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Asymptotic solutions of decoupled continuous-time random walks with superheavy-tailed waiting time and heavy-tailed jump length distributions

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Abstract

We study the long-time behavior of decoupled continuous-time random walks characterized by superheavy-tailed distributions of waiting times and symmetric heavy-tailed distributions of jump lengths. Our main quantity of interest is the limiting probability density of the position of the walker multiplied by a scaling function of time. We show that the probability density of the scaled walker position converges in the long-time limit to a non-degenerate one only if the scaling function behaves in a certain way. This function as well as the limiting probability density are determined in explicit form. Also, we express the limiting probability density which has heavy tails in terms of the Fox H-function and find its behavior for small and large distances.

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I. INTRODUCTION

Continuous-time random walks (CTRWs), introduced by Montroll and Weiss [1], constitute an important class of jump processes that are widely used to model a variety of physical, geological, biological, economic and other phenomena. In particular, these processes describe anomalous diffusion and transport in disordered media (see, e.g., Refs. [2–4] and references therein), seismic [5, 6] and financial [7, 8] data. A remarkable fact is that systems so different from one another can successfully be described within the CTRW approach. This is because two random variables that many systems have in common, the waiting time between successive jumps and the jump length, are used to model the CTRW. Therefore, even the decoupled CTRW, when these variables are independent, is rather flexible.

The probability density $P(x,t)$ of the walker position $X(t)$ is the most important characteristic of the CTRW. It satisfies the integral master equation [9–11] which in the decoupled case depends only on the probability density $p(\tau)$ of waiting times and on the probability density $w(x)$ of jump lengths. Because exact solutions of this equation are known in very few cases [12–15], there is considerable interest in studying the long-time behavior of $P(x,t)$ that is responsible for the transport and diffusion properties of objects described by the CTRW model. In this context, much attention has been paid to the probability densities $p(\tau)$ and $w(x)$ having finite second moments and/or to those having heavy tails. It has been established [16–18] that different combinations of these properties of the waiting time and jump densities lead to different long-time distributions of $X(t)$. In Ref. [19], all possible distributions were expressed in terms of the limiting distributions of the properly scaled walker position.

In some cases the waiting-time densities are assumed to be superheavy-tailed, i.e., such that all fractional moments of $p(\tau)$ are infinite. In particular, this class of densities is used to model the superslow diffusion in which the diffusion front spreads more slowly than any positive power of time [20–23]. In general, one might expect superheavy-tailed distributions to reflect extremely slow time-dependent phenomena such as may occur in some relaxation and aging processes. Such distributions are also applicable within a Langevin rather than a CTRW description when dealing with processes that are interrupted by an absorption event or by the transition of a particle to a qualitatively different state [24, 25]. The long-time behavior of the decoupled CTRWs characterized by these waiting-time densities and jump densities with finite second moments is considered in Ref. [26]. Here we focus on asymptotic solutions of the CTRWs in the case when the densities $p(\tau)$ and $w(x)$ are superheavy- and heavy-tailed, respectively.

The paper is structured as follows. In Sec. II, we formulate the main definitions and write the basic equations describing the decoupled CTRW. A one-parameter limiting probability density of the scaled walker position that corresponds to the superheavy-tailed distributions of waiting times and the symmetric heavy-tailed distributions of jump lengths is determined in Sec. III. Here, we also find the scaling function and prove the positivity and unimodality of the limiting probability density. In Sec. IV, we express the limiting density and the corresponding cumulative distribution function in terms of Fox H-functions and consider a few particular examples. The short- and long-distance behavior of the limiting density is studied in Sec. V. Our main results are summarized in Sec. VI.

II. MAIN DEFINITIONS AND BASIC EQUATIONS

The CTRW approach deals with a wide class of continuous-time jump processes $X(t)$ represented as

$$X(t) = \sum_{n=1}^{N(t)} x_n. \tag{2.1}$$

Here, $N(t) = 0, 1, 2, \ldots$ is the random number of jumps that a walker has performed up to the time t (if $N(t) = 0$ then $X(t) = 0$), and $x_n \in (-\infty, \infty)$ are the independent random variables (jump lengths) distributed with some probability density $w(x)$. In order to specify the counting process $N(t)$, the waiting times τ_n, i.e., times between successive jumps, are introduced. Like the jump lengths, the waiting times are assumed to be independent random variables distributed with probability density $p(\tau)$. If the variables x_n and τ_n are independent of each other as well, i.e., if the CTRW is decoupled, then the probability density $P(x,t)$ of the walker position $X(t)$ depends only on $w(x)$ and $p(\tau)$. According to [1], in Fourier-Laplace space this dependence has the form

$$P_{ks} = \frac{1 - p_s}{s(1 - p_s w_k)}, \tag{2.2}$$

where $w_k = \mathcal{F}\{w(x)\} = \int_{-\infty}^{\infty} dx e^{ikx} w(x)$ $(-\infty < k < \infty)$ is the Fourier transform of $w(x)$, $p_s = \mathcal{L}\{p(t)\} = \int_0^{\infty} dt e^{-st} p(t)$ (Re$s > 0$) is the Laplace transform of $p(\tau)$, and $P_{ks} = \mathcal{F}\{\mathcal{L}\{P(x,t)\}\}$.

