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Percolation of partially interdependent networks under targeted attack

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We study a system composed of two partially interdependent networks, when nodes in one network fail they cause dependent nodes in the other network to also fail. In this paper, percolation of partially interdependent networks under targeted attack is analyzed. We apply a general technique which maps a targeted-attack problem in interdependent networks to a random-attack problem in a transformed pair of interdependent networks. We illustrate our analytical solutions for two examples: (i) the probability for each node to fail is proportional to its degree, and (ii) each node has the same probability to fail in the initial time. We find that: (i) for any targeted-attack problem, for the case of weak coupling, the system shows a second order phase transition, and for the strong coupling, the system shows a first order phase transition, (ii) for any coupling strength, when the high degree nodes have higher probability to fail, the system becomes more vulnerable, and (iii) there exists a critical coupling strength, when the coupling strength is greater than the critical coupling strength, the system shows a first order transition, otherwise, the system shows a second order transition.

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I. INTRODUCTION

Complex networks exist in many different areas in the real world and are studied in the past 15 years. However, almost all researchers have been focused on properties of a single network component that does not interact and depend on other networks [1–11]. Such situations rarely, if ever, occur in reality [12–16]. In 2010, Buldyrev et al. [12] developed a theoretical framework for studying the process of cascading failures in fully interdependent networks caused by random initial failure of nodes. Surprisingly, they found a first order percolation transition and that a broader degree distribution increased the vulnerability of interdependent networks to random failure, in contrast to the behavior of a single network. Recently, five important generalizations of basic model [13–19] are proposed sequentially. (i) Parshani et al. [13] presented a theoretical framework for studying the case of partially interdependent networks. Their findings showed that reducing the coupling strength lead to a change from a first to second order percolation transition. (ii) Because in the real word, a network is attacked not always randomly, Huang et al. [14] investigated the robustness of fully interdependent networks under targeted attack. The result implied that interdependent networks are difficult to defend. (iii) In real scenarios, the assumption that one node in a network depends only on one node in the other network is not valid. Shao et al. [17] investigated a framework to study the percolation of two interdependent networks with multiple support-dependent relations. (iv) Hu et al. [18] studied percolation of a pair of coupled networks with both interdependency links and connectivity links. They found unusual discontinuous changes from second order to first order transition as a function of the dependency coupling between the two networks. (v) In the real word, more than two networks coupled together, Gao et al. [15, 19] proposed a framework to study the robustness of network of networks (NON). Their results showed that for a treelike ER NON the robustness decreases with the number of networks and for a looplike ER NON the giant component is independent on the number of networks. However, for real scenarios, two infrastructures are always partially coupled together [20, 21], such as energy and communications, power stations and transportation etc., and they might be attacked intentionally on high degree nodes. Understanding the robustness due to partially interdependency and under targeted attack is one of the major challenges for designing resilient infrastructures.

Here we develop a generalized framework to study the percolation of partially interdependent networks under targeted attack. We further develop a general technique [14] which maps the targeted-attack problem in interdependent networks to the random-attack problem in a transformed pair of interdependent networks. We find that: (i) for any targeted-attack problem, for the case of weak coupling, the system shows a second order phase transition, and for strong coupling, the system shows a first order phase transition, (ii) for any coupling strength, when the high degree nodes have more probability to fail, the system becomes more vulnerable, and (iii) there exists a critical coupling strength, when the coupling strength is greater than the critical coupling strength, the system shows a first order transition, otherwise, the system shows a second order transition. In the following

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two examples, the critical coupling strength can be explicitly derived analytically: (i) the probability for each node to fail is proportional to its degree, and (ii) each node has the same probability to fail in the initial time. Although case (ii) was solved in [15] we present here a more general case where both interdependent networks are initially attacked randomly.

II. THE MODEL

In this model, two networks A, B with the number of nodes N_A, N_B, and within each network, the nodes are connected with degree distributions $P_A(k)$ and $P_B(k)$ respectively. We suppose that the average degree of the network A is a and the average degree of the network B is b. In addition, a fraction q_A of A nodes depends on the nodes in network B and a fraction q_B of B nodes depends on the nodes in network A. That is, if node A_i of network A depends on node B_j of network B and B_j depends on node A_s of network A, then $s = i$, which satisfies the no-feedback condition [19]. Consequently, when nodes in one network fail, the interdependent nodes in the other network also fail, and we suppose that only the nodes in the giant component remain functional, which leads to further fail in the first network. This dynamic process leads to a cascade of failures. In order to study the cascading failure under targeted attack, we apply the general technique that a targeted-attack problem in networks can be mapped to a random-attack problem [14, 22]. A value $W_\alpha(k_i)$ is assigned to each node, which presents the probability that a node i with k_i links becomes inactive by targeted-attack. We focus on the family of functions [23]:

$$W_\alpha(k_i) = \frac{k_i^\alpha}{\sum_{i=1}^{N} k_i^\alpha}, -\infty < \alpha < +\infty. \tag{1}$$

When $\alpha > 0$, nodes with higher degree are more vulnerable for the targeted attack, while for $\alpha < 0$, nodes with lower degree have higher probability to fail. For $\alpha = 0$, all the nodes in a network have the same probability to fail, which is equivalent to the case of random attack.

