Second-harmonic generation of cylindrical electromagnetic waves propagating in an inhomogeneous and nonlinear medium

Hao Xiong, Liu-Gang Si, Chunling Ding, Xiaoxue Yang, and Ying Wu

Phys. Rev. E **85**, 016606 — Published 31 January 2012

DOI: 10.1103/PhysRevE.85.016606

Second-harmonic generation of cylindrical electromagnetic waves propagating in an inhomogeneous and nonlinear medium

Hao Xiong,* Liu-Gang Si,† Chunling Ding, Xiaoxue Yang, and Ying Wu
Wuhan National Laboratory for Optoelectronics and School of Physics,
Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China

A general description of cylindrical electromagnetic waves propagating in nonlinear and inhomogeneous media is given by deducing cylindrical coupled-wave equations. Base on the cylindrical coupled-wave equations, we analyze second-harmonic generation (SHG) of some special cases of inhomogeneity, and find that the inhomogeneity of the first and second order polarization can influence on the amplitude of the SHG. From a different point of view, exact solutions of cylindrical electromagnetic waves propagating in a nonlinear medium with a special case of inhomogeneity have been obtained previously. We show that cylindrical SHG in an inhomogeneous and nonlinear medium also can be deduced from exact solutions. As a verification, we compare the results obtained from the two different methods and find that descriptions of SHG by the coupled-wave equations are in good agreement with the exact solutions.

PACS numbers: 42.65.Ky, 03.50.De, 41.20.Jb

I. INTRODUCTION

Nonlinear optics plays a central role in the advancement of optical science, and significant progress has been made by using analytical and numerical methods[1–14]. Cylindrical nonlinear optics is a burgeoning research area which describes cylindrical electromagnetic waves propagation in nonlinear media and some fundamental researches have been done [15–19]. It is an extremely complicated problem to describe electromagnetic waves propagation in a medium with simultaneously inhomogeneous and nonlinear, and such problem remains poorly studied. Reference [16] has given an exact solution to describe cylindrical electromagnetic waves propagation in a medium with special polarization as $\varepsilon(E,r) = \mathrm{d}D/\mathrm{d}E = \epsilon_0\varepsilon_1 r^\beta \exp(\alpha E)$ with $\varepsilon_1, \alpha, \beta$ are certain constants and ϵ_0 is the permittivity of free space. This special polarization denotes that the medium considered is nonlinear and inhomogeneous. The nonlinear factor is $\exp(\alpha E)$, and the inhomogeneous factor is r^β. Using the exact solution obtained, Ref. [16] has discussed the initial value problem and boundary value problem, to compare the differences between homogenous and inhomogeneous conditions. However, such description is not enough because the exact solution can only be found for few cases. For most cases of inhomogeneity, the system can not be integrated exactly. Thus, finding general description of cylindrical electromagnetic waves propagating in nonlinear and inhomogeneous media is important and useful.

In this article, we give a general description of cylindrical electromagnetic waves propagating in nonlinear and inhomogeneous media. We deduce coupled-wave equations which describe the interaction between cylindrical electromagnetic waves and nonlinear inhomogeneous media. Using the coupled-wave equations, we analyze second-harmonic generation (SHG) of some special cases of inhomogeneity, and find that both the inhomogeneity of the first and second order polarization can influence on the amplitude of the SHG ($E_{2\omega}$), and the inhomogeneity of the first order polarization can change the peak values and positions of the peaks of $E_{2\omega}$. From a different point of view, we show that cylindrical SHG in an inhomogeneous and nonlinear medium also can be deduced from the exact solutions. We give a full description of using the exact solutions to deal with the problem of cylindrical SHG in an inhomogeneous and nonlinear medium, and as a verification, we compare the results obtained from the two different methods and find that descriptions of SHG by the coupled-wave equations are in good agreement with the exact solutions.

The paper is organized as follows. In Sec. II we derive the coupled-wave equations, and we analyze SHG of some special cases of inhomogeneity. In Sec. III we use the exact solutions to investigate cylindrical SHG and show that second-harmonic generation comes out quite naturally from the exact solutions. We also compare the results obtained from the two different methods. In Sec. IV our conclusions are summarized at the end of this article.

II. COUPLED-WAVE EQUATIONS OF CYLINDRICAL ELECTROMAGNETIC WAVES PROPAGATING IN AN INHOMOGENEOUS AND NONLINEAR MEDIUM

In this section, we will deduce the coupled-wave equations of cylindrical electromagnetic waves propagating in an inhomogeneous and nonlinear medium. We start by introducing our physical model. We assume that the medium possesses an axis of symmetry, and taken as the z axis of a cylindrical coordinate system (r, ϕ, z). A linear light source is placed in the axis of the nonlinear medium, and cylindrical electromagnetic waves are emitted. By considering that the fields are independent of ϕ and z, the Maxwell equations can be written as follows:

$$\frac{\partial H}{\partial r} + \frac{H}{r} = \frac{\partial D}{\partial t}, \qquad \frac{\partial E}{\partial r} = \mu_0 \frac{\partial H}{\partial t}, \tag{1}$$

*Electronic address: haoxiong1217@gmail.com
†Electronic address: siliugang@gmail.com

where $H \equiv H_\phi(r,t)$, $E \equiv E_z(r,t)$ and $D(r,t) \equiv \epsilon_0 E + P$ with P being the intensity of polarization of the medium. Such model has been used in some foregoing works [15–19]. Hereafter, we will focus on electric fields to set up a classical theory which describes cylindrical electromagnetic waves propagating in an inhomogeneous and nonlinear medium.

With regard to different medium, the connection between P and E shows in different forms. For the case $P \propto E$, viz. the medium is linear. We can solve Eq. (1) by the method of variable separation and the solution is $E = \zeta J_0(kr)\exp(-i\omega t)$, where J_m is a Bessel function of the first kind of order m, ζ is a constant, $k = \omega\sqrt{\epsilon_0\varepsilon_1\mu_0}$ with ω is the frequency of the electromagnetic wave, ϵ_0 is the permittivity of free space, μ_0 is the permeability of free space, and $\varepsilon_1 = 1 + \chi^{(1)}$. The real media are usually nonlinear and inhomogeneous, and we consider that $P = \epsilon_0\chi^{(1)}g(r)E + \sum_{n=2}^{\infty} P_{NL}^{(n)} f^{(n)}(r)$, where $g(r)$ means the inhomogeneity of the linear polarization and $f^{(n)}(r)$ means the inhomogeneity of the nth-order nonlinear polarization. For convenience, we define a function $f^{(1)}(r) = [1 + \chi^{(1)}g(r)]/\varepsilon_1$, and the polarization of the medium can be written as: $P = \epsilon_0[\varepsilon_1 f^{(1)}(r) - 1]E + \sum_{n=2}^{\infty} P_{NL}^{(n)} f^{(n)}(r)$. Some special cases can be discussed. If $f^{(1)}(r) = f^{(n)}(r) = 1$, then the medium is homogeneous but nonlinear. If $f^{(1)}(r) = 1$ and $f^{(n)}(r) = 0$, then the medium is homogeneous and linear. Here, we give a general description of the system and consider $f^{(1)}(r)$ and $f^{(n)}(r)$ can be arbitrary functions. On the basis of Eq. (1) we have:

