Hydrodynamics of a vibrated granular monolayer

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We investigate the long-standing puzzle of phase separation in a granular monolayer vibrated from below. Although this system is three-dimensional, an interesting dynamics occurs mostly in the horizontal plane, perpendicularly to the direction of vibration. Experiments [Olafsen and Urbach, Phys. Rev. Lett. 81, 4369 (1998)] demonstrated that for high amplitude of vibration the system is in the gas-like phase, but when the amplitude becomes smaller than a certain threshold, a phase separation occurs: a solid-like dense condensate of particles forms in the center of the system, surrounded by particles in the gas-like phase. We theoretically explain the experimentally observed coexistence of dilute and dense phases, employing Navier-Stokes granular hydrodynamics. We show that the phase separation is associated with negative compressibility of granular gas.

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A vertically vibrated monolayer of spherical grains is probably the simplest experimental system exhibiting regular large-scale patterns and phase separation [1], propagating fronts [2], melting transitions [3], non-Maxwellian velocity distribution, and other non-trivial statistical properties of non-equilibrium steady-state [4–7]. In the most of these experiments, a thin layer of spherical grains is energized by precise vertical vibrations. The most important dimensionless parameters affecting the dynamics are the normalized filling fraction (or number density) $f = \langle n \rangle / n_c$, aspect ratio, and dimensionless acceleration Γ, i.e the amplitude of the vibration acceleration of the bottom plate normalized by the gravity acceleration g: $\Gamma = 4\pi^2\nu^2 A_0/g$, where A_0 and ν are the displacement amplitude and the frequency of vibration, $n_c = 2/(\sqrt{3}d^2)$ is the hexagonal close packing density.

For sufficiently high magnitude of vibrations Γ, a quasi-two-dimensional granular gas with nearly uniform spatial density is observed. However, upon reducing the acceleration below a certain critical value Γ that depends on the filling fraction f, a transition to a bimodal regime occurs [1], which is characterized by a single dense cluster of closely packed almost immobile grains surrounded by a gas of agitated particles. Despite the fact that this striking phenomenon is known for almost 15 years, no consistent continuum description rooted in granular hydrodynamics was developed to date [8]. In contrast, continuum description of a phase separation and coarsening in some-what similar system, electrostatically driven granular layers, was successful [9–11].

In this paper we develop quasi-two-dimensional hydrodynamic description of vertically-vibrated granular monolayers. The description is based on the granular inelastic Navier-Stokes equation averaged over the vertical (z) component. We take into account the specifics of energy injection and momentum redistribution due to collisions of particles with the vibrating bottom plate. The model parameters are calibrated against available experimental data. We obtained analytically the domain of co-existence between two phase-separated states; the phase diagram is consistent with experimental data.

The model. Experiments show that for low enough acceleration of vibration Γ, one can observe a coexistence of a dense phase and a gas phase [1]. Since in the steady-state, the pressure of the two states should be the same, the high density phase should have a low effective horizontal temperature, T_h and the dilute gas phase should be "hot". To maintain this steady state despite inelastic collisions between the particles, one needs to continuously pump energy into the system. Since the particles take the energy from the vertically vibrating plate, we will also consider the dynamics of the (vertical) kinetic energy E_v and the vertical temperature T_v. Below are the two equations for the energy balance:

$$\frac{\partial E_v}{\partial t} = A(\Gamma) - B{E_v}^{1/2} - C\left(T_v - T_h\right) \quad (1)$$

$$\frac{\partial T_h}{\partial t} = C(T_v - T_h) - I(n, T_h) - \frac{BT_h}{{E_v}^{1/2}} + \frac{\nabla(\kappa \nabla T_h)}{n} \quad (2)$$

Let us discuss now every term in Eqs. (1)-(2). The term $A(\Gamma)$ describes the energy gain due to vibration; this coefficient will be determined later on from single particle experiments [12]. The second term in Eq. (1) describes energy losses due to inelastic particle collisions with a plate. One can derive this term, considering a single bouncing particle on a plate [2]. If the maximal kinetic energy of a particle before the collision is E_v, then after the collision it becomes $\alpha^2 E_v$, where α is the particle-plate restitution coefficient. Since the time between the consecutive collisions is $2\sqrt{2E_v}/g$, the energy loss term is proportional to $\sqrt{E_v}$ and $B = (1-\alpha^2)g/(2\sqrt{2})$.

