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We consider turbulence within the Gross-Pitaevsky model and look into the creation of a coherent condensate via an inverse cascade originated at small scales. The growth of the condensate leads to a spontaneous breakdown of statistical symmetries of over-condensate fluctuations: first, isotropy is broken, then series of phase transitions mark changing symmetry from two-fold to three-fold to four-fold. We describe respective anisotropic flux flows in the **k**-space. At the highest level reached, we observe a short-range positional and long-range orientational order (like in a hexatic phase). In other words, the more one pumps the system the more ordered the system becomes. The phase transitions happen when the system is pumped by an instability term and does not occur when pumped by a random force. We thus demonstrate for the first time non-universality of an inverse-cascade turbulence with respect to the nature of small-scale forcing.

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Probably the most unexpected discovery made in studying turbulence is an inverse cascade [1]. Flying at the face of an intuitive picture of turbulence as the process of fragmentation, inverse cascade is self-organization, i.e. appearance of large-scale motions out of a small-scale noise. Inverse cascade culminates in the creation of spectral condensate, a mode spatially coherent across the whole system. Understanding the interaction of turbulence with a coherent flow is a central problem in turbulence studies in fluid mechanics and beyond, both from fundamental and practical perspectives. In fluids, condensates are system-size vortices or zonal flows [2–4]. Turbulence with the condensate shares many properties with quantum systems as displaying both fluctuations and coherence. This closeness shows perhaps most vividly within the framework of Nonlinear Schrödinger (Gross-Pitaevsky) Equation, called here NSE:

$$i\psi_t = -\Delta\psi + |\psi|^2\psi \ . \qquad (1)$$

In classical physics, NSE describes spectrally narrow distribution of nonlinear waves; respective non-equilibrium states are called optical turbulence [1, 5]. NSE conserves the wave action $N = \overline{|\psi|^2}$ and the Hamiltonian $\mathcal{H} = \overline{|\nabla\psi|^2 + |\psi|^4/2}$, overline being the space average. For weak nonlinearity, when $\mathcal{H} \approx \overline{|\nabla\psi|^2}$ i.e. quadratic in ψ like N, one can rigorously show that pumping at some scale produces two cascades, a direct one of $\mathcal{H}$ towards smaller scales and an inverse one of N towards larger scales [1]. Generally, one may question the very existence of the inverse cascade [6]. Here we show that the inverse cascade of N exists even for $\overline{|\nabla\psi|^2} \ll \overline{|\psi|^4/2}$ and analyze the ways N flows from pumping towards the condensate in **k**-space. The condensate solution of (1) is just a constant, $\psi = \sqrt{N_0}\exp(-iN_0t)$, while turbulence can consist of weakly interacting waves. This simple system demonstrates unexpectedly rich behavior with features never before observed in turbulence.

Small over-condensate fluctuations follow the Bogolyubov dispersion relation: $\Omega_k^2 = 2N_0k^2 + k^4$ [7]. As $N_0 = |\overline{\psi}|^2$ grows, the dispersion relation approaches the acoustic one with the sound speed $c = \sqrt{2N_0}$. Acoustic waves running at the same direction interact strongly, producing shocks. On the other hand, the matrix element of the three-wave interaction decreases as $N_0^{-1/2}$ (see e.g. [5]). The nonlinearity parameter for over-condensate fluctutions can be estimated as $(N - N_0)/N_0$, using the weak-turbulence approximation [1, 8]. That suggests that when the number of waves with zero momentum outgrows that with nonzero momentum, nonlinear interaction must be effectively weak despite the fact that the nonlinear term in (1) is dominant. That explains the observation made in [9] and confirmed here for much higher N_0 that the statistics of the fluctuations approaches Gaussian, as N_0 grows. Still, the nature of even weakly nonlinear turbulence coexisting with a strong condensate is very much a mystery. This work is intended to elucidate the most salient features of such turbulence: non-universality with respect to excitation mechanism and statistical symmetries.

