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Abstract

Charged particle interaction with localized wave packets in a magnetic field is formulated using the canonical perturbation theory and the Lie transform theory. An electrostatic wave packet characterized by a wide range of group and phase velocities as well as spatial extent along and across the magnetic field is considered. The averaged changes in the momentum along the magnetic field, the angular momentum, and the guiding center position for an ensemble of particles due to their interaction with the wave packet are determined analytically. Both resonant and ponderomotive effects are included. For the case of a Gaussian wave packet, closed-form expressions include the dependency of the ensemble averaged particle momenta and gc position variations on wave packet parameters and particle initial conditions. These expressions elucidate the physics of the interaction which is markedly different from the well known case of particle interaction with plane waves and are relevant to a variety of applications ranging from space and astrophysical plasmas to laboratory and fusion plasmas as well as accelerators and microwave devices.

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I. INTRODUCTION

The interaction of charged particles with electromagnetic waves under the presence of a magnetic field is a ubiquitous phenomenon in a variety of natural and technological systems. The wave-particle interactions are common occurrence in astrophysical and space plasmas, and also have useful applications in beam physics and accelerators [1] as well as in laboratory and fusion plasmas [2–4]. The charged particles, through their interaction, can collectively exchange energy and momentum with the waves. In accelerators particles gain energy from the electromagnetic fields, while in microwave sources and amplifiers energetic electrons give up some fraction of their energy to waves [5]. In fusion plasmas, radio frequency waves are used to heat the plasma, and also to generate currents in plasmas by imparting momentum to particles. In addition to electromagnetic waves, lower hybrid (LH) electrostatic modes can also be used for heating and especially for current drive in fusion plasmas [6]. In general, the waves – either electromagnetic or electrostatic – are not in the form of plane waves. Rather, they are wave packets that are localized in space and could also be of finite duration in time. This is commonly the case in fusion plasmas where the externally applied rf waves have a finite spatial extent as well as in space plasmas where LH solitary structures occur [7]. The spatial or temporal extent of the wave pulses could be as small as a few cycles or even sub-cycles, differing significantly from ordinary adiabatically modulated wave packets. So the wave-particle interaction is a finite transit-time interaction which is qualitatively different from the continuous interaction in the case of a plane wave.

In a uniform, static magnetic field, there are two basic mechanisms by which particles exchange energy and momentum with wave packets – resonant and non-resonant. Consider a Fourier component of a wave packet representing one plane wave of frequency ω and parallel wave vector $k_\|$. By parallel or perpendicular we mean the components of a vector parallel or perpendicular, respectively, to the direction of the imposed magnetic field. For resonant interactions, the parallel velocity of the particle has to be such $v_\| = (\omega - n\Omega_c)/k_\|$ where n is an integer and Ω_c is the cyclotron frequency of the particle. Since a wave packet is composed of many plane waves, the resonant interaction occurs only for those particles in the distribution function that satisfy the above condition for any plane wave. On the other hand, the non-resonant interaction is between particles and the envelope of the wave packet and is referred to as the ponderomotive force [8]. The resonant and non-resonant interactions are clearly different. The condition for resonant interaction is satisfied in a restricted domain of the dynamical phase space of the particle. The nonlinear ponderomotive effect depends on the average force seen by a charged particle as it traverses the wave packet, and depends on the particle velocity and the spatial profile of electric field of the wave packet [9]. The bulk of the particles is affected by the ponderomotive force due to their interaction with the spatially localized wave packet.

Since wave-particle interactions are of fundamental importance in physics and a paradigm for dynamical chaos in Hamiltonian systems [9], the interaction with spatially or temporarily modulated waves has been studied for many different, and special,cases. The motion of particles in the presence of adiabatically varying waves has been studied in [10]. The interaction of particles, moving along the magnetic field, with periodic, spatially localized, static, coherent, electrostatic wave packets has been studied in [11] while the single-pass interaction was discussed in [12, 13]. The energy transfer between particles and wave packets has been analytically formulated, in the Born approximation, in [14] as well as in [15]. An extensive study of the ponderomotive force on particles has been carried out, for the adiabatic case, by Cary and Kaufman [8], and, for the non-adiabatic, by Dodin [16].

In this paper, we study the dynamics of charged particles, in a uniform magnetic field, interacting with an electrostatic field localized in space and time. The realm of validity of the widely used electrostatic approximation is that of short-wavelength plasma modes; for example LH waves fall into that category. The form of the field is assumed to be quite general with no restrictions on the phase and group velocities of the wave packet. The temporal and spatial extent of the wave packet is arbitrary, except for the requirement that the perpendicular width of the wave packet is much larger than the Larmor radius of the particle. Since we make no adiabatic approximations, the wave packets can range from ordinary slowly modulated wave packets to wave fields that either span few cycles or are sub-cycle. The main aim of this paper is the study of the finite transit-time interaction effects on the collective particle dynamical behavior.

Our approach is based on a Hamiltonian action-angle formulation with the canonical perturbation method [9] and Lie transform techniques [17] being utilized for the calculation of angle averaged variations of the actions corresponding to the particle parallel and angular momentum as well as its guiding center position. The method naturally couples analytical information on single particle dynamics to the collective particle behavior as described by the aforementioned averaged action variations. The analytical results focus on the dynamical aspects of the finite transit-time interaction and on the dependence of the particle collective behavior on both particle and wave packet characteristics. Therefore, taking advantage of the generality of the results, rather than specifying a particular plasma mode we present characteristic cases of qualitatively different particle collective behavior. Depending on the specific application aiming either to parallel or angular momentum or guiding center position variation (or a mixture of them), the wave packet characteristics can be chosen appropriately in order to optimize the desired effect. For the case of plasma particles the wave packet parameters are obtained from a self-consistent plasma dispersion relation such as that of LH waves exhibiting a wide range of phase and group velocity values. On the other hand, for the case of a particle beam in the vacuum, as in accelerators or microwave devices, the wave packet parameters are determined simply by the geometry and the wave launching conditions.