When the particles collide, part of the energy goes from vertical degrees of freedom to the horizontal motion. This is described by the collision term $C\left(T_v - T_h\right)$, which enters Eq. (1) with minus sign and enters Eq. (2) with plus sign. Here C is proportional to the frequency of collisions: $C = c\sqrt{T_h}/\lambda$, where $\sqrt{T_h}$ is the horizontal thermal velocity, λ is the mean free path, and c is unknown constant of order unity. The second term in Eq. (2) represents energy losses due to inelastic collisions between particles.

As we show below, this term is not responsible for phase separation, in contrast to instabilities in other granular systems [13, 14]. The third term represents energy losses due to inelastic collisions with a plate [15], while the last term describes thermal conduction in a horizontal plane.

In order to describe the high-density phase, we need to employ constitutive relations valid up to the densities close to the close packing density n_c. Various expressions for pressure, inelastic heat losses and transport coefficients were recently suggested [16]. Here we use the interpolation between the dilute and dense limits, proposed in Ref. [17] and successfully employed in Ref. [18]. The pressure, inelastic heat losses, coefficient of thermal conductivity and the mean free path are given by:

$$p = n\,T\,\frac{n_c+n}{n_c-n}, \quad I = \frac{2\mu(1-r)}{\gamma}\frac{T_h^{3/2}}{\lambda}$$
$$\kappa = \frac{\sigma\, n(a_1\lambda+d)^2\, T_h^{1/2}}{\lambda}, \quad \lambda = \frac{1}{\sqrt{8}nd}\frac{n_c-n}{n_c-a_2\,n} \quad (3)$$

where $a_2 = 1-\sqrt{3/8}$, and σ, γ, and a_1 are numerical factors of order unity.

Finally, we need to distinguish between the vertical kinetic energy E_v, measured in experiments [12] and the vertical temperature T_v, which is related to the vertical velocity fluctuations. The difference between E_v and T_v becomes crucial for high densities. Indeed, a dense cluster bouncing on a plate as a whole might have substantial vertical kinetic energy, but at the same time its vertical temperature is extremely low. Clearly, T_v tends to zero at the density of close packing. Therefore, we assume $T_v = [(n_c-n)/n_c]^\delta E_v$. The power δ describes how fast T_v tends to zero when $n \to n_c$. Therefore, δ is a decreasing function of the vibration acceleration Γ: for the same density the synchronization in the vertical motion of a cluster is smaller for larger values of Γ.

The coefficient A before the gain term in Eq. (1) can be found from experiments with single particle [12]. In a steady state for a single particle one obtains $A(\Gamma) = B\,{E_g}^{1/2}$, where we use an empirical expression for the maximal kinetic energy of the bouncing particle [12, 19]:

$$E_g = \frac{g^2}{\omega^2}\frac{c_1\Gamma(\Gamma-\Gamma_1)}{1-\alpha+c_2(1-\alpha)^2},$$

where $c_1 = 3.8$, $c_2 = 4.45$, α is the particle plate restitution coefficient, and ω is the frequency of vibrating plate. This expression is meaningful only for the values of acceleration that exceed a certain threshold, $\Gamma > \Gamma_1$; there are various estimates of the value of this threshold acceleration: $\Gamma_1 = 0.85$ [12] or $\Gamma_1 = 0.79$ [19].