To simulate turbulence, we add pumping and dissipation. Waves in fluids and plasma are usually excited by instability, while in quantum physics and optics by an external force. Therefore, we treat two types of excitation, adding to the right-hand side of the Fourier-transformed NSE either i) instability term $\gamma_k\psi_k$ or ii) random complex force $F_{\mathbf{k}}(t)$ with a constant amplitude and phases uncorrelated both in time and **k**-space. The (real) growth rate γ_k and $|F_{\mathbf{k}}(t)|$ are positive within the narrow shell at the middle of the spectral domain, $k_1 < k < k_2$, negative at $k > k_2$, and zero at $k < k_1$. To study steady state, we interrupt the simulation and restart it with additional friction, replacing γ_k by $\tilde{\gamma}_k = \gamma_k - \alpha$ (we choose α empirically to stabilize N_0 at a desirable level). The results presented below are obtained in $8\pi \times 8\pi$ domain at spectral resolution 512×512 points (excluding dealiased modes), more details can be found in [8]. Numerics run long past the time when all vortices disappear, so we are left only with a large condensate and small-amplitude waves running

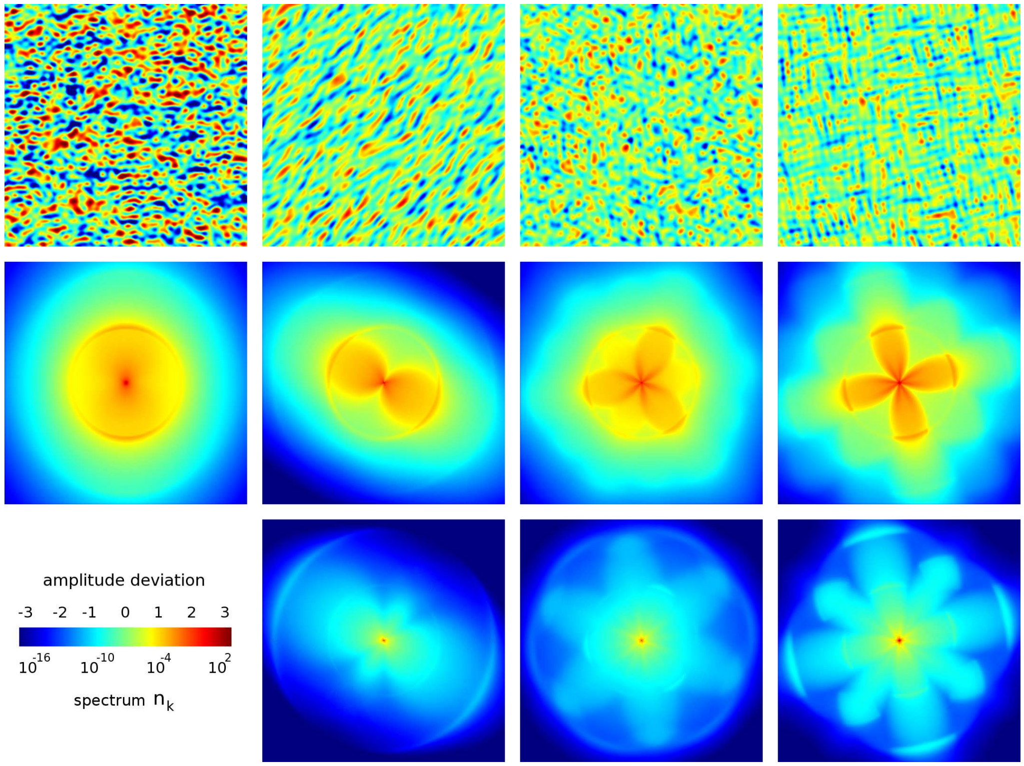

FIG. 1: Color. Top row: amplitude deviation from the mean in the quarter of the domain; middle row: average spectrum; bottom row: coherent mode spectrum. Columns, from left to right: $\epsilon = 2.40$, 8.45, 12.77, 46.08; $\overline{|\psi|} = 19.9$, 30.9, 37.4, 67.5.

over it. The pumping wavelength $\lambda = 4\pi/(k_1 + k_2)$ is effectively the smallest scale in the system since dissipation dominates at smaller scales, and we use $\epsilon = N\lambda^2/4$ as a dimensionless measure of nonlinearity. All simulations are done with $\lambda \approx 2\pi/30$, so $\epsilon \approx 0.01N$.