Without loss of generality, we begin by studying the generating function and the giant component of network A after targeted attack, which can be directly applied to network B. Next we study the generating functions of network A at each iteration step.

(i) The generating function of network A is defined as

$$G_{A0}(x) = \sum_k P_A(k) x^k. \tag{2}$$

The generating function of the associated branching process, $G_{A1}(x) = G'_{A0}(x)/G'_{A0}(1)$ [12, 13, 24, 25]. The average degree of network A is defined as $a = \bar{k} = \sum_k P_A(k)k$.

(ii) We intentionally remove $1 - p_1$ fraction of nodes from network A according to Eq. (1) and remove the links between the removed nodes. Thus, we obtain that the generating function of the nodes left in network A is [14, 25, 26]

$$G_{Ab}(x) = \sum_k P_A^{p_1}(k) x^k = \frac{1}{p_1} \sum_k P_A(k) h_1^{k^\alpha} x^k, \tag{3}$$

where the new degree distribution of the remaining p_1 fraction of nodes $P_A^{p_1}(k) \equiv \frac{1}{p_1} P_A(k) h_1^{k^\alpha}$, and h_1 satisfies

$$p_1 = G_\alpha(h_1) \equiv \sum_k P_A(k) h_1^{k^\alpha}, h_1 \equiv G_\alpha^{-1}(p_1). \tag{4}$$

The average degree of the remaining nodes in network A in this step is $\bar{k}(p_1) = \sum_k P_A^{p_1}(k)k$.

(iii) We remove the links between the removal nodes and the remaining nodes. Thus we obtain that the generating function of the network composed by the remaining nodes is [26]

$$G_{Ac}(x) = G_{Ab}(1 - \tilde{p}_1 + \tilde{p}_1 x), \tag{5}$$

where $\tilde{p}_1$ is the fraction of the original links that connect to the nodes left, which satisfies

$$\tilde{p}_1 = \frac{p_1 N_A \bar{k}(p_1)}{N_A \bar{k}} = \frac{\sum_k P_A(k) k h_1^{k^\alpha}}{\sum_k P_A(k) k}. \tag{6}$$

Then we can find the equivalent network A' with generating function $\tilde{G}_{A0}(x)$, such that after a fraction $1 - p_1$ of nodes random removed, the new generating function of nodes left in A' is the same as $G_{Ac}(x)$. By solving the equation $\tilde{G}_{A0}(1 - p_1 + p_1 x) = G_{Ac}(x)$, and Eq. (5) , we can get

$$\tilde{G}_{A0}(x) = G_{Ab}(1 - \frac{\tilde{p}_1}{p_1} + \frac{\tilde{p}_1}{p_1} x). \tag{7}$$

And the generating function of the associated branching process, $\tilde{G}_{A1}(x) = \tilde{G}'_{A0}(x)/\tilde{G}'_{A0}(1)$.

(iv) Thus, the targeted-attack problem on network A can be mapped to random-attack problem on network A'. For network A, $1-p_1$ fraction of nodes in network A is intentionally removed according to Eq. (1), the fraction of nodes that belongs to the giant component is [14, 26, 27]:

$$p_A(p_1) = 1 - \tilde{G}_{A0}[1 - p_1(1 - f_A)], \tag{8}$$

where $f_A \equiv f_A(p_1)$ satisfies a transcendental equation

$$f_A = \tilde{G}_{A1}[1 - p_1(1 - f_A)]. \tag{9}$$

For network B, $1-p_2$ fraction of nodes in network B is intentionally removed according to Eq. (1), the fraction of nodes that belongs to the giant component $p_B(p_2)$ is similar to Eq. (8), but change p_1 to p_2 and A to B.