II. HAMILTONIAN FORMULATION FOR THE PARTICLE DYNAMICS

The Hamiltonian of a particle with charge q and mass M, moving in a homogeneous, static, magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ is

$$H_0 = \frac{1}{2M}\left|\mathbf{p} - \frac{q}{c}\mathbf{A_0}\right|^2 \tag{1}$$

where $\mathbf{A_0}(\mathbf{r}) = -B_0 y\hat{\mathbf{x}}$ is the vector potential corresponding to the prescribed magnetic field, $\mathbf{p} = (p_x, p_y, p_z)$ is the momentum of the particle with its components written out in Cartesian geometry, and c is the speed of light. The canonical momenta are $p_x = Mv_x - M\Omega_c y$, $p_y = Mv_y$, and $p_z = Mv_z$, where $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity of the particle. The Hamiltonian H_0 describes the motion of a gyrating particle with cyclotron frequency $\Omega_c = qB_0/Mc$. We transform to the guiding center variables (gc) using the generating function

$$F_1 = M\Omega_c\left[\frac{1}{2}(y - Y)^2\cot\phi - xY\right] \tag{2}$$

The transformed Hamiltonian is

$$H_0 = \frac{P_z^2}{2M} + P_\phi\Omega_c \tag{3}$$

where (P_z, z), (P_ϕ, ϕ) and $(M\Omega_c X, Y)$ are the new pairs of canonical coordinates. (X, Y, z) are the appropriate Cartesian components of the guiding center position vector. P_z is the component of the gc momentum along $\mathbf{B}$, $P_\phi = Mv_\perp^2/2\Omega_c = (Mc/q)\mu = M\Omega_c\rho^2/2$ is the magnitude of the gc angular momentum, $\mu = Mv_\perp^2/2|\mathbf{B}|$ and $\rho = v_\perp/\Omega_c$ are the magnetic moment and Larmor radius, respectively, of the particle, and $\phi = \tan^{-1}(v_x/v_y)$ is its gyration angle. If we perform another canonical transformation using the generating function $F_1 = (1/2)M\Omega_c Y^2 \cot\theta$, the Hamiltonian H_0 in Eq. (3) remains the same, but the gc position, in the plane perpendicular to $\mathbf{B}$, is expressed in terms of polar canonical coordinates $\theta = \tan^{-1}(Y/X)$ and $P_\theta = (M\Omega_c/2)R_{gc}^2$ with $R_{gc}^2 = X^2 + Y^2$. Finally, the dynamical phase space of the particle is spanned by a set of canonically conjugate coordinates $\mathbf{z} = (\mathbf{J}, \boldsymbol{\theta})$, where $\mathbf{J} = (P_z, P_\phi, P_\theta)$ and $\boldsymbol{\theta} = (z, \phi, \theta)$ are the canonical momenta or actions, and positions or angles, respectively. The actions P_z and P_ϕ correspond to momentum-like variables while the action P_θ corresponds to a space-like variable.

We consider the interaction of the charged particle with a spatially localized wave packet described by an electrostatic potential of the form

$$\Phi = \Phi_0(\mathbf{r} - \mathbf{V}t; t) \sin(\mathbf{kr} - \omega t) \tag{4}$$

where $\mathbf{V}$ is the group velocity of the wave packet. The fast variation within the wave packet is given in terms of the angular frequency ω and the wave vector $\mathbf{k}$. Without loss of generality we can assume that $\mathbf{k} = k_\perp\hat{\mathbf{y}} + k_\parallel\hat{\mathbf{z}}$ is in the $y - z$ plane. In the form for Φ_0, the argument $\mathbf{r} - \mathbf{V}t$ gives the spatial modulation while the argument t gives the temporal modulation of the wave packet. This form implies that no significant spreading of the wave packet takes place. The important consequences of spreading wave packets have long been identified experimentally and theoretically [18]. However, the interaction of the particles with the wave packet depends on the transit time of the particles through the wave packet. For a large variety of applications it is realistic to assume that this time is small compared to the characteristic time for the spreading of the wave packet.

The Hamiltonian, in gc coordinates, that includes the interaction of the particle with the electrostatic potential is

$$H = H_0 + H_1 \tag{5}$$

where

$$H_1 = \left(\frac{1}{2i}\right) q\Phi_0(\mathbf{r} - \mathbf{V}t; t)\, e^{i(k_\parallel z - \omega t)} e^{-ik_\perp R_{gc}\sin\theta} e^{ik_\perp\rho\sin\phi} + c.c. \tag{6}$$

Since H_1 is a periodic function of ϕ, a Fourier expansion leads to

$$H_1 = \left(\frac{1}{2i}\right) q\Phi_0\left(\mathbf{r} - \mathbf{V}t; t\right) e^{i(k_\parallel z - \omega t)} e^{-ik_\perp R_{gc}\sin\theta} \sum_{m=-\infty}^{+\infty} (-1)^m J_m(k_\perp \rho) e^{-im\phi} + c.c. \tag{7}$$

The Hamiltonian H in Eq. (5) with H_0 given in Eq. (3) and H_1 in Eq. (7) is quite general since we have not specified either the profile of the electrostatic potential or the group velocity of the wave packet. The Hamiltonian H is, in general, non-integrable and so it is difficult to analytically determine the effect of the wave packet on particles interacting with it. In order to proceed analytically, we study the effect of the wave packet on particles perturbatively with the wave amplitude being the perturbation parameter. We consider the general case where the wave packet propagates obliquely with respect to the magnetic field. In general, the wave packet characteristics, for example, its phase velocity, group velocity, spatial and temporal extent, and amplitude, are given by linear and nonlinear plasma processes. The linear characteristics are governed by the dispersion relation for prescribed plasma parameters. We will calculate the average change in the momentum and transverse gc position of an ensemble of particles due to their interaction with the wave packet. The particle ensemble is assumed to be a distributed set of initial conditions. We further assume that the Larmor radius of any particle is small compared to the spatial width of the wave packet across $\mathbf{B}$. We do not impose any other restrictions on the form of the wave packet, so that our model applies not only to ordinary wave packets but also to few cycles and sub-cycle wave packets.