At the beginning, waves in the pumping shell are pumped isotropically. Let us now discuss what possible anisotropy may turbulence spectra acquire after the condensate appears. Standing lowest modes of the box provide spatial modulation of the condensate intensity N_0 and respective modulation of the sound speed c for shorter waves, according to the Bogolyubov dispersion relation. The regions of low N_0 (and c) must act as waveguides with short waves moving predominantly along the minima of N_0. In other words, angular maxima of $n_{\mathbf{k}} = \langle|\psi_{\mathbf{k}}|^2\rangle$ at high k would correspond to minima at low k. (Angular brackes denote time averaging.) That suggests a simple picture of possible anisotropy: lowest modes will have the symmetry of the square box and impose that symmetry on the small-scale turbulence (turned by $\pi/4$). The main result of this work is that this is not the case as a result of nonlinear interaction unaccounted in that simple picture).

We focus first on the instability-driven system. It undergoes a series of phase transitions between states of different symmetry. Initially, the condensate is fed by an inverse cascade carried by an almost isotropic spectrum of fluctuations. At this stage, the nonlinear (interaction) term in the Hamiltonian is less or comparable to the quadratic term (that describes diffraction and dispersion for waves or kinetic energy for particles). As condensate grows, the first symmetry breaking appears at $\epsilon = \epsilon_1 \simeq 1$ when the system becomes anisotropic and the spectrum turns into an oval. Further transformation happens gradually, in the interval $2 \lesssim \epsilon \lesssim 5$, as the oval becomes thinner at the waist, turning into a dumbbell. As N_0 grows further, the symmetry changes from two-fold to three-fold, and then to four-fold, as seen in Figure 1. Comparing slow-evolving runs with no friction with those stabilized at different values of N_0, we find the same symmetry states (which means that the evolution is slow enough so that the system is close to a steady state an any moment). The phase transitions thus happen between phases which are steady states. To avoid slow evolution from initial thermal noise to higher levels of condensate, we made separate runs taking initial conditions for ψ as a thermal noise plus a large constant (pre-set condensate). In these cases, the same states (with 2,3,4 petals) appear i.e. they are true attractors, independent of the way one reaches given N_0. Further transitions are possible: we observe 6-petal spectra with triangular spatial pattern in the amplitude, and transient 8-petal spectrum, with patches of square patterns oriented at 45 degree angle, see [8] for more details.

In evolving systems, the transitions occur sharply yet the threshold values of ϵ fluctuate from run to run, being

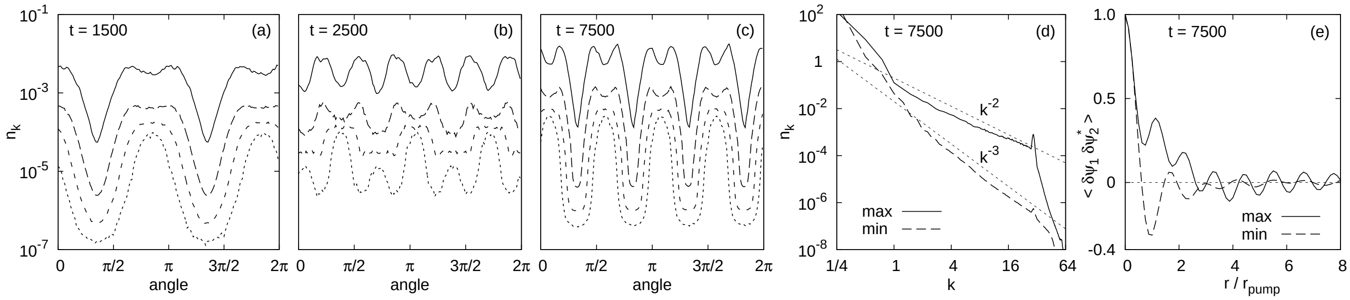

FIG. 2: Angular dependence of spectra at $|k| = 8, 24, 48, 64$ (top to bottom), directional spectra, and correlation function of the amplitude.