According to the definition of the fraction of nodes that belongs to the giant component, we perform the dynamic of cascading failures as follows: Initially, $1-p_1$ and $1-p_2$ fraction of nodes are intentionally removed from network A and network B respectively. The remaining fraction of network A nodes after an initial removal of $1-p_1$ is $\psi'_1 = p_1$, and the remaining fraction of network B nodes after an initial removal of $1-p_2$ is $\phi'_0 = p_2$. The remaining functional part of network A contains a fraction $\psi_1 = \psi'_1 p_A(\psi'_1)$ of network nodes. Accordingly, for the same reason, the remaining fraction of network B is $\phi'_1 = p_2[1 - q_B(1 - p_A(\psi'_1)p_1)]$, and the fraction of nodes in the giant component of network B is $\phi_1 = \phi'_1 p_B(\phi'_1)$. Then the sequence, ψ_n and ϕ_n, of giant components, and the sequence ψ'_n and ϕ'_n, of the remaining fraction of nodes at each stage of the cascading failures, are constructed as follows:

$$\begin{aligned}
&\psi'_1 = p_1, \psi_1 = \psi'_1 p_A(\psi'_1),\\
&\phi'_0 = p_2, \phi'_1 = p_2[1 - q_B(1 - p_A(\psi'_1)p_1)], \phi_1 = \phi'_1 p_B(\phi'_1),\\
&\psi'_2 = p_1[1 - q_A(1 - p_B(\phi'_1)p_2)], \psi_2 = \psi'_2 p_A(\psi'_2),\\
&\phi'_2 = p_2[1 - q_B(1 - p_A(\psi'_2)p_1)], \phi_2 = \phi'_2 p_B(\phi'_2),\\
&\cdots\\
&\psi'_n = p_1[1 - q_A(1 - p_B(\phi'_{n-1})p_2)], \psi_n = \psi'_n p_A(\psi'_n),\\
&\phi'_n = p_2[1 - q_B(1 - p_A(\psi'_n)p_1)], \phi_n = \phi'_n p_B(\phi'_n).
\end{aligned} \tag{10}$$

Fig. 1 shows the giant component ψ_n and ϕ_n as functions of time step n for different values of $a = b$, p_1, p_2, q_A, q_B and α. The simulation results show excellent agreement with the theory, system (10). Fig. 1(a) shows that a finite giant component exists for $p_2 > p_2^c$, and fig. 1(b) shows the case when $p_2 < p_2^c$, the two networks collapse.

Next, we study the steady state of system (10) after the cascading failures, which can be represented by ψ'_n, ϕ'_n at the limit of $n \to \infty$. The limit must satisfy the equations $\psi'_n = \psi'_{n+1}, \phi'_n = \phi'_{n+1}$ since eventually the clusters stop fragmenting and the fractions of randomly removed nodes at step n and $n+1$ are equal. Denoting $\psi'_n = x$, $\phi'_n = y$, we arrive at a system of two symmetric equations:

$$\begin{aligned}
x &= p_1[1 - q_A(1 - p_B(y)p_2)],\\
y &= p_2[1 - q_B(1 - p_A(x)p_1)].
\end{aligned} \tag{11}$$

III. ANALYTICAL SOLUTION

In this section we present two examples that can be explicitly solved analytically: (i) $\alpha = 1$ and (ii) $\alpha = 0$ for two interdependent Erdös-Rényi (ER) networks. Case (ii) is similar to that of Parshani et al [13] but more general. For the ER [28–30] networks, characterized by the Poisson degree distribution, $G_{A0}(x) = G_{A1}(x) = \exp[a(x-1)]$, $G_{B0}(x) = G_{B1}(x) = \exp[b(x-1)]$.

(i) For the case of $\alpha = 1$, substituting $\alpha = 1$ into Eqs. (3)-(7), we obtain that $G_{Ab}(x)$, $G_{Ac}(x)$ and $\tilde{G}_{A0}(x)$ can be represented by $G_{A0}(x)$, which reflects the mapping from a targeted-attack problem to random-attack problem. Then we get $\tilde{G}_{A0}(x) = \tilde{G}_{A1}(x) = \exp[ah_1^2(x-1)]$, $\tilde{G}_{B0}(y) = \tilde{G}_{B1}(y) = \exp[bh_2^2(y-1)]$. Thus, from Eq. (9) we obtain

$$f_A = \exp[-ah_1^2 x(1 - f_A)], f_B = \exp[-bh_2^2 y(1 - f_B)]. \tag{12}$$

Substituting Eqs. (8), (9), (11) into Eqs. (12), by eliminating x and y, we obtain