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} = \frac{1}{v(r)^2}\frac{\partial^2 E}{\partial t^2} + \mu_0\sum_{n=2}^{\infty} f^{(n)}(r)\frac{\partial^2 P_{NL}^{(n)}}{\partial t^2}, \quad (2)$$

where $1/v(r)^2 = \epsilon_0\varepsilon_1\mu_0 f^{(1)}(r)$, and hereafter, $k_i = \omega_i\sqrt{\epsilon_0\varepsilon_1\mu_0}$. Following the example of the plane nonlinear optics, we present the electric field as

$$E = \frac{1}{2}\sum_i E(\omega_i), \qquad P_{NL} = \frac{1}{2}\sum_q P_{NL}(\omega_q) \quad (3)$$

where $E(\omega_i) = A_i J_0(k_i r)\exp(-i\omega_i t)$, $E(-\omega_i) = E^*(\omega_i)$, $P_{NL}(\omega_q) = P_q J_0(k_q r)\exp(-i\omega_q t)$, $P_{NL}(-\omega_q) = P_{NL}^*(\omega_q)$ with A_i is the amplitude of the cylindrical electromagnetic wave carry a frequency ω_i and P_q present the amplitude of the polarized cylindrical electromagnetic wave carry a frequency ω_q. It should be noted that the summation runs over all frequencies, including $\omega > 0$ and $\omega < 0$. Using these presentations, we can simplify Eq. (2) and obtain the coupled-wave equations of cylindrical electromagnetic waves propagating in an inhomogeneous and nonlinear medium:

$$\frac{\partial^2 E(\omega_i)}{\partial r^2} + \frac{1}{r}\frac{\partial E(\omega_i)}{\partial r} + k_i^2 f^{(1)}(r)E(\omega_i) = -\mu_0\omega_i^2\sum_{n=2}^{\infty} f^{(n)}(r)P_{NL}^{(n)}(\omega_q = \omega_i). \quad (4)$$

These equations describe cylindrical electromagnetic waves with frequency ω_i propagating in an inhomogeneous and nonlinear medium, which are coupled by P_{NL} and the inhomogeneous factor $f(r)$. In what follows, we will use these equations to study SHG.

We set $\omega_1 = \omega$, $\omega_2 = 2\omega$ and hereafter n=1, 2. Using Eq. (4) we can obtain:

$$\frac{\partial^2 E(\omega)}{\partial r^2} + \frac{1}{r}\frac{\partial E(\omega)}{\partial r} + k^2 f^{(1)}(r)E(\omega) = -\mu_0\omega^2 f^{(2)}(r)P_{NL}^{(2)}(\omega_q = \omega),$$
$$\frac{\partial^2 E(2\omega)}{\partial r^2} + \frac{1}{r}\frac{\partial E(2\omega)}{\partial r} + 4k^2 f^{(1)}(r)E(2\omega) = -4\mu_0\omega^2 f^{(2)}(r)P_{NL}^{(2)}(\omega_q = 2\omega). \quad (5)$$

Here we only consider the second-order nonlinear polarization of the medium and P_{NL} is used as the secondary nonlinear polarization $P_{NL}^{(2)}$:

$$P_{NL}^{(2)}(\omega_q = \omega) = 2\epsilon_0\chi^{(2)}(-\omega, 2\omega, -\omega) : E(2\omega)E^*(\omega),$$
$$P_{NL}^{(2)}(\omega_q = 2\omega) = \epsilon_0\chi^{(2)}(-2\omega, \omega, \omega) : E(\omega)E(\omega). \quad (6)$$

Using effective nonlinear optical coefficient, we can rewrite Eqs. (5) as:

$$\frac{\partial^2 E_\omega}{\partial r^2} + \frac{1}{r}\frac{\partial E_\omega}{\partial r} + k^2 f^{(1)}(r)E_\omega = -2\epsilon_0\mu_0\omega^2 d_{\text{eff}} f^{(2)}(r)E_{2\omega}E_\omega^*,$$
$$\frac{\partial^2 E_{2\omega}}{\partial r^2} + \frac{1}{r}\frac{\partial E_{2\omega}}{\partial r} + 4k^2 f^{(1)}(r)E_{2\omega} = -4\epsilon_0\mu_0\omega^2 d_{\text{eff}} f^{(2)}(r)E_\omega^2, \quad (7)$$

where d_{eff} is the effective nonlinear optical coefficient of the nonlinear medium, $E_\omega = A_\omega J_0(kr)$ and $E_{2\omega} = A_{2\omega}J_0(2kr)$. It is difficult to give analytical results of features of cylindrical electromagnetic waves propagating in an inhomogeneous and nonlinear medium by using Eqs. (7). However, equations (7) can be solved numerically by considering A_ω and $A_{2\omega}$ are only functions of r, then Eqs. (7) become

$$\frac{d^2 E_\omega}{dr^2} + \frac{1}{r}\frac{dE_\omega}{dr} + k^2 f^{(1)}(r)E_\omega = -2\epsilon_0\mu_0\omega^2 d_{\text{eff}} f^{(2)}(r)E_{2\omega}E_\omega^*,$$
$$\frac{d^2 E_{2\omega}}{dr^2} + \frac{1}{r}\frac{dE_{2\omega}}{dr} + 4k^2 f^{(1)}(r)E_{2\omega} = -4\epsilon_0\mu_0\omega^2 d_{\text{eff}} f^{(2)}(r)E_\omega^2. \quad (8)$$

The initial condition of equations (8) is

$$E_\omega|_{r=0} = A_\omega(0), \frac{dE_\omega}{dr}|_{r=0} = 0, E_{2\omega}|_{r=0} = 0, \frac{dE_{2\omega}}{dr}|_{r=0} = 0. \quad (9)$$

It means that at $r = 0$, there is only cylindrical electromagnetic wave with fundamental frequency ω whose amplitude is $A_\omega(0)$, and equations (8) describe the amplitude of cylindrical SHG at arbitrary r. In this case, equations (8) is a set of ordinary differential equations, which can be solved by Runge-Kutta method, and one can discuss the features of cylindrical SHG in a nonlinear medium with arbitrary inhomogeneity. Here, for example, we consider some special cases.

(i) We consider the inhomogeneity of the medium can be described by a sine function $f^{(n)}(r) = \sin(\omega_0 r)$, where ω_0 is a constant and the nth-order nonlinear polarization of the medium is fluctuant periodically with r. The spatial period is $2\pi/\omega_0$. Such medium with the periodic inhomogeneity is something like photonic crystal, or nonlinear photonic crystal.