sensitive to the phases of initial noise. This shows that the phenomenon is not caused by a linear instability with a well-defined threshold. Indeed, the condensate is stable with respect to the infinitesimal perturbations. It is likely that our transitions are of probabilistic nature similar to transitions in a pipe flow [10] or fiber laser [11]: with the change of the control parameter (condensate amplitude in our case) one changes the probability that a finite-amplitude perturbation will lead the system away to a new state.

One may suggest a possible physical mechanism of the transitions as follows: acoustic waves effectively interact only within the angle $k/\sqrt{N_0}$, which decreases as N_0 grows, so the turbulence tends to be broken into jets; the number of jets j increases as an inverse of the interaction angle i.e. as $N_0^{1/2} \propto \epsilon^{1/2}$. We observe that the transitions happen approximately at $\epsilon_{2\to3} \approx 10$, $\epsilon_{3\to4} \approx 43$ which may roughly correspond to $\epsilon_j \sim j^2$. Even if this is indeed the basic mechanism of the transitions, we still lack any understanding of whether there is a unifying principle that can predict which turbulent state is realized the way variational principle does it for thermal equilibrium. One direction worth exploring is whether one can develop an approach similar to the weak crystallization theory [12] despite the fact the we have developed turbulence with power-law spectra carrying flux in **k**-space, as seen in Figs 2, 3.

With ϵ growth, two-petal spectra concentrate in a more narrow angle and broaden again just before the transition. The three-petal spectra broaden with ϵ growth, especially at high k. For ϵ just below the transition value, the spectra are almost uniform, with seeds of new symmetry visible at low k in the coherent part of averaged spectra. The population of the pumping shell increases between the transitions and falls during the transitions [8], which leads to a complicated evolution of $N(t)$, different from the simple laws of condensate growth suggested before, from $N(t) \propto \sqrt{t}$ and $N(t) \propto t$ [9] to $N \propto t^2$ [13].

For all states (except possibly very close to transition events), and for all symmetries, the angular width of the spectra decreases towards larger wavenumbers, as seen in Figures 1, 2. This may be analogous to spectrum anisotropization observed for acoustic turbulence [1, 14]. It is worth stressing though that in the above references the direct energy cascade (realized by pumping-generated long waves turning into shocks) was studied, while we are dealing with the inverse cascade. One can therefore say that as the inverse cascade proceeds towards larger wavelength, the spectra are getting wider as weak turbulence theory predicts [1]; yet presently we see no way it can predict the spontaneous appearance of anisotropy and changes in symmetry, particularly, significant anisotropy at the pumping scale.

For the two- and three-fold spectra, maxima and minima are distributed randomly in space. The most remarkable finding is seen in the last image of the top row in Figure 1: on top of a long-range orientational order, a short-range positional order appears. The symmetry of this state is between that of a solid and an isotropic liquid, very much like a hexatic phase in 2d melting [15]. It requires future studies (and larger domains) to establish whether this is indeed a turbulent analog of the Berezinski-Kosterlitz-Thouless transition; to avoid misunderstanding, remind that there are no vortices (holes) in the condensate left at this stage, see Fig. 1, the phase fluctuates weakly. It is the amplitude of small condensate perturbations which is getting ordered into a lattice with defects. The spatial correlations are also seen in Figure 2e which shows the correlation function $\overline{\psi(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{r})} - \overline{\psi}^2$, overline being average over $\boldsymbol{x}$. The crystallization which we observe is very much different from that externally imposed by a cut-off in 2d incompressible turbulence [16]. Likewise, in the work [17] symmetry change was caused by a change in the domain shape (from square to rectangle); in our case, the symmetry of the environment is not changed, the control parameter is the overall excitation level (the number of waves) in the system.