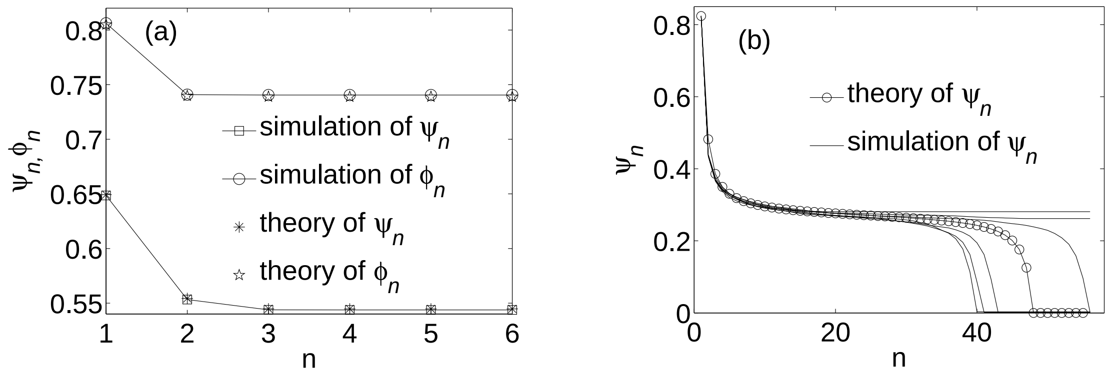

FIG. 1: (a) The giant component of both networks A and B, ψ_n and ϕ_n, after time n cascading failures for the case when $a = b = 3$, $p_1 = 0.8$, $p_2 = 0.9 > p_2^c$, $q_A = 0.45$, $\alpha = 1$ and $q_B = 0.15$. The simulation results show excellent agreement with the theory, system (10). All estimates are the results of averaging over 40 realizations. (b) The giant component of network A, ψ_n, after time n cascading failures for the case when $a = b = 3$, $p_1 = 0.9$, $q_A = 0.65$, $q_B = 0.7$, $\alpha = 0$, and $p_2 = 0.6726 < p_2^c = 0.673$. The simulation results show excellent agreement with the theory, system (10). In both (a) and (b), $N_A = N_B = 2 \times 10^5$.

$$
\begin{aligned}
f_A &= e^{-ap_1h_1^2(1-f_A)[1-q_A+p_2q_A(1-f_B)]},\\
f_B &= e^{-bp_2h_2^2(1-f_B)[1-q_B+p_1q_B(1-f_A)]}.
\end{aligned}
\tag{13}
$$

According to the definition of $\psi_\infty = p_A(x)x$ and $\phi_\infty = p_B(y)y$, we obtain the giant component of networks A and B at the end of the cascading failure respectively as

$$
\begin{aligned}
\psi_\infty =& p_1(1-f_A)[1-q_A+p_2q_A(1-f_B)],\\
\phi_\infty =& p_2(1-f_B)[1-q_B+p_1q_B(1-f_A)].
\end{aligned}
\tag{14}
$$

Solving the Eqs. (13), we obtain f_A and f_B, and then we obtain ψ_∞ and ϕ_∞ by substituting f_A and f_B into Eqs. (14).

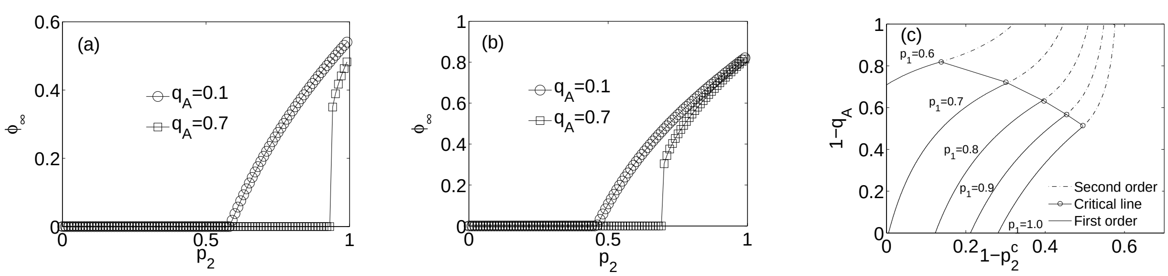

FIG. 2: (a) The giant component ϕ_∞ of network B as a function of initial attack on network B, $1-p_2$, when $p_1 = 0.7$, $a = 3$, $b = 4$, $q_B = 0.7$ and $\alpha = 1$ for two different q_A. (b) The giant component ϕ_∞ of network B as a function of initial attack on network B, $1-p_2$, when $p_1 = 0.9$, $a = 3$, $b = 4$, $q_B = 0.7$ and $\alpha = 1$ for two different q_A. For the weak coupling strength ($q_A = 0.1$), the system shows a second order phase transition, and for the strong coupling strength ($q_A = 0.7$), the system shows a first order phase transition. From (a) and (b), we find that the changes of the critical threshold depends on the changes of p_1. (c) The coupling strength $1-q_A$ as a function of $1-p_2^c$, for different values of remaining fraction of nodes after initial attack on network A, p_1, when $a = 3$, $b = 4$, $q_B = 0.7$. For each p_1, $1-q_A$ as a function of $1-p_2^c$ is divided into two region by a symbol $\circ$. The dash-dot curve above a symbol $\circ$ represents the second order phase transition and the solid curve below the symbol $\circ$ represents the first order phase transition. All the circles are connected to form a critical line, which represents the $1-q_A^c$ as a function of $1-p_2^c$. It also shows that q_A^c increases as p_1 increases.