Every term in Eq. (2) is measured in units T/t, where the temperature T is measured in units of g^2/ω^2 and time t in units of $1/\omega$. Assuming a steady state, we multiply every term by ω/g^2 and obtain the following equations

$$\bar{b}\left(E_g^{1/2}-E_v^{1/2}\right) - \frac{c\sqrt{T_h}}{\lambda}(T_v-T_h) = 0, \quad (4)$$

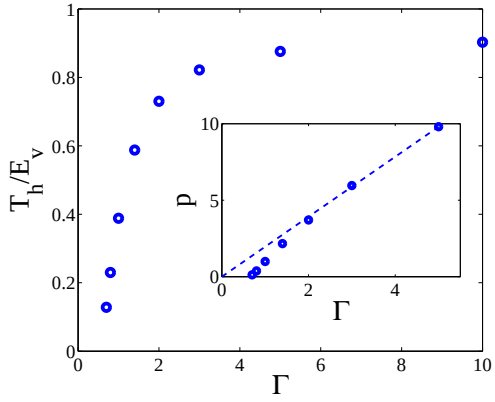

FIG. 1: (Color online) The ratio of vertical energy and the horizontal temperature the ratio E_v/T_h versus driving Γ, compare to Ref. [20]. An inset shows the steady state pressure versus driving Γ, compare to Ref. [19]. The parameters are: $R = 0.0885$, $\bar{b} = 0.4335$ ($\omega = 40Hz$, $d = 1mm$, $\alpha = 0.9$), $\Gamma_1 = 0.6$, $\delta = 1.85/\Gamma^2$, and $z = n_c/n = 2$.

$$- \frac{\bar{b}T_h}{E_v^{1/2}} - \frac{RT_h^{3/2}}{\lambda} + \frac{c\sqrt{T_h}}{\lambda}(T_v-T_h)$$
$$+ \frac{\sqrt{3}z}{2}\frac{d}{dx}\left(\bar{\kappa}\,T_h^{1/2}\frac{dT_h}{dx}\right) = 0. \quad (5)$$

Two parameters that enter Eqs. (5) are $\bar{b} = Bd\omega^2/g^2$ and $R = 2\mu(1-r)/\gamma$. One can estimate these dimensionless parameters independently from experiments [1, 12]. Roughly, R is of the order of 0.1 and $\bar{b}$ is of the order of unity for intermediate frequencies of vibration.

Homogeneous steady-state. Let us find first the uniform solution of Eqs. (4)-(5). From Eq. (5) one find that $(c/\lambda)\,(T_v-T_h)/T_h = \bar{b}/(T_h^{1/2}E_v^{1/2}) + R/\lambda$ so that the ratio of the vertical and horizontal temperatures is

$$\frac{T_v}{T_h} = 1 + \frac{\bar{b}\lambda}{cT_h^{1/2}E_v^{1/2}} + \frac{R}{c}. \quad (6)$$

When Γ is large, the temperatures are large as well and the ratio T_v/T_h approaches $1+R/c \simeq 1$, so the two temperatures are almost equal for large driving. However, for small driving, the second term in Eq. (6) becomes large and the vertical temperature is much larger than the horizontal temperature, in agreement with the results of molecular dynamics simulations [20]. The ratio T_h/E_v (computed for some intermediate density) as a function of driving is shown in Figure 1. The inset shows the steady state pressure as a function of vibration acceleration. For large Γ, the pressure increases linearly with Γ in agreement with experiments [19].

We also computed the steady state pressure p as a function of the normalized average area fraction $f = \langle n\rangle/n_c$, see Fig. 2. For the values of f above a certain threshold (to the right of the dashed line), the pressure decreases with f. This is the negative compressibility that drives the instability of the homogeneous state and leads to