From Eq. (2.2) one can get

$$P_s(x) = \frac{(1-p_s)p_s}{s}\,\mathcal{F}^{-1}\bigg\{\frac{w_k}{1-p_s w_k}\bigg\} + \frac{1-p_s}{s}\,\delta(x) \tag{2.3}$$

and

$$P(x,t) = \mathcal{L}^{-1}\bigg\{\frac{(1-p_s)p_s}{s}\mathcal{F}^{-1}\bigg\{\frac{w_k}{1-p_s w_k}\bigg\}\bigg\} + V(t)\delta(x). \tag{2.4}$$

Here, $\mathcal{F}^{-1}\{f_k\} = f(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} dk e^{-ikx} f_k$ is the inverse Fourier transform, $\delta(x)$ is the Dirac δ function, $\mathcal{L}^{-1}\{g_s\} = g(t) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} ds e^{st} g_s$ (c is a real number exceeding the real parts of all singularities of g_s) is the inverse Laplace transform, and

$$V(t) = \mathcal{L}^{-1}\bigg\{\frac{1-p_s}{s}\bigg\} = \int_t^{\infty} d\tau p(\tau) \tag{2.5}$$

with $V(0) = 1$ and $V(\infty) = 0$ is the survival or exceedance probability. Using Eq. (2.4), the integral formula $\int_{-\infty}^{\infty} dx e^{-ikx} = 2\pi\delta(k)$ and the well-known properties of the δ function, it is not difficult to show that the probability density $P(x,t)$ is properly normalized: $\int_{-\infty}^{\infty} dx P(x,t) = 1$. Since $X(0) = 0$, the initial condition for $P(x,t)$ reads $P(x,0) = \delta(x)$ and, if boundary conditions are not imposed, $P(x,t) \to 0$ as $t \to \infty$.

According to this last property, the probability density of the walker position vanishes in the long-time limit. It is therefore reasonable to introduce the scaled walker position $Y(t) = a(t)X(t)$ and find the positive scaling function $a(t)$ such that the limiting probability density

$$\mathcal{P}(y) = \lim_{t\to\infty} \frac{1}{a(t)}\,P\bigg(\frac{y}{a(t)}, t\bigg) \tag{2.6}$$

of $Y(t)$, i.e., the probability density of the random variable $Y(\infty)$, is non-vanishing and non-degenerate. The importance of the functions $a(t)$ and $\mathcal{P}(y)$ is that, since $P(x,t) \sim a(t)\mathcal{P}(a(t)x)$ as $t \to \infty$, they completely describe the long-time behavior of the original walker position $X(t)$. To satisfy the above requirements on $\mathcal{P}(y)$, the scaling function must go to zero as $t \to \infty$ in a certain way. In fact, these requirements permit one to determine $a(t)$ up to a constant factor which, however, is not important and can be chosen for convenience.

The pairs $a(t)$ and $\mathcal{P}(y)$ have been determined for all cases characterized by finite second moments and/or heavy tails of the probability densities $p(\tau)$ and $w(x)$ [19]. In contrast, the case with superheavy tails has been much less studied. In fact, the pair $a(t)$ and $\mathcal{P}(y)$ has been determined only when $p(\tau)$ has a superheavy tail and $w(x)$ has a finite second moment l_2 [26]. Because $l_2 = \infty$ if $w(x)$ is heavy tailed, one may expect that in this case the long-time behavior of the walker position changes qualitatively and thus the pair $a(t)$ and $\mathcal{P}(y)$ changes as well. More precisely, in this paper we study the long-time behavior of decoupled CTRWs whose waiting-time densities $p(\tau)$ and jump densities $w(x)$ [it is assumed that $w(-x) = w(x)$] are described by the asymptotic formulas

$$p(\tau) \sim \frac{h(\tau)}{\tau} \quad (\tau \to \infty) \tag{2.7}$$

and

$$w(x) \sim \frac{u}{|x|^{1+\alpha}} \quad (|x| \to \infty), \tag{2.8}$$

where the positive function $h(\tau)$ varies slowly at infinity, i.e., $h(\mu\tau) \sim h(\tau)$ as $\tau \to \infty$ for all $\mu > 0$, the tail index α is restricted to the interval $(0, 2]$, and $u > 0$. The waiting-time and jump densities considered here belong to the classes of superheavy- and heavy-tailed densities, respectively. The difference between these classes consists in different asymptotic behavior of the constituent probability densities that, in turn, results in different properties of their fractional moments. Specifically, while the fractional moments $\int_0^\infty d\tau \tau^\rho p(\tau)$ of $p(\tau)$ are infinite for all $\rho > 0$, the fractional moments $\int_{-\infty}^\infty dx|x|^\rho w(x)$ of $w(x)$ are infinite only if $\rho \geq \alpha$. It should also be noted that the conditions $u > 0$ and $\alpha \in (0, 2]$ are completely compatible with the normalization condition $\int_{-\infty}^\infty dx w(x) = 1$. In contrast, the normalization condition $\int_0^\infty d\tau p(\tau) = 1$ imposes an additional restriction on the asymptotic behavior of $h(\tau)$: $h(\tau) = o(1/\ln \tau)$ as $\tau \to \infty$.