III. CANONICAL PERTURBATION THEORY

We rewrite the Hamiltonian in Eq. (5) as

$$H = H_0 + \epsilon H_1 \tag{8}$$

where ϵ is a dimensionless ordering parameter. We assume that the wave packet acts perturbatively on the motion of a particle, so ϵ is used as a perturbation (order-keeping) parameter which, eventually, is set to unity.

The unperturbed particle motion is described by the zero-order Hamiltonian H_0 given in Eq. (3). The canonical momenta, or actions, P_z and P_ϕ are invariants of the motion, so that the corresponding canonical angles z and ϕ, respectively, evolve linearly with time.

The third set of canonically conjugate variables (P_θ, θ), corresponding to the transverse gc coordinates, do not appear in H_0. So they are both constants of the unperturbed motion. The effect of the wave packet on particles is included in the perturbed Hamiltonian H_1 which is a function of all the canonical actions and angles and of time. For an arbitrary wave packet, the complete H is not integrable.

We use the canonical perturbation theory to perturbatively study the effect of the wave packet on the motion of particles interacting with it [9]. The ordering parameter is ϵ, and the general strategy is to construct near-identity canonical transformations T so that, order by order, we can determine the invariants that describe the particle motion. At any order of ϵ, the transformation T leads to a new Hamiltonian K which is a function of the new canonical momenta only. These canonical momenta are the approximate invariants of the motion. The Lie canonical transform formulation results in an explicit form for the generating function. This is in contrast to the mixed-variable generating functions which result in implicit relations between the old and the new canonical variables. The Lie transformations are defined in terms of the operators $T = e^{-L}$ where $Lf = [w, f]$, w is the Lie generating function, and $[\quad,\quad]$ denotes the Poisson bracket.

In the Lie canonical perturbation scheme, the old Hamiltonian H, the new Hamiltonian K, the transformation operator T, and the Lie generator w are each expressed as a power series in ϵ

$$X(\mathbf{z}, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n X_n(\mathbf{z}, t) \tag{9}$$

where X represents any of the variables $\{H, K, T, L, w\}$ [17]. We choose w_0 such that T_0 is the identity transformation I. Then, to second order in ϵ, T and T^{-1} are

$$T = I - \epsilon L_1 + \frac{\epsilon^2}{2}(L_1^2 - L_2) \tag{10}$$

and

$$T^{-1} = I + \epsilon L_1 + \frac{\epsilon^2}{2}(L_1^2 + L_2) \tag{11}$$

The corresponding generating functions $w_{1,2}$ are obtained from

$$\frac{\partial w_1}{\partial t} + [w_1, H_0] = K_1 - H_1 \tag{12}$$

$$\frac{\partial w_2}{\partial t} + [w_2, H_0] = 2K_2 - L_1(K_1 + H_1) \tag{13}$$

The left hand sides of Eqs. (12),(13) are the total time derivatives of w_1 and w_2 along the unperturbed orbits given by H_0. Consequently, they are determined by integrating the right hand sides along these orbits H_0. The new Hamiltonians K_1 and K_2 are arbitrary and can be chosen to be either functions of the new actions or constants. Clearly, the latter is a convenient choice. Thus, in Eq. (12), we set $K_1 = 0$ and solve for w_1

$$w_1 = -\int^t H_1\left(\mathbf{J}, \boldsymbol{\theta}, s\right) ds \tag{14}$$

with the integral being along the unperturbed orbits

$$\mathbf{J} = \text{const.} \tag{15}$$

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \boldsymbol{\omega} t \tag{16}$$

where $\boldsymbol{\omega} = \partial H_0/\partial \mathbf{J} = (P_z/M, \Omega_c, 0)$ is a vector composed of the unperturbed frequencies for the three degrees of freedom. At second order in ϵ, we can also set $K_2 = 0$ in Eq. (13), and, similarly, solve for w_2. However, as we show below, there is no need to have an explicit form for w_2 in our calculations.

IV. AVERAGED ACTION VARIATIONS

The time evolution, from an initial time t_0 to time t, of any well-behaved function of phase space coordinates $f(\mathbf{z})$ is given by $f\left(\mathbf{z}(t;t_0)\right) = S_H(t;t_0) f\left(\mathbf{z}_0\right)$ where $z_0 = z(t_0;t_0)$ and $S_H(t;t_0)$ is the time evolution operator corresponding to H. The derivation of $S_H(t;t_0)$ is equivalent to solving the equations of motion which, in general, is not possible for the non-integrable system in Eq. (8). However, the Lie perturbation theory can be used to determine a transformation to a new set of canonical variables $\mathbf{z}' = (\mathbf{J}', \boldsymbol{\theta}')$ with the corresponding Hamiltonian K having a simpler evolution operator $S_K(t;t_0)$. This is the case when K is chosen to be a function of the new actions $\mathbf{J}'$ only. Then $\mathbf{J}'$ are constants of the motion and $S_K(t;t_0)$ evolves the angles $\boldsymbol{\theta}'$ such that

$$f\left(\mathbf{z}'(t;t_0)\right) = S_K(t;t_0) f\left(\mathbf{z}'_0\right) = f\left((\mathbf{J}'_0, \boldsymbol{\theta}'_0 + \omega_K(\mathbf{J}'_0)(t - t_0)\right) \tag{17}$$

where $\omega_K(\mathbf{J}'_0) = \nabla_{\mathbf{J}'_0} K(\mathbf{J}'_0)$. In other words, the evolution of $f(\mathbf{z})$ can be obtained by transforming to the new canonical variables $\mathbf{z}'$, applying the time evolution operator $S_K(t;t_0)$

to the transformed function, and then transforming back to the original canonical variables $\mathbf{z}$. Then [17],

$$f(\mathbf{z}(t;t_0)) = T(\mathbf{z}_0,t_0)S_K(t;t_0)T^{-1}(\mathbf{z}_0,t_0)f(\mathbf{z}_0) \tag{18}$$

where we have used the property that T commutes with any function of $\mathbf{z}$ [17]. The Lie generators are determined for the finite time interval $[t_0,t]$ using the fact that $w_1(\mathbf{z}_0,t_0)=0$ and $w_2(\mathbf{z}_0,t_0)=0$, so that $T(\mathbf{z}_0,t_0)=I$. The evolution of $f(\mathbf{z})$ in Eq. (18) from $t=t_1$ to $t=t_2$ is

$$f(\mathbf{J},\boldsymbol{\theta})_{t_2} = T^{-1}(\mathbf{J}_{t_1},\boldsymbol{\theta}_{t_1}+\boldsymbol{\omega}_K(\mathbf{J}_{t_1})(t_2-t_1),t_2)f(\mathbf{J},\boldsymbol{\theta})_{t_1} \tag{19}$$

where $f(\mathbf{z})_t = f(\mathbf{z}(t))$.