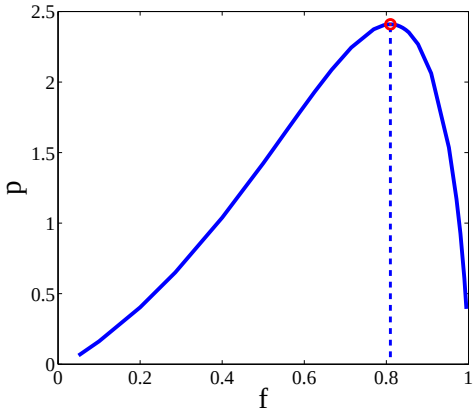

FIG. 2: (Color online) The steady state pressure p vs normalized area fraction f (solid line). For large enough f, the pressure decreases with f; this negative compressibility drives the instability of the uniform state. The parameters are $\Gamma = 0.9$, $\bar{b} = 0.9736$ ($\omega = 60$Hz, $d = 1$mm, $\alpha = 0.9$), $\Gamma_1 = 0.6$, $\delta = 1.34/\Gamma^2$.

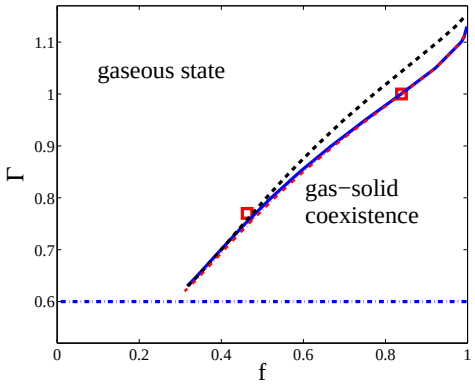

FIG. 3: (Color online) Phase diagram as computed from Eq. (8). Experimental observations [1] are denoted by squares. The dashed line corresponds to elastic particle collisions, $R = 0$, the solid line corresponds to $R = 0.0885$ (as in [1]), the dash-dotted line corresponds to $R = 0.0885$ and takes into account the density gradient term in the heat flux. The parameters are $\bar{b} = 0.9736$ ($\omega = 60$Hz, $d = 1$mm, $\alpha = 0.9$), $\Gamma_1 = 0.6$, $\delta = 1.34/\Gamma^2$.

phase separation. Similar negative compressibility phenomena was observed in granular systems, where the particles were driven by a thermal side wall [14] or for vigorously vibrated granular layers confined by a top lid [21, 22].

Phase coexistence. Equation (4) is a quadratic equation for E_v. One can solve it for E_v and substitute the result to Eq. (5). This can be rewritten as:

$$\frac{\sqrt{3}z}{2}\frac{d}{dx}\left(\bar{\kappa}\,T_h^{1/2}\frac{dT_h}{dx}\right) = \frac{RT_h^{3/2}}{\lambda} + \bar{b}\left(\frac{T_h}{E_v^{1/2}} + E_v^{1/2} - E_g^{1/2}\right), \qquad (7)$$

All quantities here are functions of the inverse density $\zeta = n_c/n$: $T_h = T_h(\zeta(x))$, $\lambda = \lambda(\zeta(x))$, $E_v = E_v(\zeta(x))$. Eq. (7) can be written as $(d/dx)(F(\zeta)\,d\zeta/dx) = \Phi(\zeta)$. Multiplication by $F(\zeta)(d\zeta/dx)$ and integration over x yields the modified area rule

$$\int_1^{z_g} F(\zeta)\Phi(\zeta)d\zeta = 0, \qquad (8)$$

which determines the conditions for coexistence of the gaseous phase ($\zeta = \zeta_g$) and the dense solid-like phase ($\zeta = 1$). This equation was solved numerically, the results are shown in Fig. 3. Figure 3 shows a phase diagram in the Γ (driving, vertical axis) - f (average area fraction, horizontal axis) plane, computed from Eq. (8) both for inelastic particle collisions ($R \neq 0$) and for elastic collisions between the particles ($R = 0$). There are three different regions in the diagram. When $\Gamma < \Gamma_1$, particles do not detach from the plate. For larger Γ, one can observe either a uniform gaseous state or a coexistence between the gaseous state and a dense solid-like condensate, depending on the average area fraction. Interestingly, this coexistence does not occur for sufficiently small values of f, $f < f_c$, this prediction can be tested experimentally.