(ii) We consider that the inhomogeneity of the medium can be described by $f^{(n)}(r) = \exp(\Delta_n r)$. Unlike the periodic case, the nth-order nonlinear polarization of the medium is increased with r when $\Delta_n > 0$ while decreasing with r when $\Delta_n < 0$.

(iii) We consider the inhomogeneity of the medium can be described by a Gaussian function, viz. $f^{(n)}(r) \propto \exp(-r^2/R_n^2)$, where R_n is the nth-order characteristic length.

With regard to inhomogeneity, the types of sine, exponent and Gaussian function are simple but useful cases. Although strict inhomogeneity of materials as $\sin(\omega_0 r)$, $\exp(\Delta_n r)$ or $\exp(-r^2/R_n^2)$ can not find in nature, some inhomogeneity of materials may be approximatively described by $\sin(\omega_0 r)$, $\exp(\Delta_n r)$ or $\exp(-r^2/R_n^2)$. On the other hand, a lot of researches have been made theoretically and experimentally by using various inhomogeneity, and all most inhomogeneity can be realized by using metamaterials [20–25] and nonlinear metamaterials [26–32].

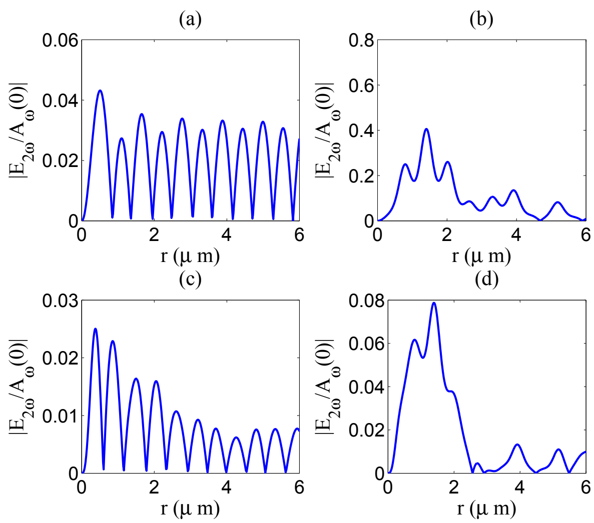

FIG. 1: (Color online) Calculation results of the amplitudes of the SHG ($E_{2\omega}$) in the medium with different inhomogeneity. We use $A_\omega(0) = 1$, $\omega = 6 \times 10^8$ MHz, $\chi^{(1)} = 1$, $\omega_0 = 10\mu m^{-1}$ and $d_{\text{eff}} = 0.1$. (a) $f^{(1)}(r) = 1$, $f^{(2)}(r) = 1$; (b) $f^{(1)}(r) = \sin(\omega_0 r)$, $f^{(2)}(r) = 1$; (c) $f^{(1)}(r) = 1$, $f^{(2)}(r) = \sin(\omega_0 r)$; (d) $f^{(1)}(r) = \sin(\omega_0 r)$, $f^{(2)}(r) = \sin(\omega_0 r)$.

First, we consider the sine function cases. Figure 1 shows the amplitudes of the SHG ($|E_{2\omega}|$) in the medium with different inhomogeneity. Figure 1(a) shows the homogeneous case; Figure. 1(b) shows the case that the first order nonlinear polarization is inhomogeneous but the second order nonlinear polarization is homogeneous; Figure. 1(c) shows the case that the first order nonlinear polarization is homogeneous but the second order nonlinear polarization is inhomogeneous, and Fig. 1(d) shows the case that the first and second order nonlinear polarizations are both inhomogeneous. It need to mention that we use $A_\omega(0) = 1$ in our calculation, which means that we use the amplitude of the fundamental frequency ω at $r = 0$ as the unit to compare other oscillations. Figure 1 (also other figures) shows the amplitudes of the SHG in this unit. So we use $|E_{2\omega}/A_\omega(0)|$ to represent the values in Fig. 1. It can be found that both $f^{(1)}(r)$ and $f^{(2)}(r)$ can influence on the amplitudes of the SHG. On the whole, the amplitudes of the SHG in Fig. 1(a) and Fig. 1(b) are larger than Fig. 1(c) and Fig. 1(d). The reason is that SHG comes from the secondary nonlinear polarization of the nonlinear medium and the existence of an inhomogeneous factor $f^{(2)}(r) \leq 1$ will reduce the amplitude of the SHG. Comparison of Fig. 1(a) and Fig. 1(b) we find that the inhomogeneity of the first order nonlinear polarization can change the peak value of $|E_{2\omega}|$ while comparison of Fig. 1(c) and Fig. 1(d) show that the inhomogeneity of the first order nonlinear polarization can alter the positions of the peaks and troughs of $|E_{2\omega}|$.

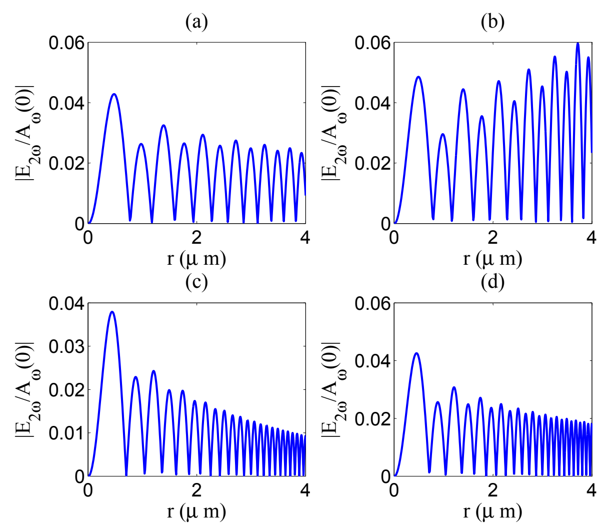

FIG. 2: (Color online) Calculation results of the amplitudes of the SHG ($|E_{2\omega}|$) in the medium with inhomogeneity of $\Delta_n > 0$. We also use $\omega = 6 \times 10^8$ MHz, $\chi^{(1)} = 1$ and $d_{\text{eff}} = 0.1$. (a) $\Delta_1 = 0.5\mu m^{-1}$, $\Delta_2 = 0.5\mu m^{-1}$; (b) $\Delta_1 = 0.5\mu m^{-1}$, $\Delta_2 = 1\mu m^{-1}$; (c) $\Delta_1 = 1\mu m^{-1}$, $\Delta_2 = 0.5\mu m^{-1}$; (d) $\Delta_1 = 1\mu m^{-1}$, $\Delta_2 = 1\mu m^{-1}$.