The three components of the action vector $\mathbf{J}=(P_z,P_\phi,P_\theta)$ represent, respectively, the linear momentum, the angular momentum, and the transverse gc position of the particles. Then, setting $f=P_\ell$ in Eq. (19), where $\ell=z$, ϕ, or θ, we obtain the variations of the actions

$$\delta P_\ell(t_2) \equiv P_\ell(t_2)-P_\ell(t_1) = \left(L_1+\frac{1}{2}L_2+\frac{1}{2}L_1^2\right)P_\ell(t_1) \tag{20}$$

We define the ensemble average of any dynamical variable ζ as

$$\langle\zeta\rangle = \frac{1}{(2\pi)^2L_z}\int_{-\infty}^{\infty}dz\int_0^{2\pi}d\theta\int_0^{2\pi}d\phi\;\;\zeta \tag{21}$$

where the initial conditions of the particles are uniformly distributed in z, θ, and ϕ over the ranges indicated by the limits of the integrals, and L_z is a normalizing length along the z direction which will be determined later. From Eqs. (7), (12), and (13) we find, upon integrating by parts, that

$$\langle L_nP_\ell\rangle = \langle[w_n,P_\ell]\rangle = 0, \qquad\qquad n=1,2 \tag{22}$$

This follows from the fact that $w_{1,2}$ are periodic in ϕ and θ, and vanish as $z\to\pm\infty$. Thus, upon ensemble averaging, only the third term in the right hand side of Eq. (20) is non-zero, so that

$$\langle\delta P_\ell\rangle = \frac{1}{2}\frac{\partial}{\partial P_\ell}\left\langle\left(\frac{\partial w_1}{\partial\ell}\right)^2\right\rangle \tag{23}$$

The averaged variation of the actions of the particles, due to a complete interaction with the wave packet, is obtained from Eq. (23) in the limit $w_1(t_1\to-\infty,t_2\to+\infty)\equiv w_1^\infty$. The averaged variations are accurate to second order in ϵ even though w_1 is accurate to first order in ϵ. This is consistent with Madey's theorem and its generalizations [19].

V. ANALYTICAL RESULTS FOR A GAUSSIAN WAVE PROFILE

We apply the general formalism developed above to study the interaction of particles with a Gaussian wave packet

$$\Phi_0(x,y,z;t) = F e^{-\left(\frac{x^2+y^2}{a_\perp^2}+\frac{z^2}{a_\parallel^2}\right)} e^{-\frac{t^2}{a_t^2}} \tag{24}$$

where $a_\perp$ and $a_\parallel$ are the perpendicular and parallel spatial widths, respectively, a_t is a measure of the temporal duration of the wave packet, and F is its maximum amplitude. Substituting Eq. (24) in Eq. (7), and using Eq. (14) we obtain,

$$w_1 = -qF\tau\frac{\sqrt{\pi}}{2i} e^{-\frac{\tau^2|\mathbf{R}|^2}{a_t^2}} e^{-\tau^2|\mathbf{R}\times\mathbf{T^{-1}}|^2} e^{-i\left(k_\perp R_{gc}\sin\theta - k_\parallel z\right)} \sum_m G_m(t) J_m(k_\perp\rho) e^{-\frac{\tau^2\Omega_m^2}{4}} e^{i\tau^2\Omega_m \mathbf{R}\cdot\mathbf{T^{-1}}} e^{-im\phi} + c.c. \tag{25}$$

where:

$$\mathbf{R} = \left(\frac{R_{gc}\cos\theta}{a_\perp}, \frac{R_{gc}\sin\theta}{a_\perp}, \frac{z}{a_\parallel}\right) \tag{26}$$

is the normalized gc position,

$$\mathbf{T^{-1}} = \left(\frac{V_x}{a_\perp}, \frac{V_y}{a_\perp}, \frac{V_z - P_z/M}{a_\parallel}\right) \tag{27}$$

and $\mathbf{V} = (V_x, V_y, V_z)$ is the group velocity of the wave packet. The components of $\mathbf{T^{-1}}$ correspond to the inverse transit times of the particle through the wave packet along each direction. The transit time vector $\mathbf{T}$ is not to be confused with the Lie operator T. τ, the autocorrelation time of the wave packet as seen by the particles, is given by

$$\tau^{-2} = |\mathbf{T^{-1}}|^2 + a_t^{-2} \tag{28}$$

τ is a measure of the effective interaction time which takes into account both the transit time of the particle through the wave packet and the finite duration of the wave packet. We have also defined

$$\Omega_m = k_\parallel P_z/M - \omega - m\Omega_c \tag{29}$$

and

$$G_m(t) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{t}{\tau} - \frac{\tau}{2}\left(2\mathbf{R}\cdot\mathbf{T^{-1}} + i\Omega_m\right)\right)\right] \tag{30}$$

For $\Omega_m = 0$, Eq. (29) gives the resonance condition between the particle and the fast oscillations within the packet. Equation (30) is the time dependence of the transient particle dynamics during its interaction with the wave packet. Note that $\lim_{t\to+\infty} G_m(t) = 1$.