Numerical simulation results of system (14) are shown in Fig. 2. As shown in Fig. 2, for fixed a, b, q_B, there exists a critical p_2^c, when $p_2 < p_2^c$, $\phi_\infty = 0$, when $p_2 > p_2^c$, $\phi_\infty > 0$. For the weak coupling case, i.e., when q_A is small ($q_A = 0.1$ in Fig. 2), $\phi_\infty(p_2^c) = 0$, which shows a second order phase transition, and the transition threshold is defined as p^{II}. For strong coupling, i.e., when q_A is large ($q_A = 0.7$ in Fig. 2), $\phi_\infty(p_2^c) > 0$, which represents a first order percolation phase transition, and the transition threshold is defined as p^I. Fig. 2(a) and (b) indicate that there exists a critical q_A^c, which corresponds to the condition when $p^I = p^{II}$, when $q_A < q_A^c$, the system shows a second order phase transition, and when $q_A > q_A^c$, the system shows a first order phase transition. Furthermore, Fig. 2(a) and (b) indicate that the critical threshold changes with p_1, i.e., q_A^c also changes with p_1. In Fig. 2(c), we studied by numerical simulation, $1-q_A$ as a function of $1-p_2^c$, for different values of p_1 when $a = 3$, $b = 4$, $q_B = 0.7$. As shown in Fig. 2(c), for each p_1, there exists a critical q_A^c ($\circ$), which corresponds to the condition $p^I = p^{II}$. Moreover, q_A^c increases as p_1 increases, which is represented by the circle curve in Fig. 2(c), and indicates that the two networks becomes more robust as q_A decreases.

Next, we study the transition threshold p^I and p^{II} analytically when $a = b = \bar{k}$, $p_1 = p_2 = p$, $q_A = q_B = q$. In this case, from Eqs. (13) and (14), we obtain that the giant components of networks A and B at the end of the cascading failure $\psi_\infty = \phi_\infty$ satisfies

$$\phi_\infty = p(1 - e^{-\bar{k}h^2\phi_\infty})[1 - q + pq(1 - e^{-\bar{k}h^2\phi_\infty})], \tag{15}$$

and $f \equiv f_A = f_B$ satisfies

$$f = e^{-\bar{k}ph^2(1-f)[1-q+pq(1-f)]}, \tag{16}$$

where $h = \ln p/\bar{k} + 1$. The condition for the first order transition ($p = p^I$) is

$$1 = f[\bar{k}p^I h^2(1-q) + 2\bar{k}(p^I)^2 qh^2(1-f)], 0 \le f < 1. \tag{17}$$

And solving Eq. (16) for $f \to 1$ yields the condition for the second order transition ($p = p^{II}$),

$$\bar{k}p^{II}(1-q)h^2 = 1. \tag{18}$$

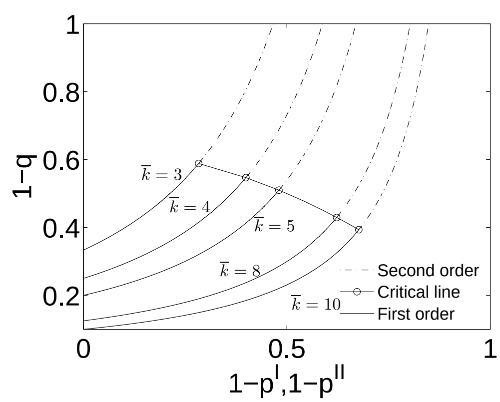

FIG. 3: The coupling strength $1-q = 1-q_A = 1-q_B$ as a function of the first order and second order phase transition threshold, $1-p^I, p^{II}$, for different values of average degree $\bar{k} = a = b$, when $\alpha = 1$. The circles curve shows the critical line, below which the system shows a first order phase transition, and above which the system shows a second order phase transition. The simulation of the critical line agrees well with the theory Eq. (20).