III. SCALING FUNCTIONS AND THE LIMITING PROBABILITY DENSITY

According to the Tauberian theorem for Laplace transforms [27], the long-time behavior of the probability density $P(x, t)$ is determined by the asymptotic behavior of the Laplace transform $P_s(x)$ when the *real* parameter s tends to zero. Because the waiting-time distribution is normalized to unity, the condition $p_s \to 1$ holds as $s \to 0$. It follows from Eq. (2.3) that we also need to find the $s \to 0$ behavior of $1 - p_s$. To this end, it is convenient to use the representation $1 - p_s = \int_0^\infty dq e^{-q} V(q/s)$ which, together with the fact [26] that the survival probability $V(t)$ varies slowly at infinity, immediately gives

$$1 - p_s \sim V(1/s) \tag{3.1}$$

as $s \to 0$. Then, taking into account that as $s \to 0$ the main contribution to $\mathcal{F}^{-1}\{w_k/(1 - p_s w_k)\}$ comes from a small vicinity of the point $k = 0$, i.e.,

$$\mathcal{F}^{-1}\left\{\frac{w_k}{1 - p_s w_k}\right\} \sim \mathcal{F}^{-1}\left\{\frac{1}{V(1/s) + 1 - w_k}\right\}, \tag{3.2}$$

Eq. (2.3) in the small-s limit yields

$$P_s(x) \sim \frac{V(1/s)}{s}\delta(x) + \frac{V(1/s)}{\pi s}\int_0^\infty dk \frac{\cos(xk)}{V(1/s) + 1 - w_k}. \tag{3.3}$$

The long-time behavior of $P(x,t)$ can be found directly from the limiting formula (3.3) by applying the above mentioned Tauberian theorem. It states that if the function $v(t)$ is ultimately monotonic and $v_s \sim s^{-\gamma}L(1/s)$ $(0 < \gamma < \infty)$ as $s \to 0$, then $v(t) \sim t^{\gamma-1}L(t)/\Gamma(\gamma)$ as $t \to \infty$. Here, $\Gamma(\gamma)$ denotes the gamma function and $L(t)$ is a slowly varying function at infinity. In our case $\gamma = 1$, therefore from Eq. (3.3) one obtains

$$P(x,t) \sim V(t)\delta(x) + \frac{V(t)}{\pi}\int_0^\infty dk \frac{\cos(xk)}{V(t) + 1 - w_k} \tag{3.4}$$

$(t \to \infty)$. Since in the long-time limit (when $V(t)$ tends to zero) the main contribution to the integral in Eq. (3.4) comes from a small vicinity of the point $k = 0$, the exact formula

$$1 - w_k = 2\int_0^\infty dx[1 - \cos(kx)]w(x) \tag{3.5}$$

can be replaced by one valid in this regime. Using Eq. (2.8) and the integral relation

$$\int_0^\infty dx \frac{1 - \cos(x)}{x^{1+\alpha}} = \frac{\pi}{2\Gamma(1+\alpha)\sin(\pi\alpha/2)} \tag{3.6}$$

$(0 < \alpha < 2)$, from Eq. (3.5) at $|k| \to 0$ we find

$$1 - w_k \sim \frac{\pi u}{\Gamma(1+\alpha)\sin(\pi\alpha/2)}|k|^\alpha. \tag{3.7}$$

Substituting this result into the asymptotic formula (3.4) and applying the definition (2.6), the limiting probability density $\mathcal{P}(y)$ can be written in the form

$$\mathcal{P}(y) = \lim_{t\to\infty}\frac{1}{\pi}\int_0^\infty dx \frac{\cos(yx)}{1 + \frac{\pi u a^\alpha(t)}{\Gamma(1+\alpha)\sin(\pi\alpha/2)V(t)}x^\alpha}. \tag{3.8}$$

It appears from this that $\mathcal{P}(y)$ is non-vanishing and non-degenerate only if the factor in front of x^α tends to a nonzero finite limit as $t \to \infty$. Assuming for convenience that this limit equals 1, we obtain the asymptotic representation of the scaling function

$$a(t) \sim \left(\frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi u}V(t)\right)^{1/\alpha} \tag{3.9}$$

($t \to \infty$) and the corresponding limiting density

$$\mathcal{P}(y) = \frac{1}{\pi} \int_0^\infty dx \frac{\cos(yx)}{1 + x^\alpha} \tag{3.10}$$

(the fact that $\mathcal{P}(y)$ is a probability density will be proved below). The symmetry condition $\mathcal{P}(-y) = \mathcal{P}(y)$, which follows from Eq. (3.10), is a consequence of the symmetry of the jump density $w(x)$.