Second, we turn to consider the case that the inhomogeneity of the medium is described by $f^{(n)}(r) = \exp(\Delta_n r)$ (n=1, 2) and the nth-order nonlinear polarization of the medium is increased with r, viz. $\Delta_n > 0$. Figure 2 shows calculation results of $|E_{2\omega}|$ in the medium. In Fig. 2(a) and Fig. 2(c), the inhomogeneity of the secondary nonlinear polarization is the same while the first order nonlinear polarization is different. Comparison of Fig. 2(a) and Fig. 2(c) gives that the larger Δ_1 leads a lower $|E_{2\omega}|$ on the whole. It means that a larger inhomogeneous factor $f^{(1)}(r)$ will reduce the amplitude of the SHG. The same result can also be obtained from the comparison of Fig. 2(b) and Fig. 2(d). We also can fix the first order nonlinear polarization and consider different secondary nonlinear polar-

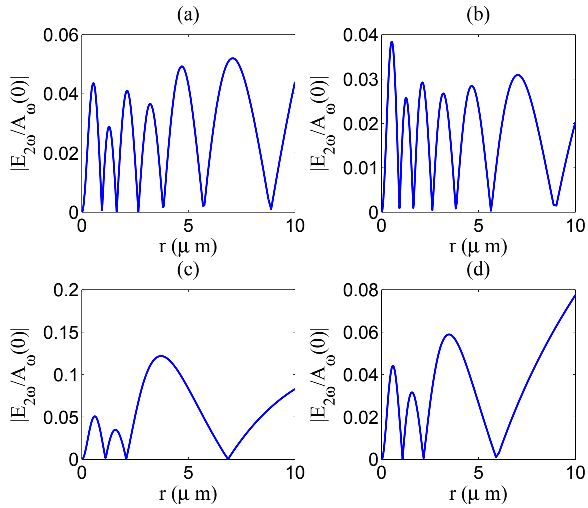

FIG. 3: (Color online) Calculation results of the amplitudes of the SHG ($|E_{2\omega}|$) in the medium with inhomogeneity of $\Delta_n < 0$. We also use $\omega = 6 \times 10^8$ MHz, $\chi^{(1)} = 1$ and $d_{\text{eff}} = 0.1$. (a) $\Delta_1 = -0.5\mu m^{-1}$, $\Delta_2 = -0.5\mu m^{-1}$; (b) $\Delta_1 = -0.5\mu m^{-1}$, $\Delta_2 = -1\mu m^{-1}$; (c) $\Delta_1 = -1\mu m^{-1}$, $\Delta_2 = -0.5\mu m^{-1}$; (d) $\Delta_1 = -1\mu m^{-1}$, $\Delta_2 = -1\mu m^{-1}$.

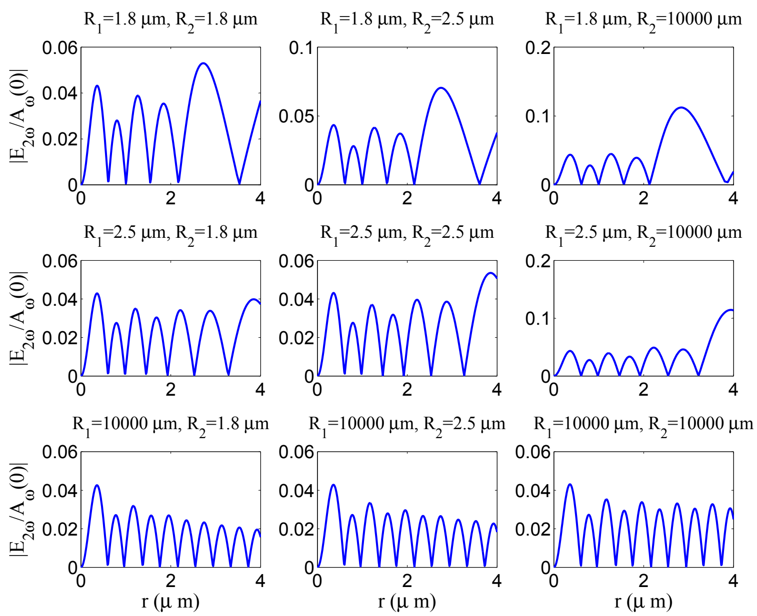

FIG. 4: (Color online) Calculation results of the amplitudes of the SHG ($|E_{2\omega}|$) in the medium with inhomogeneity of Gaussian profile. We also use $\omega = 6 \times 10^8$ MHz, $\chi^{(1)} = 1$ and $d_{\text{eff}} = 0.1$.

izations. In Fig. 2(a) and Fig. 2(b), the inhomogeneity of the first order nonlinear polarization is both $\Delta_1 = 0.5\mu m^{-1}$ while the secondary nonlinear polarization is $\Delta_2 = 0.5\mu m^{-1}$ and $\Delta_2 = 1\mu m^{-1}$ respectively. We can find that the larger Δ_2 leads a higher $|E_{2\omega}|$ on the whole and the change of Δ_2 takes less modification of the positions of the peaks and troughs of $|E_{2\omega}|$.

Similar results can be obtained for the case of $\Delta_n < 0$. Figure 3 shows calculation results of $|E_{2\omega}|$ in the medium with $\Delta_n < 0$. In Fig. 3(a) and Fig. 3(c), the inhomogeneity of the secondary nonlinear polarization is the same while the first order nonlinear polarization is different. Comparison of Fig. 3(a) and Fig. 3(c), as well as the comparison of Fig. 2(b) and Fig. 2(d), also gives the conclusion that the larger Δ_1 leads a lower $|E_{2\omega}|$ on the whole. We also can fix the first order nonlinear polarization and consider different secondary nonlinear polarizations. In Fig. 3(a) and Fig. 3(b), the inhomogeneity of the first order nonlinear polarization is both $\Delta_1 = -0.5\mu m^{-1}$ while the secondary nonlinear polarization is $\Delta_2 = -0.5\mu m^{-1}$ and $\Delta_2 = -1\mu m^{-1}$ respectively. We can find the similar results that the larger Δ_2 leads a higher $|E_{2\omega}|$ on the whole and the change of Δ_2 takes less modification of the positions of the peaks and troughs of $|E_{2\omega}|$.

Third, we consider the inhomogeneity of the medium is Gaussian type $f^{(n)}(r) = \exp(-r^2/R_n^2)$. Figure 4 shows calculation results of $|E_{2\omega}|$ in the medium with different characteristic length R_n. We can find that if R_1 is fixed, then a larger R_2 leads larger $|E_{2\omega}|$ outside the characteristic length R_1. We take the first line of Fig. 4 as an example. In this case, R_1 is fixed as 1.8 μm while R_2 is chosen as 1.8 μm, 2.5 μm and 10000 μm respectively. For larger R_2, $|E_{2\omega}|$ is enhanced greatly only within r ranges from 2 to 4 μm, while little difference within r ranges from 0 to 2 μm. So using the characteristic length features, one can enhance or reduce $|E_{2\omega}|$ at some specific region of r. If fixing R_2 and changing the value of R_1, one can find that the positions of the peaks of $|E_{2\omega}|$ will also change. As an example, R_2 is fixed as 1.8 μm while R_1 is chosen as 1.8 μm, 2.5 μm and 10000 μm respectively. When $R_1 = 1.8\ \mu m$, there are five peaks of $|E_{2\omega}|$ within r ranges from 0 to 4 μm; When $R_1 = 2.5\ \mu m$, there are seven peaks of $|E_{2\omega}|$ within r ranges from 0 to 4 μm; When $R_1 = 10000\ \mu m$, there are ten peaks of $|E_{2\omega}|$ within r ranges from 0 to 4 μm.