The first exponential term in Eq. (25) depends on the finite time duration of the wave packet. It approaches unity for wave packets which persist for long times ($a_t \to \infty$). Its effect is increased as a_t decreases, revealing the fact that particles with small $|\mathbf{R}|$ will have a significant interaction with the wave packet during the time that its amplitude is non-zero. The second exponential term in Eq. (25) reflects the dependence of the interaction on the angle between the group velocity of the wave packet and the particle gc position, as it is to be expected for scattering-like interaction. The dependence on the angle is given by a Gaussian with its width being determined by the effective duration of the interaction τ. The third exponential term Eq. (25) depends on Ω_m and signifies the resonant character of the interaction. When $k_\parallel \neq 0$, the effect of the interaction is localized in phase space to regions around $\Omega_m(P_z) = 0$. These are the Doppler-shifted resonances with harmonics of Ω_c. The width of the area in phase space depends on τ. The limit as τ goes to infinity corresponds to an interaction with plane waves having discrete spectra. The exponential terms are then replaced, as expected, by the Dirac delta functions. We must emphasize that the interaction with wave packets of finite spatial and temporal extent properly accounts for the finite transit time interaction and, furthermore, removes singularities in the vicinity of gyro resonances which plague the interaction with plane waves. The first order generating function w_1 includes all the essential information about the interaction of particles with wave packets.

By substituting w_1, as given from Eq. (25), in Eq. (23), we obtain quantitative results for the averaged momentum variations $< \delta P_\ell >$ ($\ell = z$, ϕ or θ) which are accurate to second order in ϵ. Averaging over z according to Eq. (21) involves integrating the first and second exponential terms of Eq. (25) which have a Gaussian dependence on z. This results in the appearance of a scaling factor related to the width in z of this Gaussian which can be chosen as the normalization length L_z in Eq. (21)

$$L_z = \frac{1}{\tau\sqrt{\frac{\partial^2}{\partial z^2}\left(\frac{|\mathbf{R}|^2}{a_t^2} + |\mathbf{R}\times\mathbf{T}^{-1}|^2\right)}} = \frac{a_\parallel}{\sqrt{2}\tau\sqrt{\frac{V_x^2+V_y^2}{a_\perp^2} + \frac{1}{a_t^2}}} \tag{31}$$

This normalization length directly reflects the physical fact that only a finite portion of the particles with initial positions along the z direction actually interact with the localized wave packet due to either its finite time duration ($a_t \neq \infty$) or its nonzero perpendicular group velocity ($V_x, V_y \neq 0$).

Before discussing the results obtained from our perturbation analysis, it is useful to relate our theory to previous studies on nonlinear wave-particle interactions. The case of a plane wave corresponds to a wave packet having infinite time duration (a_t) and spatial width ($a_\perp, a_\parallel$) and has been studied for perpendicular ($k_\parallel = 0$) and for oblique propagation ($k_\parallel \neq 0$) of the electrostatic wave [2]. The cases of perpendicular and oblique propagation correspond to qualitatively different dynamics since, for the former, the resonance condition does not depend on the particle momentum. This corresponds to an intrinsic degeneracy of the Hamiltonian system [9]. When $k_\parallel \neq 0$, the phase space of those particles is strongly affected by the wave for which the resonance condition is fulfilled. For a multiple number of plane waves, forming a wave packet with a discrete spectrum, the analysis is similar to that of one plane wave [3] with the spectral components of the wave packet determining the resonant parts of phase space. For parallel propagation ($k_\perp = 0$) of the wave, the interaction with a particle is along the direction of the magnetic field and independent of the gyration of the particle. The resonance condition is given by $v_z = \omega/k_\parallel$ and the wave strongly affects those particles whose velocities are equal to the phase velocity of the wave along the magnetic field. For this case, particle interaction with a spatially localized wave packet has been studied [13]. Localized wave packets with a compact support, so that $\lim_{|\mathbf{r}|\to\pm\infty} \Phi_0 = 0$, have a continuous spectrum which is centered around the wave vector $\mathbf{k}$ and the resonance condition does not lead to a discrete set of momenta in the dynamical phase space. The collective effects of transit-time interactions on the wave-particle energy and momentum transfer have been studied analytically for a particular set of wave packets that have continuous spectra [14, 15]. These studies are special cases of the more general particle interaction with wave packets described in this paper.

In the following, we investigate the dependence of the averaged variations of particle transverse position (P_θ), parallel momentum (P_z), and the angular momentum (P_ϕ), on parameters describing the wave packet. We normalize the various parameters as follows: time is expressed in units of $1/\Omega_c$, distances in units of $1/|\mathbf{k}|$, speeds in units of $\Omega_c/|\mathbf{k}|$, and the amplitude of the wave potential F in units of $q|\mathbf{k}|^2/(M\Omega_c^2)$. We assume that, initially, all particles have $P_\theta = 1$ so that they can actually be approached by the wave packet. For larger values of P_θ, from the first and the second exponential term in Eq. (25), the strength of the particle interaction with the wave packet is weaker. We are primarily interested in the effect of a spatially localized wave packet so we consider long duration times and set

$a_t = 10^5$. In all cases, for a complete interaction of the particles with the wave packet, the limit $w_1^\infty \equiv \lim_{t\to\infty} w_1$ is applied to Eq. (25) and substituted in Eq. (23). From these equations it is evident that all variations are proportional to the square of $(F\tau)$, which has the dimensions of an action (energy $\times$ time), so that the normalized variations are defined as

$$\Delta P_\ell \equiv \frac{\langle \delta P_\ell \rangle}{(F\tau)^2}, \qquad \ell = z, \phi, \theta \tag{32}$$

Moreover, the amplitude F of the wave packet is normalized with respect to its spatial extent as $F = (\pi^{3/2} a_\perp^2 a_\parallel^1)^{-1}$.