Since at $\alpha = 2$ the integral in Eq. (3.6) diverges, the limiting formula (3.7) is not applicable to this case. Therefore, in order to find $1 - w_k$ at $\alpha = 2$ and $|k| \to 0$, we first split the interval of integration in Eq. (3.5) into two parts, $(0, b)$ and (b, ∞) with $b \sim 1$. Then, taking into account that as $|k| \to 0$ the contribution of the first interval to the right-hand side of Eq. (3.5) can be approximated by $k^2 \int_0^b dx x^2 w(x)$ and the second one by $uk^2 \ln(1/|k|)$, we get

$$1 - w_k \sim uk^2 \ln \frac{1}{|k|} \tag{3.11}$$

($|k| \to 0$). In accordance with this, the limiting probability density when $\alpha = 2$ takes the form

$$\mathcal{P}(y) = \lim_{t\to\infty} \frac{1}{\pi} \int_0^\infty dx \frac{\cos(yx)}{1 + \frac{ua^2(t)\ln[1/a(t)]}{V(t)} x^2}. \tag{3.12}$$

As before, we choose the long-time limit of the factor in front of x^2 to be equal to unity. In this case the asymptotic behavior of the scaling function $a(t)$ is determined by the relation $ua^2(t) \ln[1/a(t)] \sim V(t)$ ($t \to \infty$). Assuming that $a(t) \sim \sqrt{V(t)/u}\, a_1(t)$, where the positive function $a_1(t)$ satisfies the conditions $a_1(t) \to 0$ and $\sqrt{V(t)} = o(a_1(t))$ as $t \to \infty$, from this relation we obtain $a_1(t) \sim \sqrt{2/\ln[1/V(t)]}$, and thus

$$a(t) \sim \sqrt{\frac{2V(t)}{u \ln[1/V(t)]}} \tag{3.13}$$

($t \to \infty$). The limiting probability density (3.12) which corresponds to this scale function is given by

$$\mathcal{P}(y) = \frac{1}{\pi} \int_0^\infty dx \frac{\cos(yx)}{1 + x^2} = \frac{1}{2} e^{-|y|}, \tag{3.14}$$

showing that Eq. (3.10) is valid for $\alpha = 2$ as well. We note that the same two-sided exponential density (3.14) describes the limiting distribution when the jump density $w(x)$ has a finite second moment $l_2 = \int_{-\infty}^\infty dx x^2 w(x)$ [26]. However, because at $l_2 < \infty$ the asymptotic behavior of the scaling function, $a(t) \sim \sqrt{2V(t)/l_2}$, is quite different from that given in Eq. (3.13), the long-time behaviors of the walker position in these cases are also quite different.

A. Positivity and unimodality of $\mathcal{P}(y)$

To be a probability density, the function $\mathcal{P}(y)$ must be normalized and positive (non-negative). The normalization condition $\int_{-\infty}^{\infty} dy \mathcal{P}(y) = 1$ can easily be proved using Eq. (3.10), which represents $\mathcal{P}(y)$ as a cosine Fourier transform, and the integral representation $\delta(x) = (1/2\pi)\int_{-\infty}^{\infty} dy \cos(yx)$ of the δ function. However, except for the case $\alpha = 2$, where according to Eq. (3.14) $\mathcal{P}(y) > 0$, the use of Eq. (3.10) to prove the positivity of $\mathcal{P}(y)$ is impractical because of the oscillating character of the integrand. On this point, the representation of $\mathcal{P}(y)$ in the form of a Laplace transform would be preferable. In order to find it, we first define the function

$$f(z) = \frac{1}{\pi}\frac{e^{i|y|z}}{1+z^{\alpha}} \tag{3.15}$$

$(0 < \alpha < 2)$ of the complex variable $z = x + iu$. This function is analytic in the first quadrant of the z-plane (when $|z| > 0$ and $0 \leq \arg z \leq \pi/2$), and so from the Cauchy integral theorem [28] we have $\oint_C dz f(z) = 0$, where C is a simple closed contour that lies in the domain of analyticity of $f(z)$. Then, choosing the contour C to be the boundary of the first quadrant (we emphasize that the branch point $z = 0$ is outside the contour) and applying the Jordan lemma [28], the above integral reduces to

$$\int_0^{\infty} dx f(x) - i \int_0^{\infty} du f(iu) = 0. \tag{3.16}$$

Finally, taking into account that $\mathcal{P}(y) = \mathrm{Re}\left[\int_0^{\infty} dx f(x)\right]$ and $i^{\alpha} = \cos(\pi\alpha/2) + i\sin(\pi\alpha/2)$, from the real part of Eq. (3.16) we obtain

$$\mathcal{P}(y) = \frac{1}{\pi}\int_0^{\infty} dx e^{-|y|x} \frac{\sin(\pi\alpha/2)x^{\alpha}}{1 + 2\cos(\pi\alpha/2)x^{\alpha} + x^{2\alpha}}. \tag{3.17}$$