The phase diagram shows that above some critical value of the vibration acceleration Γ, the phase coexistence is not possible for any average area fraction: the only possible state for large Γ is a gas phase. If we now slowly quench the vibration for $f > f_c$, a condensate will form, surrounded by the particles in a gas phase. The model suggests that for smaller f the phase coexistence occurs at a smaller threshold Γ, in a agreement with experimental observations [1, 19]. Surprisingly, the results do not significantly depend on the inelasticity of collisions between the particles; one does not need inelastic particle collisions to reproduce experimental observations. Instead, the phase separation mechanism is related to the nontrivial interplay between energy injection and the vertical temperature of the particles.

For inelastic particle collisions, there is an additional contribution to the heat flux from the density gradient term $Q \sim \mu\nabla n$ [23, 24]. The transport coefficient μ was computed for small and moderate densities, but the expression for high densities remains unknown. Below we heuristically compute this term in the spirit of Ref. [17]. Suppose, the temperature in the system is the same, but there is a density gradient ∇n in the x-direction. Consider the number of particles crossing a line perpendicular to this direction in a time interval δt. During this time a particle collides just once on average, in every collision it loses energy $(1 - r^2)\,v^2$. The energy flux is due to this single collision of every particle during this mean free path λ: the number of particles moving from left to right is not equal to the number of particles moving from right to left. The number of particles moving from right to left (or from left to right) per unit cross section area equals $n_{right}(\lambda + d)/2$ (or $n_{left}(\lambda + d)/2$). The differ-

ence between these two quantities should be multiplied by energy loss in one collision. Choosing $\delta t = \lambda/\sqrt{T}$ (the ratio of the mean free path to the thermal velocity) yields $Q = (0.5(\lambda+d)^2/\lambda)(1-r^2)\,T^{3/2}\,dn/dx$. Therefore, $\mu = \mu_0(r)[(\lambda+d)^2/\lambda]T^{3/2}$, where $\mu_0(r)$ is a function of r that becomes zero for elastic collisions. The factor $\mu_0(r)$ is chosen such that the expression for μ is consistent with the dilute limit [23]. Figure 3 shows that the effect of this term is small both due to relatively high restitution coefficient and due to the structure of the density gradient term: at high densities it is negligible compared to the temperature gradient term in the heat flux.

In conclusion, we demonstrated that the bistability and phase separation in vibrated granular monolayers observed experimentally in [1] can be captured in the framework of granular hydrodynamics equations. The bistability results from non-trivial energy injection mechanism from vibrating plate to a granular gas. The model, calibrated against available experimental data and simulations [1, 12, 19, 20] is in a good agreement with the observations: it yields, in particular, a phase diagram consistent with that of Ref. [1]. While we have studied phase separation in one-dimensional geometry, Eqs. (1)-(2) are also valid in two dimensions. In this case, however, the dynamics of the front between dense and dilute phase will also be controlled by the curvature, i.e. the radius of the cluster [25].

A phase separation in quasi-one-dimensional granular system was studied in Ref. [21, 22]. The phase separation was attributed to negative compressibility of granular gas in a certain range of concentrations. The experimental system considered in Ref. [22], a vibrated granular gas confined on a horizontal plate with a top lid, is different from ours: the most interesting regimes of phase separation in Ref. [22] occur for acceleration $\Gamma > 4$, when the motion of particles is strongly affected by collisions with the confining lid. In contrast, for these parameters our system will be in a uniform gas state. However, the fundamental mechanism of phase separation, negative compressibility of granular gas, is similar.

It is well-known that in large enough system phase separation results in the formation of a large number of clusters; the cluster consequently coarsen [25]. In a similar system, granular layer energized by electric field, the number of clusters decays inversely proportional to time [9–11]. We anticipate similar scaling behavior for vertically vibrated granular monolayers. However, due to relatively small aspect ratio and number of particles in the experimental cell, only a few clusters (typically 1-3) can be observed simultaneously [1].

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