of the transient period depends on the specific problem. In the transverse Ising model case, it is on the order of $1/W$, with W being the width of the single particle spectrum. However, in the hard-core boson case, as we see in Fig. 2, it is much longer. The reason is that the bosons need to expand from the central region to the outer regions site by site.

C. Comparison with the Bose-Hubbard model

The two integrable models above indicate that the GGE is virtually indistinguishable from the true, constantly evolving system, neither by static nor by dynamical means. It is necessary to point out that a similar situation holds for non-integrable systems. Of course, for non-integrable systems, GGE is irrelevant and it is the time-averaged density matrix that plays the role of GGE [24]. In the previous study of quenched dynamics of the (generically non-integrable) Bose-Hubbard model [18], the double quench protocol has already been implemented. There, a similar observation is that after a transient time, the system relaxes to agree with the time-averaged density matrix well on the physical observables, and moreover, its response to further quenches can be reproduced by using the time-averaged density matrix as a pesudo initial state. Therefore, for both integrable and non-integrable systems, it is possible to construct some state simpler than the true state of the system, yet able to mimic the latter in all practical aspects.

III. CONCLUSIONS AND DISCUSSIONS

In summary, we have investigated and verified the relevance of the GGEs in the dynamical response of the two integrable models of transverse Ising model and one-dimensional hard-core bosons. Once having relaxed to have its properties correctly predicted by the GGE, the system behaves as if it were indeed in the GGE state—its response to the second quench can be accurately reproduced by the GGE even to details. On one hand, this result is a welcome complement to previously established result that the GGEs are relevant in predicting the static properties of the systems after the first quench. The two now combine to present a more complete story of the GGE and beckon more confidence on it. On the other hand, this result also gives us a sense of "dynamical typicality" [25], which is also observed in the (non-integrable) Bose-Hubbard model previously [18]. Finally, though here we have been dealing with integrable systems only, a lesson may also be drawn for non-integrable systems. A closed non-integrable system might well be a pure state yet virtually indistinguishable neither by static nor by dynamic criterions from a canonical ensemble.

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[1] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
[2] Though the notion of quantum integrability is still a subject of debate, see J.-S. Caux and J. Mossel, J. Stat. Mech. P02023 (2011).
[3] T. Kinoshita, T. Wenger, and D. S. Weiss, Nature (London) **440**, 900 (2006).
[4] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. **98**, 050405 (2007); M. Rigol, A. Muramatsu, and M. Olshanii, Phys. Rev. A **74**, 053616 (2006).
[5] E. T. Jaynes, Phys. Rev. **106**, 620 (1957); **108**, 171 (1957).
[6] M. A. Cazalilla, Phys. Rev. Lett. **97**, 156403 (2006).
[7] P. Calabrese and J. Cardy, J. Stat. Mech. P06008 (2007).
[8] M. Kollar and M. Eckstein, Phys. Rev. A **78**, 013626 (2008).
[9] M. Eckstein and M. Kollar, Phys. Rev. Lett. **100**, 120404 (2008).
[10] T. Barthel and U. Schollwöck, Phys. Rev. Lett. **100**, 100601 (2008).
[11] D. Fioretto and G. Mussardo, New J. Phys. **12**, 055015 (2010).
[12] P. Calabrese, F. H. L. Essler, and M. Fagotti, Phys. Rev. Lett. **106**, 227203 (2011).
[13] T. Caneva, E. Canovi, D. Rossini, G. E Santoro, A. Silva, J. Stat. Mech. P07015 (2011).
[14] M. A. Cazalilla, A. Iucci, and M.-C. Chung, Phys. Rev. E **85**, 011133 (2012).
[15] V. I. Yukalov, Laser Phys. Lett. **8**, 485 (2011).
[16] Of course, the GGE does have its limitations, e.g., it can not capture the possible correlations present in a generic initial state and thus may fail to predict the asymptotic

values of physical relevant observables or correlations between the eigenmodes. See [1], D. M. Gangardt and M. Pustilnik, Phys. Rev. A **77**, 041604(R) (2008); J. Lancaster and A. Mitra, Phys. Rev. E **81**, 061134 (2010).
[17] E. Fermi, Ricerca Scientifica **7**, 13 (1936); J. C. Phillips and L. Kleinman, Phys. Rev. **116**, 287 (1959).
[18] J. M. Zhang, C. Shen, and W. M. Liu, Phys. Rev. A **83**, 063622 (2011).
[19] F. Lefloch, J. Hammann, M. Ocio, and E. Vincent, Europhys. Lett. **18**, 647 (1992).
[20] E. Lieb, T. Shultz, and D. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961); P. Pfeuty, Ann. Phys. **57**, 79 (1970).
[21] See Supplemental Material.
[22] Here and below the sign $\rightarrow$ does not mean the limit but just the typical value. Actually, for a finite N, the basic correlators are *almost periodic functions* of t and they have no limits at all. However, for N large enough, the probability of recurrence is practically irrelevant.
[23] K. Sengupta, S. Powell, and S. Sachdev, Phys. Rev. A **69**, 053616 (2004).
[24] Note that the GGE is generally not the time-averaged density matrix of an integrable system. This is readily seen from the fact that in the GGE, by construction, the eigenmodes are independent, yet the correlation between them will be preserved in the time-averaged density matrix.
[25] C. Bartsch and J. Gemmer, Phys. Rev. Lett. **102**, 110403 (2009).