The analysis of Eqs. (17) and (18) shows the first order transition at $p = p^I$ occurs for networks with strong coupling ($q > q_c$), whereas the second order transition at $p = p^{II}$ occurs for networks with weak coupling ($q < q_c$). This behavior is shown in Fig. 3, where the solid curves show the case of first order phase transition and the dash-dot curves show the case of second order phase transition. The critical value of q_c (and p_c) for which the phase transition changes from first order to second order is obtained when the conditions for both the first and second order transitions are satisfied simultaneously. Applying both conditions, Eqs. (17) and (18), we obtain

$$\bar{k}[1 + \ln(\frac{1-q_c}{2q_c})/\bar{k}]^2 = \frac{2q_c}{(1-q_c)^2}. \tag{19}$$

Solving Eq. (19), we obtain q_c, and then we can get p_c by

$$p_c = \frac{1 - q_c}{2q_c}. \tag{20}$$

(ii) For the case when $\alpha = 0$, $W_0 = 1/N$, thus the targeted-attack problem is equivalent to the random-attack problem studied in Ref. [13]. For the case of two Erdös-Rényi (ER) [28–30] networks with average degrees a and b, we can easily get $p_A(x) = 1 - f_A$, $p_B(y) = 1 - f_B$ from the Eqs. (8) and (9), and system (11) becomes

$$\begin{aligned} x &= p_1[1 - q_A + p_2 q_A(1 - f_B)], \\ y &= p_2[1 - q_B + p_1 q_B(1 - f_A)]. \end{aligned} \tag{21}$$

According to Eqs. (9), (21), f_A, f_B satisfy

$$\begin{aligned} f_A &= e^{-a p_1 (1-f_A)[1-q_A+p_2 q_A(1-f_B)]}, \\ f_B &= e^{-b p_2 (1-f_B)[1-q_B+p_1 q_B(1-f_A)]}. \end{aligned} \tag{22}$$

Thus, we obtain the fraction of nodes in the giant components of networks A and B at the end of the cascading process

$$\begin{aligned} \psi_\infty &= p_1(1 - f_A)[1 - q_A + p_2 q_A(1 - f_B)], \\ \phi_\infty &= p_2(1 - f_B)[1 - q_B + p_1 q_B(1 - f_A)]. \end{aligned} \tag{23}$$

Our framework is equivalent to Ref. [13] when $p_2 = 1$. In detail, when $p_2 = 1$, Eqs. (22) are the same as Eqs. (7) in Ref. [13]. Here we study the more general case where $p_2 < 1$.

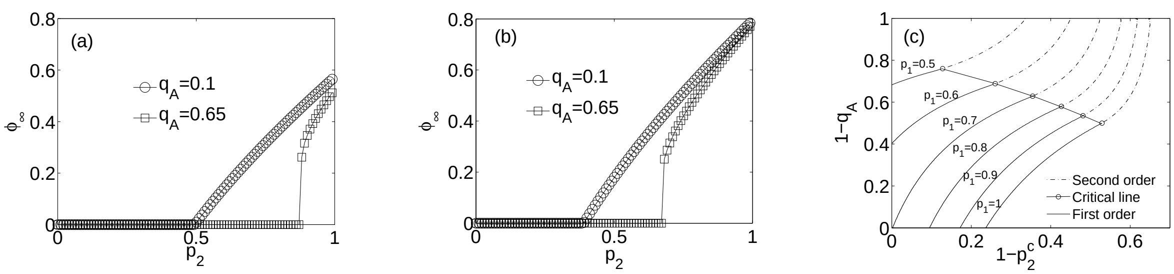

FIG. 4: (a) The giant component ϕ_∞ of network B as a function of initial attack on network B, $1 - p_2$, when $p_1 = 0.7$, $a = b = 3$, $q_B = 0.7$ and $\alpha = 0$ for two different q_A. (b) The giant component ϕ_∞ of network B as a function of initial attack on network B, $1 - p_2$, when $p_1 = 0.9$, $a = b = 3$, $q_B = 0.7$ and $\alpha = 0$ for two different q_A. For the weak coupling strength ($q_A = 0.1$), the system shows a second order phase transition, and for the strong coupling strength ($q_A = 0.65$), the system shows a first order phase transition. (c) The coupling strength $1 - q_A$ as a function of $1 - p_2^c$, for different values of remaining fraction of nodes after initial attack on network A, p_1, when $a = b = 3$, $q_B = 0.7$. For each p_1, $1 - q_A$ as a function of $1 - p_2^c$ is divided into two region by a symbol $\circ$. The dash-dot curve above a symbol $\circ$ represents the second order phase transition and the solid curve below the symbol $\circ$ represents the first order phase transition. All the circles are connected to form a critical line, which represents $1 - q_A^c$ as a function of $1 - p_2^c$. It also shows that q_A^c increases as p_1 increases.