The main advantage of this representation of $\mathcal{P}(y)$ is that it clearly shows that $\mathcal{P}(y) > 0$ when $0 < \alpha < 2$. Thus, since $\mathcal{P}(y)$ is positive for $\alpha = 2$ as well, we can conclude that the function $\mathcal{P}(y)$ is indeed the probability density for all α in the interval $(0, 2]$. Another important property of $\mathcal{P}(y)$, which follows directly from Eq. (3.17), is that $d\mathcal{P}(y)/dy < 0$ when $y > 0$. Together with the condition $\mathcal{P}(-y) = \mathcal{P}(y)$, it shows that the limiting probability density is symmetric, unimodal and centered at the origin. In contrast to the scaling function, which depends on all the parameters characterizing the asymptotic behavior of the waiting time and jump densities, the limiting density depends only on the tail index

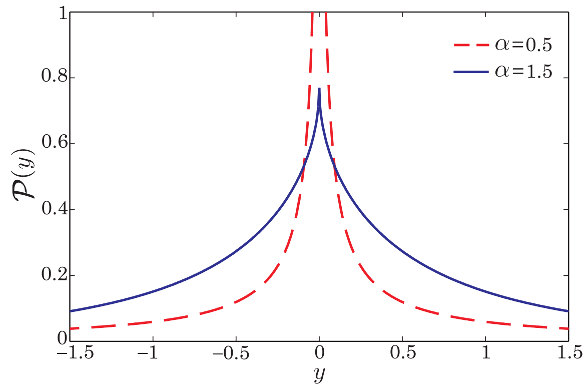

FIG. 1: Plots of the probability density $\mathcal{P}(y)$ for two values of the tail index α belonging to the intervals $(0, 1]$ and $(1, 2]$.

α. According to Eqs. (3.10) and (3.17), this parameter strongly influences the properties of $\mathcal{P}(y)$. In particular, the behaviors of $\mathcal{P}(y)$ in the vicinity of the origin differ substantially from one another when $\alpha \in (0, 1]$ and $\alpha \in (1, 2]$, as illustrated in Fig. 1 (for details, see Sec. V).

IV. LIMITING DISTRIBUTION IN TERMS OF SPECIAL FUNCTIONS

To get more insight into the mathematical structure of the limiting probability density $\mathcal{P}(y)$, it is reasonable to express it in terms of well-known special functions. Toward this end, we first represent $\mathcal{P}(y)$ as the inverse Mellin transform. The Mellin transform of a function $f(y)$ is defined by $f_r = \mathcal{M}\{f(y)\} = \int_0^\infty dy f(y) y^{r-1}$. Therefore, for the function $f(y) = \int_0^\infty dx u(yx) v(x)$ one gets $f_r = u_r v_{1-r}$ [29]. If $f(y)$ is associated with $\mathcal{P}(y)$ from Eq. (3.10), then the functions $u(x)$ and $v(x)$ can be chosen as $u(x) = \pi^{-1}\cos(x)$ and $v(x) = (1 + x^\alpha)^{-1}$ whose Mellin transforms are given by [29]

$$u_r = \frac{1}{\pi}\Gamma(r)\cos\Big(\frac{\pi r}{2}\Big) \quad (0 < \mathrm{Re}\, r < 1) \tag{4.1}$$

and

$$v_r = \frac{1}{\alpha}\Gamma\Big(\frac{r}{\alpha}\Big)\Gamma\Big(1 - \frac{r}{\alpha}\Big) \quad (0 < \mathrm{Re}\, r < \alpha). \tag{4.2}$$

Using the reflection formula [30] $\Gamma(1/2-r/2)\Gamma(1/2+r/2) = \pi/\cos(\pi r/2)$ to replace $\cos(\pi r/2)$ in Eq. (4.1), the Mellin transform $\mathcal{P}_r = u_r v_{1-r}$ of $\mathcal{P}(y)$ takes the form

$$\mathcal{P}_r = \frac{\Gamma(r)\Gamma(1 - 1/\alpha + r/\alpha)\Gamma(1/\alpha - r/\alpha)}{\alpha\Gamma(1/2 - r/2)\Gamma(1/2 + r/2)}, \tag{4.3}$$

where $\max(1-\alpha,0) < \mathrm{Re}\, r < 1$. Finally, introducing the inverse Mellin transform as $\mathcal{M}^{-1}\{f_r\} = f(y) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} dr f_r y^{-r}$ and utilizing the fact that $\mathcal{P}(-y) = \mathcal{P}(y)$, we find

$$\mathcal{P}(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dr \mathcal{P}_r |y|^{-r}. \tag{4.4}$$