III. CYLINDRICAL SHG IN AN INHOMOGENEOUS AND NONLINEAR MEDIUM DESCRIBED BY EXACT SOLUTIONS

In this section, we will deal with the problem of cylindrical SHG in an inhomogeneous and nonlinear medium from a different point of view. Reference [15] has presented a new method for deriving exact solutions and obtained an exact solution to describe the propagation of cylindrical electromagnetic waves in a nonlinear nondispersive medium. In a recent work [17] we show that the solution can be used to discuss the cylindrical SHG in a nonlinear nondispersive medium very well. Moreover, reference [16] shows that this important technique can be extended to deal with problems of cylindrical electromagnetic waves propagating in a medium with nonlinear and inhomogeneous. Here, we will give a full description of using exact solutions to deal with the problem of cylindrical SHG in an inhomogeneous and nonlinear medium, and as a verification, we will compare the results obtained from these two methods.

We begin our discussion by rewrite Eq. (1) in the form

$$\frac{\partial H}{\partial r} + \frac{H}{r} = \varepsilon(E,r)\frac{\partial E}{\partial t}, \qquad \frac{\partial E}{\partial r} = \mu_0 \frac{\partial H}{\partial t}, \tag{10}$$

where $\varepsilon(E,r) = \mathrm{d}D/\mathrm{d}E$. In reference [16], the function $\varepsilon(E,r)$ is chosen in the form:

$$\varepsilon(E,r) = \epsilon_0 \varepsilon_1 r^{\beta} \exp(\alpha E), \tag{11}$$

where $\varepsilon_1, \alpha, \beta$ are certain constants. It has been shown that such system can be integrated exactly and admits exact solutions in this case. Here we also use the same inhomogeneous factor. In what follows, we will show that the exact solution is a new way to deal with SHG.

On account of dimension, the system can be written as

$$\varepsilon(E,r) = \epsilon_0 \varepsilon_1 (r/r_0)^{\beta} \exp(\alpha E), \tag{12}$$

where r_0 is an arbitrary constant with the dimension of length. If setting $r_0 = 1$, then Eq. (12) go into Eq. (11). We define $\rho = r/a$, $\tau = t(\epsilon_0\varepsilon_1\mu_0)^{-1/2}/a$, with a being an constant with the dimension of length, Considering that if a solution of the homogenous and linear problem has been obtained and recorded as E_0 and H_0 in the form

$$E_0 = \mathcal{E}(\rho,\tau), \qquad H_0 = \varepsilon_1^{1/2} Z_0^{-1} \mathcal{H}(\rho,\tau), \tag{13}$$

where $\mathcal{E}$ and $\mathcal{H}$ satisfying the linear system:

$$\frac{\partial \mathcal{H}}{\partial \rho} + \frac{\mathcal{H}}{\rho} = \frac{\partial \mathcal{E}}{\partial \tau}, \qquad \frac{\partial \mathcal{E}}{\partial \rho} = \frac{\partial \mathcal{H}}{\partial \tau}. \tag{14}$$

then the exact solution can be obtained as [15, 16]

$$\begin{aligned} E &= \mathcal{E}\Big(2(a/r_0)^{1+\beta/2}\rho^{1+\beta/2}e^{\alpha E/2}, (\beta+2)(a/r_0)\tau + g_0 H\Big), \\ H &= \frac{\sqrt{\varepsilon_1}e^{\alpha E/2}}{Z_0}(a/r_0)^{\beta/2}\rho^{\beta/2} \\ &\quad \times \mathcal{H}\Big(2(a/r_0)^{1+\beta/2}\rho^{1+\beta/2}e^{\alpha E/2}, (\beta+2)(a/r_0)\tau + g_0 H\Big). \end{aligned} \tag{15}$$

The choice of a and r_0 are limited to the case that if $\alpha \to 0$ and $\beta \to 0$, then the solution will go into the homogenous and linear case. This implies that:

$$\begin{aligned} &\lim_{\beta\to 0} 2(a/r_0)^{1+\beta/2} = 1, \\ &\lim_{\beta\to 0} (\beta+2)(a/r_0) = 1. \end{aligned} \tag{16}$$

Reference [16] used $a/r_0 = 2^{-2/(\beta+2)}$ while in the present work we use $a/r_0 = 1/(\beta+2)$. Both of the choices can lead exact solution of Eqs. (10), however, describe propagation of cylindrical electromagnetic waves with different frequencies. If choosing $a/r_0 = 1/(\beta+2)$, it can verify that the exact solution obtained describes cylindrical electromagnetic waves whose frequency is the same as propagation in an homogeneous and linear medium. Then we can obtain:

$$\begin{aligned} E &= \mathcal{E}\Big(2(\beta+2)^{-(1+\beta/2)}\rho^{1+\beta/2}e^{\alpha E/2}, \tau + g_0 H\Big), \\ H &= \frac{\sqrt{\varepsilon_1}e^{\alpha E/2}}{Z_0}(\beta+2)^{-\beta/2}\rho^{\beta/2} \\ &\quad \times \mathcal{H}\Big(2(\beta+2)^{-(1+\beta/2)}\rho^{1+\beta/2}e^{\alpha E/2}, \tau + g_0 H\Big). \end{aligned} \tag{17}$$

These expressions give an exact solution of Maxwell equations in such an inhomogeneous nonlinear medium, and in what follows, we will give a traveling wave solution which describes cylindrical wave propagation in an infinite nonlinear and inhomogeneous medium from these expressions.

We begin our discussion by considering cylindrical wave propagation in an infinite medium and the solution of linear problem is: $\mathcal{E}(r,t) = \zeta J_0(kr)\cos(\omega t)$ and $\mathcal{H}(r,t) = -\zeta J_1(kr)\sin(\omega t)$. Rewriting it in variable (ρ,τ), and using exact solution (17) we can obtain the solution of nonlinear problem:

$$E = \zeta J_0(kr^{1+\beta/2}e^{\alpha E/2}\Omega)\cos\Big(\omega t + \frac{\alpha\mu_0\omega r H}{\beta+2}\Big), \tag{18}$$

$$\begin{aligned} H &= -\zeta\frac{\sqrt{\varepsilon_1}e^{\alpha E/2}}{Z_0}\Big(\frac{r}{r_0}\Big)^{\beta/2} \\ &\quad \times J_1(kr^{1+\beta/2}e^{\alpha E/2}\Omega)\sin\Big(\omega t + \frac{\alpha\mu_0\omega r H}{\beta+2}\Big), \end{aligned} \tag{19}$$

where $\Omega = 2(\beta+2)^{-1}r_0^{-\beta/2}$, and we set $r_0 = 1\mu m$. The solution shows that the electric field and magnetic field of the cylindrical electromagnetic wave in a nonlinear medium are not separate, but coupling with each other. It is obviously that if $\beta \to 0$ exact solutions (18) and (19) will go into the homogeneous case, which have been discussed in Ref. [17].