For the case $\alpha = 0$, numerical simulation results of system (23) are shown in Fig. 4. For given $a = b$, q_B, p_1, there exists a critical p_2^c, when $p_2 < p_2^c$, $\phi_\infty = 0$, when $p_2 > p_2^c$, $\phi_\infty > 0$. For weak coupling, i.e., when q_A is small ($q_A = 0.1$ in Fig. 4), $\phi_\infty(p_2^c) = 0$, representing a second order phase transition, and the percolation threshold is defined as p^{II}. For strong coupling, i.e., when q_A is large (e.g., $q_A = 0.65$ in Fig. 4), $\phi_\infty(p_2^c) > 0$, representing a first order phase transition, and the percolation threshold is defined as p^I. Similar to the case when $\alpha = 1$, Fig. 4 indicates that there exists a critical q_A^c, which corresponds to the condition when $p^I = p^{II}$. When $q_A < q_A^c$, the system shows a second order phase transition, and when $q_A > q_A^c$, the system shows a first order phase transition. Furthermore, Fig. 4 indicates that the critical threshold changes with p_1, i.e., q_A^c also changes with p_1. In Fig. 4(c), we investigate using numerical calculations, $1 - q_A$ as a function of $1 - p_2^c$, for different values of p_1 when $a = b = 3$, $q_B = 0.7$. As shown in Fig. 4(c), for each p_1, there exists a critical q_A^c ($\circ$), which corresponds to the condition $p^I = p^{II}$. Moreover, q_A^c increases as p_1 increases, which is represented by the curve with circles in Fig. 4(c), and indicates that the two networks become more robust as q_A decreases.

By substituting $a = b = \bar{k}$, $p_1 = p_2 = p$, $q_A = q_B = q$ into Eqs. (22) and (23), we obtain that the giant component of networks A and B at the end of the cascading failure $\psi_\infty = \phi_\infty$ satisfies

$$\phi_\infty = p(1 - e^{-\bar{k}\phi_\infty})[1 - q + pq(1 - e^{-\bar{k}\phi_\infty})], \tag{24}$$

and $f \equiv f_A = f_B$ satisfies

$$f = e^{\bar{k}p(f-1)[1-q+pq(1-f)]},\ 0 \leq f < 1. \tag{25}$$

Thus we obtain the condition for the first order transition $(p = p^I)$

$$1 = f[\bar{k}p^I(1-q) + 2\bar{k}(p^I)^2 q(1-f)]. \tag{26}$$

Solving Eq. (25) for $f \to 1$ yields the condition for the second order transition $(p = p^{II})$,

$$\bar{k}p^{II}(1-q) = 1. \tag{27}$$

Similar to the case of $\alpha = 1$, the analysis of Eqs. (26) and (27) shows that the first order transition at $p = p^I$ occurs for networks with strong coupling ($q > q_c$), whereas the second order transition at $p = p^{II}$ occurs for networks with weak coupling ($q < q_c$). This behavior is shown in Fig. 5, where the solid curves show the case of first order phase transition and the dashed-dotted curves show the case of second order phase transition. The critical values of q_c (and p_c) for which the phase transition changes from first order to second order is obtained when the conditions for both the first and second order transitions are satisfied simultaneously. Applying both conditions Eqs. (26) and (27), we obtain

$$\begin{aligned} p_c &= \frac{\bar{k} + 1 - \sqrt{2\bar{k}+1}}{\bar{k}}, \\ q_c &= \frac{\sqrt{2\bar{k}+1}+1}{2\bar{k}}. \end{aligned} \tag{28}$$

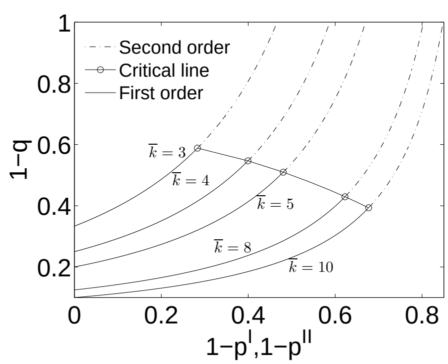

FIG. 5: The coupling strength $1-q = 1-q_A = 1-q_B$ as a function of the first order and second order phase transition threshold, $1-p^I, 1-p^{II}$, for different values of average degree $\bar{k} = a = b$, when $\alpha = 0$. The curve of circles shows the critical line, below which the system shows a first order phase transition, and above which the system shows a second order phase transition. The simulation of the critical line agrees well with the theory Eq. (28).