For $k_\perp = 0$, i.e., a wave packet with just a parallel phase velocity, $\Delta P_\phi = 0$. In Figs. 1(a) and 1(b) we plot ΔP_z as a function of P_z for various parameters. The variation of ΔP_z is significant around $P_z = 1$, which corresponds to the resonance condition $\Omega_0 = k_\parallel P_z/M - \omega = 0$. The width in P_z around $P_z = 1$ where the variation is significant, depends on the spatial width of the wave packet along the magnetic field ($a_\parallel$). Comparing Fig. 1(a) and 1(b), we note that a smaller $a_\parallel$ leads to a broader interaction region in P_z. However, as the parallel group velocity V_z is increased the magnitude of the variation in ΔP_z increases. For narrower wave packets, with $a_\parallel$ being small, the profile of ΔP_z is asymmetric with respect to the exact resonance value $P_z = 1$. Narrow wave packets have a few periods of the oscillating wave. The asymmetry with respect to $P_z = 1$ indicates that the interaction between particles with parallel velocities greater than the phase velocity of the wave packet is different from that of particles with parallel velocities smaller than the phase velocity. From Eqs. (28) and (27) we note that for larger $a_\parallel$, the dependence on $V_z - P_z/M$ weakens. So that in Eq. (25), the exponential term $\exp(-\tau^2\Omega_m^2/4)$ depends on P_z through Ω_m only. This gives a symmetric profile around $P_z = 1$. In contrast, for a narrower wave packet the dependence on P_z through $V_z - P_z/M$ in Eq. (28) cannot be ignored. Then the exponential term depends on P_z through both Ω_m and τ leading to an asymmetry around $P_z = 1$.

Figures 2(a) and 2(b) show the variation ΔP_θ as a function of P_z. Since $P_\theta = (M\Omega_c/2)R_{gc}^2$, the ensemble averaged transverse position of the gc, R_{gc}, can be deduced from these figures. The transverse drift of the gc requires that the perpendicular group velocity of the wave packet be non-zero. In comparison with ΔP_z (as shown in Fig. 1), for the same wave packet parameters, ΔP_θ is smaller by at least two orders of magnitude. This is due to the fact that the underlying mechanisms of the two variations are essentially different: The parallel momentum variation, when $k_\parallel$ is nonzero, corresponds to a resonant effect that would take place whether the wave is localized or not, while the gc position variation effect depends on the transverse localization of the wave packet and increases with decreasing $a_\perp$.

The variations ΔP_ϕ, ΔP_z, and ΔP_θ as functions of P_z for a wave packet with $k_\parallel = 0$ are shown in Fig. 3. The resonance condition $\Omega_m = \omega + m\Omega_c = 0$ is independent of P_z. So the variation of ΔP_ϕ, ΔP_z, and ΔP_θ with P_z is through the first and second exponential terms in Eq. (25) taking into account and depending strongly on the finite spatial width and temporal duration of the wave packet. Note that these variations would be independent of P_z for the case of an infinite wave spatial extent, as in the case of a plane wave. While for a wave packet with phase velocity along the magnetic field we have $\Delta P_\phi = 0$, the variation in ΔP_ϕ for a wave packet with perpendicular phase velocity is due to the cyclotron resonance. On the other hand, the variation in ΔP_z is a result of the ponderomotive force, and it would be zero for the case of a plane wave. Figures 3(a), 3(b), and 3(c) correspond to the case where the frequency of the wave packet is not exactly resonant with the cyclotron frequency ($\omega \neq 1$). Comparing these with the results shown in Figs. 3(d), 3(e), and 3(f) for the exactly resonant wave frequency, we note that, apart from some form differences in the variations of ΔP_ϕ and ΔP_θ with P_z, the most significant differences are between the parallel momentum variations ΔP_z (Figs. 3(b) and (e)). As the wave frequency becomes closer to the exact resonant value ΔP_z becomes higher and narrower in P_z due to the exponential dependence on P_z provided by the term $\exp(-\tau^2\Omega_m^2/4)$ in Eq. (25). In this case the mismatch between the wave frequency and the gyrofrequency determines the range of P_z for which the variation ΔP_z is significant. The maximum values of ΔP_z in the resonant case decrease rapidly with increasing perpendicular group velocity and/or parallel spatial width of the wave packet. All three variations in the resonance and off-resonance cases depend on the initial values of P_ϕ (related to the Larmor radius) for the particles, which is the common case whenever $k_\perp \neq 0$.

The interaction of particles with a wave packet having an oblique direction of phase velocity is shown in Fig. 4. The variations in ΔP_ϕ, ΔP_z, and ΔP_θ are localized in the vicinity of the resonances $\Omega_m(P_z) = 0$. The various peaks correspond to different integers m. For broad wave packets with large $a_\parallel$, the resonances in P_z are well separated and well localized as seen in Figs. 4(a), 4(b), and 4(c). Also, the variation of ΔP_ϕ, ΔP_z, and ΔP_θ with P_z is sensitive to the initial value of P_ϕ of the particles. For narrower wave packets, the neighboring resonances can overlap leading to a broader profile. This is evident in Figs.

4(d), 4(e), and 4(f). In all cases, very narrow resonance appears in the vicinity of $P_z = 0$ (shown out of scale in the plots in Fig. 4). Figure 5 shows the variation of ΔP_ϕ, ΔP_z, and ΔP_θ with P_z in a narrow range around $P_z = 0$. The interesting behavior and the large amplitude variations in the narrow range of P_z displayed in Fig. 5, show the strength of the fundamental interaction between particles and wave packets when the velocity of the particles matches the group velocity of the wave packet, i.e., particles are stationary in the frame moving with the group velocity $P_z = MV_z$. Such particles interact with the wave packet for the duration time a_t. The third exponential term in Eq. (25) is maximum when $k_\parallel V_z - \omega - m\Omega_c = 0$ so that these particles feel a wave that has constant amplitude and phase. This type of interaction is important not only due to the large values of parallel momentum variation ΔP_z (two orders of magnitude larger than the other resonances shown in Fig. 4(b)) but also due to its strong localization with respect to particle initial parallel momentum P_z. However, in a given distribution function, the density of such particles is usually small.

VI. SUMMARY AND CONCLUSIONS

We have developed a general formulation for the interaction of charged particles with an electrostatic wave packet in a magnetic field. The magnetic field is assumed to be uniform and stationary and the wave packet propagates at any arbitrary angle to the magnetic field. The Larmor radius of the particles is assumed to be small compared to the spatial dimensions of the wave packet. The change in ensemble averaged transverse guiding center position, parallel momentum, and angular momentum of the particles is determined using Lie transform canonical perturbation theory. The formalism includes resonant and non-resonant wave-particle interactions. The resonant interaction is between harmonics of the cyclotron frequency of the particles and the Doppler-shifted frequency of the rapid oscillations within the wave packet. The non-resonant interaction is due to the ponderomotive force which arises from the finite spatial extent of the wave packet. The general formalism allows for wave packets with a wide range of phase and group velocities as well as spatial widths.