Here, we will justify the calculations with the solution obtained in Ref. [16] and show that the same results can be obtained by using the solution in Ref. [16]. The exact solution obtained in Ref. [16] is

$$E = \mathcal{E}\Big(\rho^{1+\beta/2}e^{\alpha E/2}, 2^{\frac{-2}{\beta+2}}[(\beta+2)\tau + \frac{Z_0\alpha\rho H}{\sqrt{\varepsilon_1}}]\Big),$$
$$H = \frac{\sqrt{\varepsilon_1}e^{\alpha E/2}\rho^{\beta/2}}{Z_0 2^{\frac{\beta}{\beta+2}}}\mathcal{H}\Big(\rho^{1+\beta/2}e^{\alpha E/2}, 2^{\frac{-2}{\beta+2}}[(\beta+2)\tau + \frac{Z_0\alpha\rho H}{\sqrt{\varepsilon_1}}]\Big), \tag{20}$$

We also consider the cylindrical wave propagating in an infinite medium and the solution of linear problem is: $\mathcal{E}(r,t) = \zeta J_0(kr)\cos(\omega t)$ and $\mathcal{H}(r,t) = -\zeta J_1(kr)\sin(\omega t)$. Rewriting it in variable (ρ,τ), and using exact solution (20) we can obtain the solution of nonlinear problem:

$$E = \zeta J_0(kr^{1+\beta/2}e^{\alpha E/2}2^{\beta/(\beta+2)}) \times \cos\Big(2^{-2/(\beta+2)}(\beta+2)\omega t + 2^{-2/(\beta+2)}\alpha\mu_0\omega r H\Big), \tag{21}$$
$$H = -\zeta\frac{\sqrt{\varepsilon_1}e^{\alpha E/2}}{Z_0}\Big(\frac{r}{r_0}\Big)^{\beta/2}J_1(kr^{1+\beta/2}e^{\alpha E/2}2^{\beta/(\beta+2)}) \times \sin\Big(2^{-2/(\beta+2)}(\beta+2)\omega t + 2^{-2/(\beta+2)}\alpha\mu_0\omega r H\Big) \tag{22}$$

It should be note that the relation between variable (ρ,τ) and (r,t) are used as $\rho = r \cdot 2^{2/(\beta+2)}$ and $\tau = t(\epsilon_0\varepsilon_1\mu_0)^{-1/2} \cdot 2^{2/(\beta+2)}$. Equations (21) and (22) are very different from Eqs. (18) and (19). However, if writing $2^{-2/(\beta+2)}(\beta+2)$ as Λ, then Eqs. (21) and (22) become

$$E = \zeta J_0(\Lambda kr^{1+\beta/2}e^{\alpha E/2}\Omega) \times \cos\Big(\Lambda\omega t + \Lambda\alpha\mu_0\omega r H/(\beta+2)\Big), \tag{23}$$
$$H = -\zeta\frac{\sqrt{\varepsilon_1}e^{\alpha E/2}}{Z_0}\Big(\frac{r}{r_0}\Big)^{\beta/2}J_1(\Lambda kr^{1+\beta/2}e^{\alpha E/2}\Omega) \times \sin\Big(\Lambda\omega t + \Lambda\alpha\mu_0\omega r H/(\beta+2)\Big). \tag{24}$$

If define $\Lambda\omega = \omega'$, then $k' = \omega'\sqrt{\epsilon_0\varepsilon_1\mu_0} = \Lambda k$. So we can write Eqs. (23) and (24) as follows:

$$E = \zeta J_0(k'r^{1+\beta/2}e^{\alpha E/2}\Omega)\cos\Big(\omega' t + \frac{\alpha\mu_0\omega' r H}{\beta+2}\Big), \tag{25}$$
$$H = -\zeta\frac{\sqrt{\varepsilon_1}e^{\alpha E/2}}{Z_0}\Big(\frac{r}{r_0}\Big)^{\beta/2} \times J_1(k'r^{1+\beta/2}e^{\alpha E/2}\Omega)\sin\Big(\omega' t + \frac{\alpha\mu_0\omega' r H}{\beta+2}\Big). \tag{26}$$

Equations (25) and (26) are exactly the same as Eqs. (18) and (19) except the frequency of the wave. Both exact solutions correctly describe cylindrical electromagnetic wave propagation in an inhomogenous and nonlinear medium. Equations (18) and (19) describe cylindrical electromagnetic wave with frequency ω while Eqs. (25) and (26) with frequency ω'.

In what follows, we will show that SHG comes out quite naturally from the exact solution and the descriptions of SHG by the exact solution are in good agreement with the coupled-wave equations. Considering α is small, then H can be approximately written as

$$H \approx -\zeta\frac{\sqrt{\varepsilon_1}}{Z_0}r^{\beta/2}J_1(kr^{1+\beta/2}\Omega)\sin(\omega t), \tag{27}$$

Substitution Eq. (27) into Eq. (18) and using the approximation $J_0(xe^{\alpha E/2}) \approx J_0(x) - \alpha E x J_1(x)/2$ leads:

$$E \approx \zeta J_0(kr^{1+\beta/2}\Omega)\cos(\omega t) - \zeta J_0(kr^{1+\beta/2}\Omega)\Theta\cos(2\omega t), \tag{28}$$

where

$$\Theta = \frac{\zeta\alpha\mu_0\omega r}{\beta+2}\frac{\sqrt{\varepsilon_1}}{Z_0}r^{\beta/2}J_1(kr^{1+\beta/2}\Omega), \tag{29}$$

and higher-harmonic generation (for example, 3ω) are ignored. From Eq. (28) we can find that SHG comes out quite naturally from exact solution and the inhomogeneity of the medium can influence on the efficiency of the SHG. The amplitudes of the fundamental frequency and SHG then reads

$$E_\omega = \zeta J_0(kr^{1+\beta/2}\Omega),$$
$$E_{2\omega} = \zeta J_0(kr^{1+\beta/2}\Omega)\Theta, \tag{30}$$

so $\Theta = E_{2\omega}/E_\omega$ describes the efficiency of SHG. In what follows, we will give comparison of using coupled-wave equations and exact solution to describe SHG.