The structure of $\mathcal{P}_r$ suggests that the probability density $\mathcal{P}(y)$ is a particular case of the Fox H-function which can be defined by means of a Mellin-Barnes integral as follows (see, e.g., Ref. [31]):

$$H_{p,q}^{m,n}\left[y \middle| \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right] = \frac{1}{2\pi i}\int_L dr \Theta_r y^{-r}. \tag{4.5}$$

Here,

$$\Theta_r = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j r)\prod_{j=1}^{n}\Gamma(1 - a_j - A_j r)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j r)\prod_{j=n+1}^{p}\Gamma(a_j + A_j r)}, \tag{4.6}$$

m, n, p, q are whole numbers, $0 \le m \le q$, $0 \le n \le p$, a_j and b_j are real or complex numbers, $A_j, B_j > 0$, and L is a suitable contour in the complex r-plane which separates the poles of the gamma functions $\Gamma(b_j + B_j r)$ from the poles of the gamma functions $\Gamma(1 - a_j - A_j r)$. It is also assumed that an empty product equals 1. Comparing Eqs. (4.3) and (4.4) with Eqs. (4.6) and (4.5), respectively, we see that

$$\mathcal{P}(y) = \frac{1}{\alpha} H_{2,3}^{2,1}\left[|y| \middle| \begin{matrix} (1-1/\alpha, 1/\alpha), (1/2, 1/2) \\ (0,1), (1-1/\alpha, 1/\alpha), (1/2, 1/2) \end{matrix} \right]. \tag{4.7}$$

It should be noted that the cumulative distribution function $F(y) = 1/2 + \int_0^y dy' \mathcal{P}(y')$ of the random variable $Y(\infty)$ can also be expressed through the H-function. To show this, we write $\widetilde{F}(y) = \int_0^y dy' \mathcal{P}(y')$ and take into account the following property of the Mellin transform [29]: $\mathcal{M}\left\{\int_0^y f(y')dy'\right\} = -f_{r+1}/r$. According to this, $\widetilde{F}_r = -\mathcal{P}_{r+1}/r$ and, from Eq. (4.3) and the functional equation $\Gamma(1+x) = x\Gamma(x)$, one gets

$$\widetilde{F}_r = -\frac{\Gamma(r)\Gamma(r/\alpha)\Gamma(1-r/\alpha)}{\alpha\Gamma(1-r/2)\Gamma(r/2)} \tag{4.8}$$

$[-\min(1,\alpha) < \mathrm{Re}\, r < 0]$. Therefore, using Eqs. (4.5) and (4.6), we obtain

$$F(y) = \frac{1}{2} - \frac{\mathrm{sgn}(y)}{\alpha} H_{2,3}^{2,1}\left[|y| \middle| \begin{matrix} (0, 1/\alpha), (0, 1/2) \\ (0,1), (0, 1/\alpha), (0, 1/2) \end{matrix} \right]. \tag{4.9}$$

A. Particular examples

For some special values of the tail parameter α the H-functions in Eqs. (4.7) and (4.9) can be reduced to more familiar special (or even elementary) functions. Because the probability density $\mathcal{P}(y)$ and the distribution function $F(y)$ provide equivalent descriptions of the long-time behavior of the scaled walker position $Y(t)$, next we consider only the properties of $\mathcal{P}(y)$. The simplest situation occurs when $\alpha = 2$. In this case both reduction formulas [31] can be applied, yielding

$$\begin{aligned}\mathcal{P}(y) &= \frac{1}{2} H^{2,1}_{2,3}\left[|y| \,\middle|\, \begin{matrix} (1/2,1/2),(1/2,1/2) \\ (0,1),(1/2,1/2),(1/2,1/2) \end{matrix} \right] \\ &= \frac{1}{2} H^{2,0}_{1,2}\left[|y| \,\middle|\, \begin{matrix} (1/2,1/2) \\ (0,1),(1/2,1/2) \end{matrix} \right] \\ &= \frac{1}{2} H^{1,0}_{0,1}\left[|y| \,\middle|\, \begin{matrix} \\ (0,1) \end{matrix} \right]. \end{aligned} \tag{4.10}$$

Since the last H-function equals $e^{-|y|}$ [31], this ascertains that Eq. (4.7) at $\alpha = 2$ reduces to Eq. (3.14).