The formalism is applied to a Gaussian wave packet in order to provide closed-form expressions elucidating the physics of the interaction. These expressions include all the essential features of the interaction in terms of the ensemble averaged particle momenta and gc position variations as well as their dependencies on wave packet parameters and particle initial conditions. The effect of the finite spatial and temporal width of the wave packets are taken into account through parameters such as the effective duration of the interaction. The latter corresponds to the autocorrelation time of the wave packet as seen by the particles and determines the width of the resonance in the momentum space. The effect of non-zero group velocity of the wave packet is also included in these expressions taking into account the scattering character of the interaction. Characteristic cases have been considered for the study of particle momentum and spatial transport across the magnetic field showing marked differences with the well known case of particle interaction with plane waves. The respective results are relevant to a variety of plasmas ranging from laboratory fusion plasmas to space and astrophysical plasmas as well as to applications related to accelerators and microwave devices.

VII. ACKNOWLEDGMENTS

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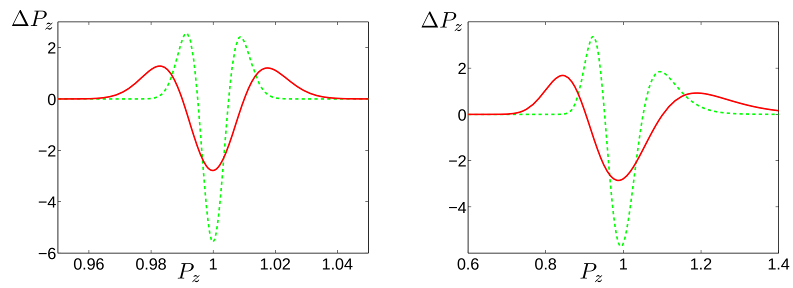
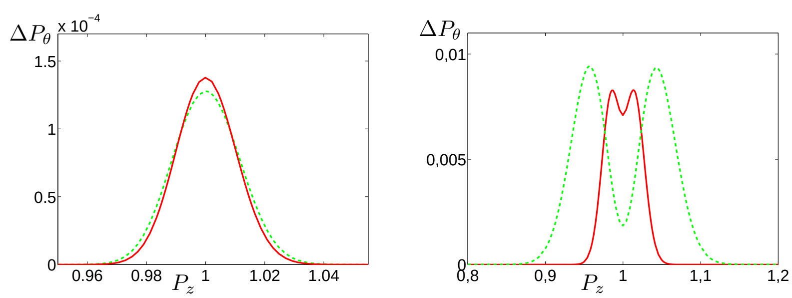


(a) (b)

FIG. 1: (Color online) Parallel momentum variation for the case of a wave packet with phase velocity parallel to the magnetic field ($k_\perp = 0$). The wave packet parameters are: $k_\parallel = 1$, $\omega = 1$, $V_x = V_y = 0$, $V_z = 0$ (red, solid), 0.5 (green, dashed), $a_\perp = 100$, $a_t = 10^5$. (a) $a_\parallel = 100$, (b) $a_\parallel = 10$.



(a) (b)

FIG. 2: (Color online) Guiding center position variation for the case of a wave packet with phase velocity parallel to the magnetic field ($k_\perp = 0$). The wave packet parameters are: $k_\parallel = 1$, $\omega = 1$, $V_y = V_z = 0$, $V_x = 0.1$ (red, solid), 0.3 (green, dashed), $a_\parallel = 100$, $a_t = 10^5$. (a) $a_\perp = 100$, (b) $a_\perp = 10$.