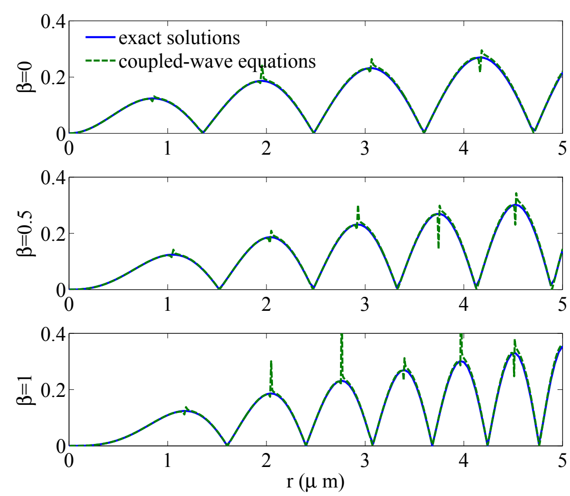

FIG. 5: (Color online) Efficiencies of cylindrical SHG in an inhomogeneous and nonlinear medium based on coupled-wave equations and exact solutions.

Using $\varepsilon(E,r) = \mathrm{d}D/\mathrm{d}E = \epsilon_0\varepsilon_1 r^\beta\exp(\alpha E)$, we obtain:

$$P = P_0 + \epsilon_0(\varepsilon_1 r^\beta - 1)E + \frac{\epsilon_0\varepsilon_1 r^\beta\alpha}{2}E^2 + \cdots, \tag{31}$$

where $P_0 = \frac{\epsilon_0\varepsilon_1 r^\beta}{\alpha}[1+C(r)]$ with $C(r)$ is the constant of integration. Then $d_{\text{eff}} = \chi^{(2)}/2 = \alpha/2$, $f^{(1)}(r) = r^\beta$ and $f^{(2)}(r) = r^\beta$. Figure 5 shows the efficiencies of cylindrical SHG in an inhomogeneous and nonlinear medium based on the coupled-wave equations (we use Runge-Kutta method to solve Eq. (8) and plot $|E_{2\omega}/E_\omega|$) and the exact solutions (we use the expression of Θ, viz. Eq. (29) which describes the efficiency of SHG, and plot $|\Theta|$). We use $\omega = 6\times 10^8$ MHz, $\chi^{(1)} = 1$, $\zeta = 1$ and $d_{\text{eff}} = 0.1$, as same as in Fig. 1 and Fig. 2. From Fig. 5 we can find that descriptions of SHG by the coupled-wave equations are in good agreement with the exact solutions. There are some peaks on the curve of using the coupled-wave equations method. In what follows, we will give an analysis on the origin of these peaks. Figure 6 shows amplitudes of fundamental frequency and cylindrical SHG in an inhomogeneous and nonlinear medium based on the coupled-wave equations and Eq. (30) which derived from the exact solutions. We can find that descriptions of both $|E_\omega|$ and $|E_{2\omega}|$ by the coupled-wave equations are in good agreement with the exact solutions, and these peaks are completely absent. However, Figure 5 shows that when we plot $|E_{2\omega}/E_\omega|$, these peaks occur. To further analyze the origin of these peaks, we plot amplitudes of fundamental frequency and cylindrical SHG and the efficiencies of cylindrical SHG based on the coupled-wave equations in Fig. 7. We can find some features of these peaks. First, around a zero point of $|E_\omega|$, both $|E_\omega|$ and $|E_{2\omega}| \to 0$. Second, these peaks appear at the fixed points. From both Fig. 5 and Fig. 7 we can find that these peaks appear at every peak of the efficiencies. Third, these peaks appear at $|E_\omega| = 0$. This fact can be found in Fig. 7. We take Fig. 7(b) as an example. We point out the five peaks, and we can find that them indeed appear at $|E_\omega| = 0$.

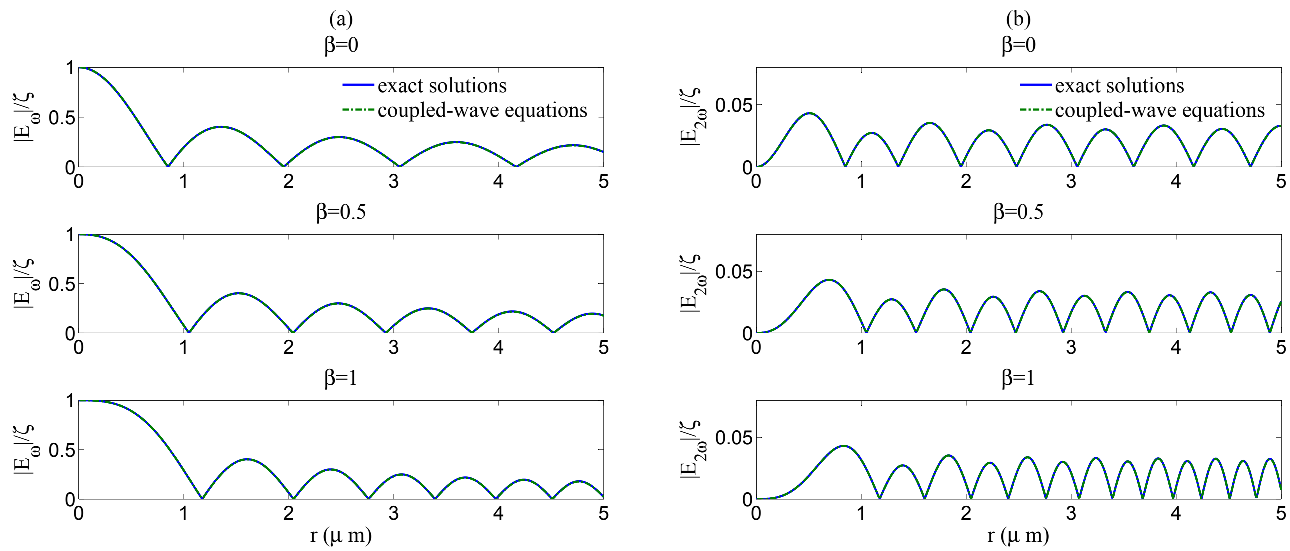

FIG. 6: (Color online) Amplitudes of (a) fundamental frequency and (b) cylindrical SHG in an inhomogeneous and nonlinear medium based on coupled-wave equations and exact solutions.

All these features of these peaks can be explained by Eqs. (30) which deduced from the exact solutions. First, Equations (30) shows that when $|E_\omega| = 0$ then $|E_{2\omega}| = 0$. However, the efficiency $|E_{2\omega}/E_\omega|$ at these points is finite, which described by the expression of Θ, viz. Eq. (29). Second, one can find that $|\Theta|$ reaches its maximums at the position $|E_\omega| = 0$. This result can be obtained by solving the equation $d\Theta/dr = 0$ with Θ is given in Eq. (29). We consider that according to the definition of SHG efficiency $|E_{2\omega}/E_\omega|$, a zero point of $|E_\omega|$ will cause significant distortions because the numerical calculation provides extreme sensitivity through a small range around the zero point. For example, around a zero point of $|E_\omega|$, both $|E_\omega|$ and $|E_{2\omega}| \to 0$. If $|E_\omega|$ is small enough that $|E_\omega|$ is recorded as 0 while $|E_{2\omega}|$ is recorded as a nonzero number in computer, then a peak arises. So we can see that these peaks appear at every zero points of $|E_\omega|$, which also corresponding to every peak of the efficiencies.