If the parameter α is rational, then the probability density $\mathcal{P}(y)$ can, in principle, be expressed in terms of the Meijer G-function as well. The G-function, which is a particular case of the H-function, is defined as

$$G^{m,n}_{p,q}\left[y \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix} \right] = \frac{1}{2\pi i}\int_L dr \Psi_r y^{-r} \tag{4.11}$$

with $\Psi_r = \Theta_r|_{A_j,B_j=1}$. As a first illustrative example, we consider the case when $\alpha = 1$. Changing the variable of integration in Eq. (4.4) from r to $2r$, one readily obtains $\mathcal{P}(y) = (\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} dr \mathcal{P}_{2r}(y^2)^{-r}$, where $\max(1-\alpha,0)/2 < c < 1/2$. Then, using Eq. (4.3) with $\alpha = 1$ and the duplication formula [30] $\Gamma(2r) = \pi^{-1/2}2^{2r-1}\Gamma(r)\Gamma(1/2+r)$, the Mellin transform $\mathcal{P}_{2r}$ can be written in the form

$$\mathcal{P}_{2r} = \frac{2^{2r}}{4\pi^{3/2}}\Gamma^2(r)\Gamma(1/2+r)\Gamma(1-r). \tag{4.12}$$

Therefore, in accordance with the definition (4.11), the limiting probability density (4.7) at

$\alpha = 1$ has the following G-function representation:

$$\mathcal{P}(y) = H^{2,1}_{2,3}\left[|y|\,\middle|\,\begin{matrix}(0,1),(1/2,1/2)\\(0,1),(0,1),(1/2,1/2)\end{matrix}\right]$$
$$= \frac{1}{2\pi^{3/2}}G^{3,1}_{1,3}\left[y^2/4\,\middle|\,\begin{matrix}0\\0,0,1/2\end{matrix}\right]. \qquad (4.13)$$

Remarkably, the limiting probability density $\mathcal{P}(y)$ at $\alpha = 1$ can be expressed not only in terms of the Fox and Meijer functions, but also in terms of the well-known sine, $\mathrm{si}(y) = -\int_y^\infty dx \sin(x)/x$, and cosine, $\mathrm{Ci}(y) = -\int_y^\infty dx \cos(x)/x$, integral functions. Indeed, using the exact result for the cosine Fourier transform of $(1+x)^{-1}$ [32], we obtain

$$\mathcal{P}(y) = -\frac{1}{\pi}[\sin(|y|)\,\mathrm{si}(|y|) + \cos(y)\,\mathrm{Ci}(|y|)]. \qquad (4.14)$$

Finally, in our last example we consider the case $\alpha = 1/2$. Following straightforward calculations similar to those described above, for the Mellin transform $\mathcal{P}_{2r}$ we obtain the expression

$$\mathcal{P}_{2r} = \frac{2^{2r}}{8\pi^{7/2}}\Gamma(-1/4+r)\Gamma^2(r)\Gamma(1/4+r)\Gamma(1/2+r)$$
$$\times\Gamma(5/4-r)\Gamma(1-r)\Gamma(3/4-r) \qquad (4.15)$$

from which it follows that

$$\mathcal{P}(y) = 2H^{2,1}_{2,3}\left[|y|\,\middle|\,\begin{matrix}(-1,2),(1/2,1/2)\\(0,1),(-1,2),(1/2,1/2)\end{matrix}\right]$$
$$= \frac{1}{4\pi^{7/2}}G^{5,3}_{3,5}\left[y^2/4\,\middle|\,\begin{matrix}-1/4,0,1/4\\-1/4,0,0,1/4,1/2\end{matrix}\right]. \qquad (4.16)$$

V. ASYMPTOTIC BEHAVIOR OF $\mathcal{P}(y)$

Using Eq. (4.7), the behavior of the limiting probability density $\mathcal{P}(y)$ for small and large values of $|y|$ can, in principle, be found from the expansions obtained for the H-function in different limits (for details, see Ref. [31] and references therein). However, because $\mathcal{P}(y)$ is a very particular case of the H-function, it is reasonable and convenient to derive the corresponding limiting formulas directly from the source representation (3.10).

A. Short-distance behavior

There are three regions of the tail index α, which we consider separately, where the limiting behaviors of $\mathcal{P}(y)$ as $|y| \to 0$ differ from one another.

$\alpha \in (0, 1)$. In this case Eq. (3.10), after changing the variable of integration from x to $x/|y|$, as $|y| \to 0$ yields

$$\mathcal{P}(y) \sim \frac{1}{\pi |y|^{1-\alpha}} \int_0^\infty dx \frac{\cos(x)}{x^\alpha}. \tag{5.1}$$

Then, since $\int_0^\infty dx \cos(x)/x^\alpha = \Gamma(1-\alpha)\sin(\pi\alpha/2)$, one gets

$$\mathcal{P}(y) \sim \frac{\Gamma(1-\alpha)\sin(\pi\alpha/2)}{\pi |y|^{1-\alpha}}. \tag{5.2}$$