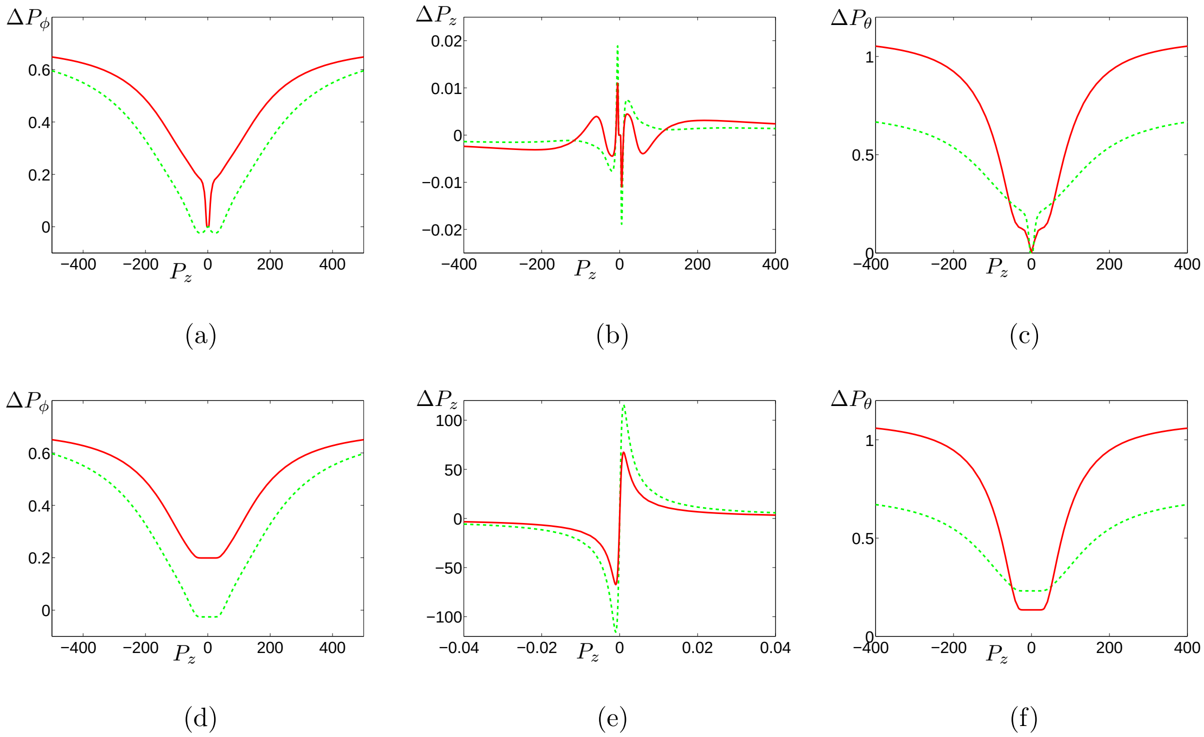


FIG. 3: (Color online) Angular/parallel momentum and guiding center position variations for the case of a wave packet with phase velocity perpendicular to the magnetic field ($k_\parallel = 0$) for particles having $P_\phi = 0.5$ (red, solid) and 2 (green, dashed). The wave packet parameters are: $k_\perp = 1$, $V_x = V_y = V_z = 0$, $a_\parallel = 100$, $a_\perp = 100$, $a_t = 10^5$. (a,b,c) $\omega = 1.1$, (d,e,f) $\omega = 1$.

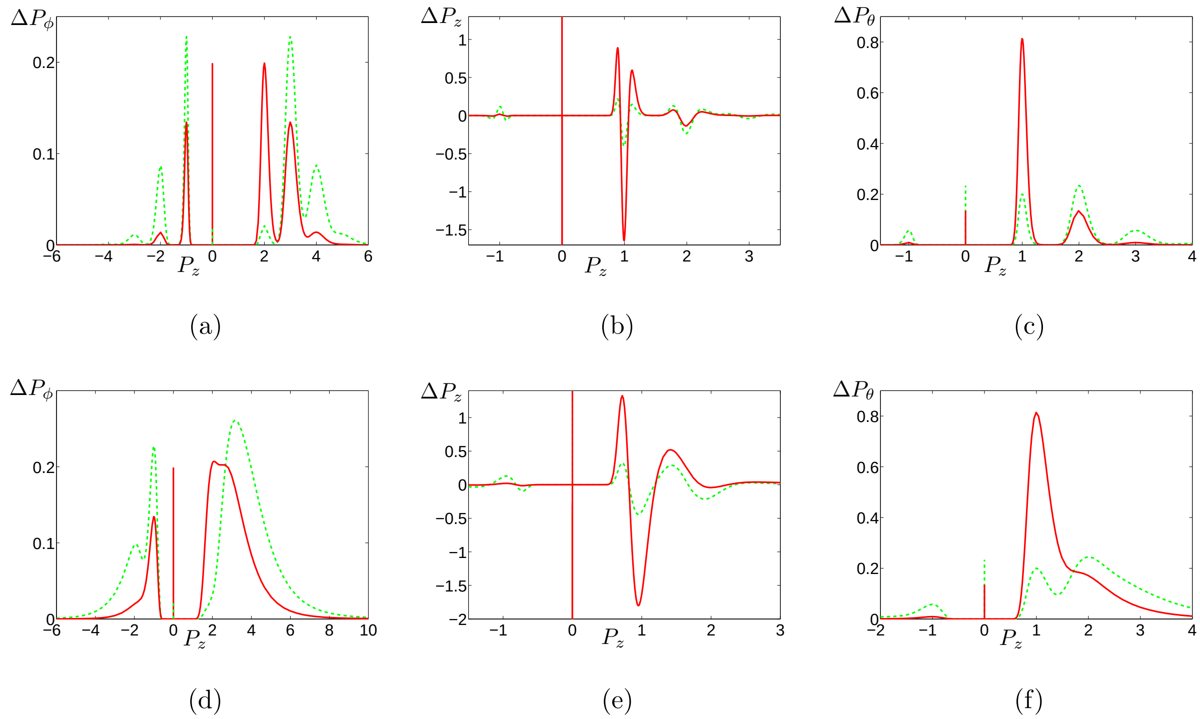


FIG. 4: (Color online) Angular/parallel momentum and guiding center position variations for the case of a wave packet with phase velocity oblique to the magnetic field for particles having $P_\phi = 0.5$ (red, solid) and 1.5 (green, dashed). The wave packet parameters are: $k_\parallel = 1$, $k_\perp = 1$, $\omega = 1$, $V_x = V_y = V_z = 0$, $a_\perp = 100$, $a_t = 10^5$. (a,b,c) $a_\parallel = 15$, (d,e,f) $a_\parallel = 5$.

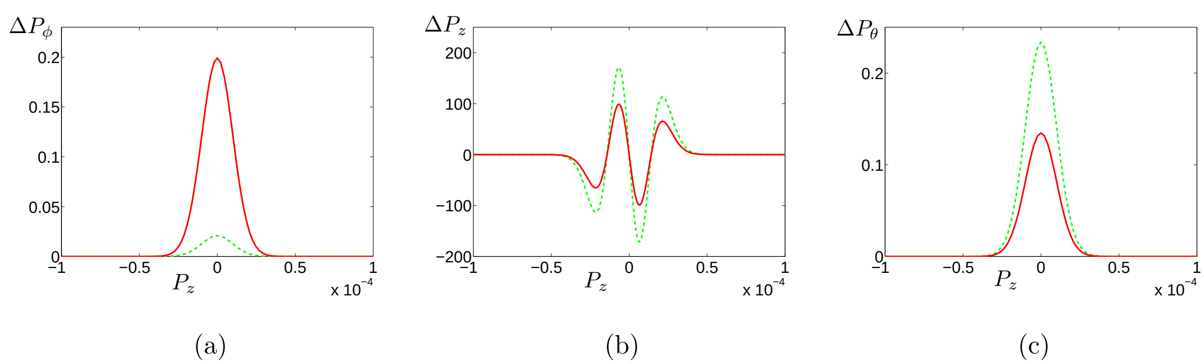


FIG. 5: (Color online) Angular/parallel momentum and guiding center position variations for the same case depicted in Fig. 4(a),(b),(c) in a very narrow area around $P_z = 0$ (shown out of scale in Fig. 4(a)-(c)).