IV. CONCLUSION

In conclusion, we have used two methods to deal with the problem of SGH of cylindrical electromagnetic wave propagating in an inhomogeneous and nonlinear medium. One method is using traditional coupled-wave equations. We have set up coupled-wave equations of cylindrical electromagnetic waves interacting with nonlinear and inhomogeneous media. Using the coupled-wave equations we have analyzed features of cylindrical SHG. The other method is using exact solutions of the Maxwell equations. We use a simple method to deduce SHG from this exact solution and found that the results are in good agreement with which are obtained by using the coupled-wave equations.

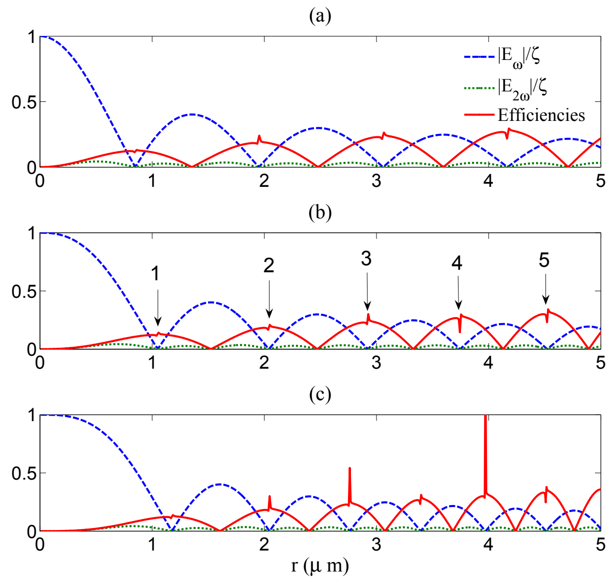

FIG. 7: (Color online) Amplitudes of fundamental frequency, cylindrical SHG and efficiencies of cylindrical SHG in an inhomogeneous and nonlinear medium based on coupled-wave equations. (a) $\beta = 0$, (b) $\beta = 0.5$, (c) $\beta = 1$.

Acknowledgments

The work was supported in part by the National Science Foundation (NSF) of China (Grant Nos. 10874050, 10975054, and 91021011),the National Fundamental Research Program of China (Grant No. 2012CB922103) and the Research Fund for the Doctoral Program of Higher Education of China (Grant No. 200804870051). H. X. and C. D are also sponsored in part by the Scholarship Award for Excellent Doctoral Student granted by the Ministry of Education China, Doctorate Innovation Foundation, and the Graduate Practice Base of Innovation and Enterprise (Grant No. HF-09-05-2011-012) of the Huazhong University of Science and Technology.

[1] P. A. Franken, A. E. Hill, C. W. Peters, and G. Weinreich, Phys. Rev. Lett. **7**, 118 (1961).
[2] J. A. Armstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan, Phys. Rev. **127**, 1918 (1962).
[3] N. Bloembergen and P. S. Pershan, Phys. Rev. **128**, 606 (1962).
[4] N. Bloembergen, Rev. Mod. Phys. **54**, 685 (1982).
[5] Y. R. Shen, *Principles of Nonlinear Optics* (Wiley, New York, 1984).
[6] Y. Nakayama, P. J. Pauzauskie, A. Radenovic, R. M. Onorato, R. J. Saykally, J. Liphardt, and P. Yang, Nature **447**, 1098 (2007).
[7] M. A. Belkin, F. Capasso, A. Belyanin, D. L. Sivco, A. Y. Cho, D. C. Oakley, C. J. Vineis, and G. W. Turner, Nature Photonics **1**, 288 (2007).
[8] H. Suchowski, D. Oron, A. Arie, and Y. Silberberg, Phys. Rev. A **78**, 063821 (2008).
[9] L. Fu, G. Ma, and E. C. Y. Yan, J. Am. Chem. Soc. **132**, 5405 (2010).
[10] N. Ji, V. Ostroverkhov, C. S. Tian, and Y. R. Shen, Phys. Rev. Lett. **100**, 096102 (2008).
[11] Y. R. Shen, Nature **337**, 519 (1989).
[12] T. C. Anglin and J. C. Conboy, Biochemistry **48**, 10220 (2009).
[13] Yan Fu, H. Wang, R. Shi, and J.-X. Cheng, Biophysical Journal **92**, 3251 (2007).
[14] X. Lu, N. Shephard, J. Han, G. Xue, and Z. Chen, Macromolecules **41**, 8770 (2008).
[15] E. Y. Petrov and A. V. Kudrin, Phys. Rev. Lett. **104**, 190404 (2010).
[16] H. Xiong, L.-G. Si, P. Huang, and X. Yang, Phys. Rev. E **82**, 057602 (2010).
[17] H. Xiong, L.-G. Si, J. F. Guo, X.-Y. Lü, and X. Yang, Phys. Rev. A **83**, 063845 (2011).
[18] V. A. Es'kin, A. V. Kudrin, and E. Y. Petrov, Phys. Rev. E **83**, 067602 (2011).
[19] H. Xiong, L.-G. Si, C. Ding, X. Yang and Y. Wu, Phys. Rev. A **84**, 043841 (2011).
[20] W. C. Chew, *Waves and Fields in Inhomogeneous Media*(Van Nostrand, New York, 1990).
[21] T. J. Cui, et al. *Metamaterials: Theory, Design, and Applications*(Springer, New York, 2010).
[22] Z. Ruan, M. Yan, C. W. Neff, and M. Qiu, Phys. Rev. Lett. **99**, 113903 (2007).
[23] M. Yan, Z. Ruan, and M. Qiu, Phys. Rev. Lett. **99**, 233901 (2007).
[24] Y. Lai, H. Chen, Z. Q. Zhang, and C. T. Chan, Phys. Rev. Lett. **102**, 093901 (2009).
[25] Y. You, G. W. Kattawar, and P. Yang, Opt. Express **17**, 6591 (2009).
[26] E. Poutrina, D. Huang and D. R. Smith, New Journal of Physics **12**, 093010 (2010).
[27] J. Petschulat, A. Chipouline, A. Tünnermann, and T. Pertsch, Phys. Rev. A **80**, 063828 (2009).
[28] A. Rose and D. R. Smith, Phys. Rev. A **84**, 013823 (2011).
[29] H. Xiong, L.-G. Si, C. Ding, X.-Y. Lü, X. Yang and Y. Wu, Phys. Rev. E **85**, 016602 (2012).
[30] D. A. Powell, I. V. Shadrivov, and Y. S. Kivshar, Appl. Phys. Lett. **95**, 084102 (2009).
[31] M. Lapine and M. Gorkunov, Phys. Rev. E **70**, 066601 (2004).
[32] A. Rose, D. Huang, and D. R. Smith, Phys. Rev. Lett. **107**, 063902 (2011).