$\alpha = 1$. Using the formulas $\mathrm{si}(|y|) \sim |y|$ and $\mathrm{Ci}(|y|) \sim \ln|y|$ ($|y| \to 0$) [33], Eq. (4.14) which follows from Eq. (3.10) immediately yields

$$\mathcal{P}(y) \sim -\frac{1}{\pi} \ln|y|. \tag{5.3}$$

$\alpha \in (1, 2]$. Finally, in this case it is convenient to rewrite Eq. (3.10) in the form

$$\mathcal{P}(y) = \mathcal{P}(0) - \frac{|y|^{\alpha-1}}{\pi} \int_0^\infty dx \frac{1-\cos(x)}{|y|^\alpha + x^\alpha}, \tag{5.4}$$

where $\mathcal{P}(0) = [\alpha \sin(\pi/\alpha)]^{-1}$. Then, neglecting $|y|^\alpha$ in the integrand and taking into account that $\int_0^\infty dx[1-\cos(x)]/x^\alpha = \Gamma(2-\alpha)\sin(\pi\alpha/2)/(\alpha-1)$, we obtain

$$\mathcal{P}(y) \sim \frac{1}{\alpha\sin(\pi/\alpha)} - \frac{\Gamma(2-\alpha)\sin(\pi\alpha/2)}{\pi(\alpha-1)} |y|^{\alpha-1}. \tag{5.5}$$

It should be noted that, since $\lim_{x\to 0}\Gamma(x)\sin(\pi x/2) = \pi/2$, the limiting formula (5.5) at $\alpha = 2$ reduces to $\mathcal{P}(y) \sim (1-|y|)/2$, in accordance with Eq. (3.14).

B. Long-distance behavior

The asymptotic behavior of $\mathcal{P}(y)$ as $|y| \to \infty$ can easily be found by a single (if $0 < \alpha < 1$) or double (if $1 < \alpha < 2$) integration by parts of Eq. (3.10) with a subsequent change of the integration variable from x to $x/|y|$. In particular, for $\alpha \in (0, 1)$ this yields

$$\begin{aligned}\mathcal{P}(y) &= \frac{\alpha}{\pi |y|^{1+\alpha}} \int_0^\infty dx \frac{\sin(x)}{x^{1-\alpha}[1 + (x/|y|)^\alpha]} \\ &\sim \frac{\alpha}{\pi |y|^{1+\alpha}} \int_0^\infty dx \frac{\sin(x)}{x^{1-\alpha}}\end{aligned} \tag{5.6}$$

and so

$$\mathcal{P}(y) \sim \frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi|y|^{1+\alpha}}. \tag{5.7}$$

It is not difficult to verify that the asymptotic formula (5.7) also holds for $\alpha \in (1,2)$. Moreover, since Eq. (4.14) leads to $\mathcal{P}(y) \sim \pi^{-1}|y|^{-2}$ as $|y| \to \infty$, this formula is valid for $\alpha = 1$ as well.

Thus, according to Eq. (5.7), the limiting probability density $\mathcal{P}(y)$ when $\alpha \in (0,2)$ is heavy-tailed with the same tail index α as in the jump density $w(x)$. In contrast, at $\alpha = 2$ the limiting density has exponential tails, while the jump density is still heavy-tailed, see Eq. (2.8). We also note that the same tail index α characterizes the limiting probability density when both the waiting-time and jump distributions are heavy-tailed [19]. However, this does not mean that the long-time behaviors of the CTRWs with heavy- and superheavy-tailed distributions of waiting times are identical. This is because the scaling functions for these CTRWs are quite different. Specifically, while in the former case the scaling functions are power functions of time [19], in the latter case they vary more slowly, see Eqs. (3.9) and (3.13).

VI. CONCLUSIONS

We have determined a new class of asymptotic solutions of the CTRWs characterized by superheavy-tailed distributions of waiting times and symmetric heavy-tailed distributions of jump lengths. These solutions represent the probability densities of the scaled walker position, i.e., the random walker position multiplied by a time-dependent deterministic scaling function, in the long-time limit. We have found both the limiting probability densities and the corresponding scaling functions which completely describe the long-time behavior of the reference CTRWs. It turns out that the scaling functions depend on the survival probability characterizing the long-time behavior of the waiting-time density and on the tail index $\alpha \in (0,2]$ describing the asymptotic behavior of the jump density. In contrast, the limiting densities, which have been represented in the form of Fourier and Laplace transforms, depend only on α.

The limiting probability densities $\mathcal{P}(y)$ form a class of symmetric and unimodal functions centered at the origin. Among other things, we have determined the limiting behavior of these densities for small and large distances. We find that while at $\alpha = 2$ the function $\mathcal{P}(y)$

has exponential tails, at $\alpha \in (0, 2)$ the tails are heavy and are characterized by the same tail index α as the jump density. In the vicinity of the origin, the behavior of $\mathcal{P}(y)$ for $\alpha \in (0, 1]$ is quite different from that for $\alpha \in (1, 2]$. Specifically, $\mathcal{P}(0)$ is infinite in the former case and is finite in the latter. Finally, we have expressed the limiting probability densities in terms of the Fox H-function for the general case of arbitrary α and, for a few values of α, in terms of the Meijer G-function.

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