The time-spectral filtering [18] shows that the mode coherently oscillating with the frequency N is not a simple constant, it has an intricate spatial structure and multiscale correlations whose anisotropy is directly related to the anisotropy of over-condensate fluctuations, see Figure 1. The spatial spectrum of the coherent mode has the doubled number of angular maxima and is generally more symmetric than the whole spectrum. For nonzero k, the intensity of the temporally coherent mode

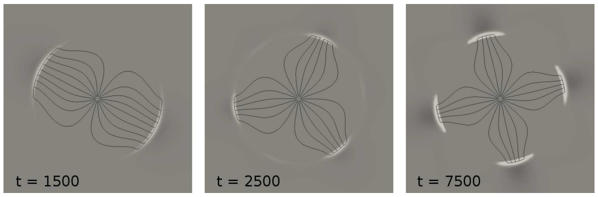

FIG. 3: Source distribution and flux flow lines.

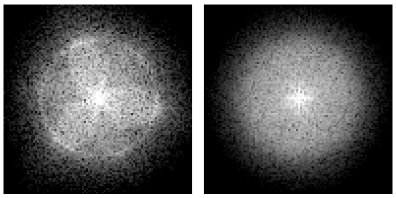

FIG. 4: Spectra for an instability (left) and the random force (right) at the same levels of condensate. The force-driven turbulence remains almost isotropic, while the instability-driven turbulence develops a three-petal spectrum.

is much smaller than overall $n_{\mathbf{k}}$, which decay approximately by power laws inside the petals, as seen in the fourth panel of Figure 2. Note that $1/k^2$ is the spectrum of the (isotropic weakly turbulent) inverse cascade which coincides in this case, up to logarithmic factor, with thermal equilibrium, while $1/k^3$ is the spectrum due to shock waves [1]. Conservation of N by (1) allows one to write NSE as a continuity equation and define the flux: $\mathrm{div}\mathbf{J}_{\mathbf{k}} = \langle|\psi_{\mathbf{k}}|^2\rangle\gamma_k \equiv Q_{\mathbf{k}}$. We compute $Q_{\mathbf{k}}$ and solve the above equation for $\mathbf{J}_{\mathbf{k}}$. Figure 3 shows the flux lines while positive/negative source $Q_{\mathbf{k}}$ is shown by dark/light regions. We have also calculated the flatness which is slightly below Gaussian value 3 for over-condensate fluctuations [8], i.e. the condensate effectively suppresses strong fluctuations contrary to what condensate vortices do to 2D incompressible turbulence [2].

A central issue in non-equilibrium physics is that of universality: what properties of a state depend on the excitation mechanism? We find that the phase transitions happen only when the waves are excited by an instability. If we pump by an additive random force concentrated in the same ring in $\mathbf{k}$-space and producing the same energy flux, turbulence spectrum at $k \gg 1$ remains isotropic at all N_0, see Fig 4. This shows that the structure of turbulence is not solely determined by the energy/action input but depends on the nature of the forcing.

We thus described a new phenomenon of spontaneous symmetry changes in the turbulence state under the change of a single parameter, the condensate level. To put it into perspective, let us remind briefly the main difference between equilibrium and turbulence: the former is known to realize an extremum of some thermodynamic potential (say, free energy), while for the latter no variational principle was ever discovered, despite much effort. Phase transitions provide such an important window into equilibrium statistical physics because they show how order and disorder (or different types of order) compete via different contributions into the thermodynamic potential (e.g. energy competes against entropy). In all cases of turbulence known so far, the symmetry of the state was completely determined by the symmetry of the forcing and boundary. To put it simply, one had no different turbulent states under similar conditions to compare. Now we have. We conclude by asking: Is there any quantity that the system tries to optimize by undergoing the series of phase transitions discovered here?

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