IV. NUMERICAL SOLUTIONS OF THE GENERAL CASE

Our theoretical study can be applied to any case of α. In this section, we investigate the solutions for the general cases of α. Fig. 6 shows the giant component ϕ_∞ of network B as a function of initial attack on network B, $1-p_2$ for $\alpha = 2$ [Fig. 6(a)] and $\alpha = -1$ [Fig. 6(b)]. For given a, b, q_B, p_1, there exists a critical p_2^c, when $p_2 < p_2^c$, $\phi_\infty = 0$, when $p_2 > p_2^c$, $\phi_\infty > 0$. For weak coupling, i.e., when q_A is small ($q_A = 0.1$ in Fig. 6), $\phi_\infty(p_2^c) = 0$, which shows a second order phase transition. For strong coupling, i.e., when q_A is large ($q_A = 0.8$ in Fig. 6), $\phi_\infty(p_2^c) > 0$, which shows a first order phase transition. Fig. 6 indicates that there exists a critical q_A^c, when $q_A < q_A^c$, it shows a second order phase transition, and when $q_A > q_A^c$, the system shows a first order phase transition. Furthermore, Fig. 6 indicates that the critical threshold changes with α, i.e., q_A^c also changes with α.

In Fig. 7, we investigate the numerical simulation of $1-q_A$ as a function of $1-p_2^c$, for different values of α when $a = 3$, $b = 4$, $p_1 = 0.8$, $q_B = 0.7$. As shown in Fig. 7, for each α, there exists a critical q_A^c ($\circ$). Moreover, $1-q_A^c$ increases as α increases, which is represented by the circle curve in Fig. 7, and indicates that the two networks becomes more robust as α decreases.

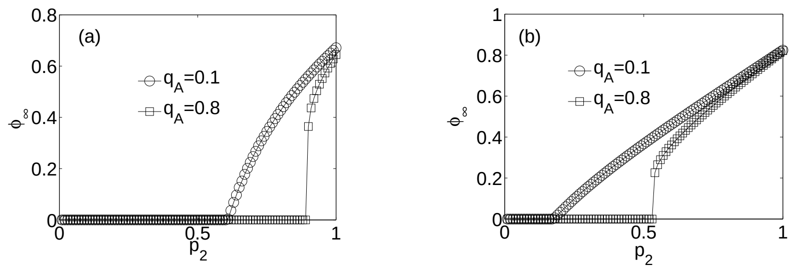

FIG. 6: (a) The giant component ϕ_∞ of network B as a function of initial attack on network B, $1-p_2$, when $p_1 = 0.8$, $a = 3$, $b = 4$, $q_B = 0.7$ and $\alpha = 2$ for two different q_A. (b) The giant component ϕ_∞ of network B as a function of initial attack on network B, $1-p_2$, when $p_1 = 0.8$, $a = 3$, $b = 4$, $q_B = 0.7$ and $\alpha = -1$ for two different q_A. For weak coupling strength ($q_A = 0.1$), the system shows a second order phase transition, and for strong coupling strength ($q_A = 0.8$), the system shows a first order phase transition.

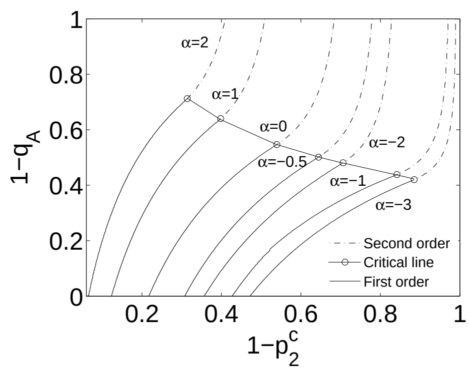

FIG. 7: The coupling strength $1-q_A$ as a function of $1-p_2^c$, for different values of α, when $a = 3$, $b = 4$, $q_B = 0.7$ and $p_1 = 0.8$. The circles curve shows the critical line, below which the system shows a first order phase transition, and above which the system shows a second order phase transition.

V. CONCLUSIONS

In summary, we developed a framework for studying percolation of two partially interdependent ER networks under targeted attack for the cases of high degrees attack $\alpha = 1$ and random attack, $\alpha = 0$. For any value of α, the system shows a second order phase transition when q is small, and a first order phase transition when q is large. We find the critical q_c and critical threshold p_c, when the percolation of the system changes from first to second order, for the case when $\alpha = 1$ and $\alpha = 0$. Moreover, we find that when α increases, i.e. the high degree nodes have more probability to fail, the system becomes more vulnerable.

VI. ACKNOWLEDGMENTS

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