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Nonlinear ion modes in a dense plasma with strongly coupled ions and degenerate electron fluids

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The properties of solitary and shock structures associated with nonlinear ion modes in a dense plasma with strongly coupled non-degenerate ions and degenerate electron fluids are presented. For this purpose, we have used the viscoelastic fluid model for the ions, inertialess electron momentum equation with weakly- and ultra-relativistic pressure laws for degenerate electron fluids, and Poisson's equation to derive the Burgers and K-dV equations. Possible stationary solutions of the latter are the shock and solitary structures, respectively. It is found that the speed, amplitude, and width of the shock and solitary waves critically depend on the strong coupling between ions and electron degeneracy effects. The relevance of our investigation to the role of localized excitations in dense astrophysical objects is briefly discussed.

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I. INTRODUCTION

In the past, a number of authors has been concerned with the study of matter under extreme conditions [1–6], which occur in compact astrophysical objects and planetary systems. Examples of the latter are white and brown dwarf stars [7–9], as well as massive Jupiter [10] (serving as the benchmark for giant planets), and super-Earth terrestrial planets around other stars [11]. White dwarf stars have low luminosity and high surface emissivity, with masses typically less than $1M_\odot$ and radii typically less than $10^{-2}R_\odot$. The average bulk densities of white dwarf stars are typically $\sim 10^{30}$ cm^{-3}. Astrophysical aspects of high density have been recently discussed by Fortov [12].

In high density plasmas (the particle number density of the order mentioned above), electron fluids are degenerate, and non-degenerate ions are strongly coupled since the ion Coulomb coupling parameter $\Gamma_i = Z_i e^2/a_i k_B T_i$ is larger than unity. Here, Z_i is the ion charge state, e the magnitude of the electron charge, $a_i = (3/4\pi n_i)^{1/3}$ the inter-ion spacing, n_i the ion number density, k_B the Boltzmann constant, and T_i the ion temperature. Since electrons at high densities are degenerate, one must use the Fermi-Dirac statistics to deduce the electron pressure. In his classical papers, Chandrasekhar [1, 3] presented a general expression for the relativistic electron degeneracy pressure $P_e = (\pi m_e c^5/3\hbar^3)\left[\alpha(2\alpha^2-3)(\alpha^2+1)^{1/2} + 3\sinh^{-1}\alpha\right]$, where m_e is the electron rest mass, c the speed of light in vacuum, $\hbar$ the Planck constant divided by 2π, and $\alpha = p_e/m_e c$, with $p_e = (3\hbar^2 n_e/8\pi)^{1/3}$ being the momentum of an electron on the Fermi surface. One can obtain explicitly expressions for P_e in the weakly and ultra-relativistic limits, which are characterized by $\alpha \ll 1$ and $\alpha \gg 1$, respectively. We have

$$P_e = K n_e^\gamma, \tag{1}$$

where

$$\gamma = \frac{5}{3};\quad K = \frac{3}{5}\left(\frac{\pi}{3}\right)^{\frac{1}{3}}\frac{\pi\hbar^2}{m_e} \simeq \frac{3}{5}L_c\hbar c \tag{2}$$

for the weakly-relativistic degenerate electron fluids (where $L_c = \pi\hbar/m_e c = 1.2\times 10^{-10}$ cm), and

$$\gamma = \frac{4}{3};\quad K = \frac{3}{4}\left(\frac{\pi^2}{9}\right)^{\frac{1}{3}}\hbar c \simeq \frac{3}{4}\hbar c \tag{3}$$

for the ultra-relativistic degenerate electron fluids. Here n_e is the electron number density.

Recently, several authors [13–22] have used the pressure laws (1) and (2) to investigate the linear and nonlinear properties of electrostatic and electromagnetic waves, by using the non-relativistic quantum hydrodynamic (QHD) [13] and quantum-magnetohydrodynamic (Q-MHD) [15] models and by assuming either immobile ions or non-degenerate uncorrelated mobile ions. It turns out that the presence of the latter and degenerate ultra-relativistic electrons with the pressure law (3) admits one-dimensional localized ion modes (IMs) supported by

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linear and nonlinear ion inertial forces and the pressure of degenerate electron fluids in a dense quantum plasma that is unmagnetized. Furthermore, modified Volkov solutions of the Dirac equation for electrostatic and electromagnetic waves in relativistic quantum plasmas have been discussed by Mendonça and Serbeto [23].

In the present paper, we study the properties of weakly nonlinear IMs in a dense quantum plasma composed of degenerate electron fluids and strongly coupled non-degenerate ion-fluids. To describe the dynamics of the IMs, we use the ion continuity and visco-elastic ion momentum equation (similar to those in Refs. [24–26]), as well as inertialess electron momentum equation with the pressure laws (2) and (3) and Poisson's equation. Strong ion coupling effects enter in the generalized ion momentum equation through the viscoelastic relaxation time for the decay of ion correlations and the bulk ion viscosity. By using the standard reductive perturbation technique [27, 28], we derive the Burgers and Kortweg-de Vries (K-dV) equations from the governing nonlinear equations for IMs. Stationary solutions of the Burgers and K-dV equations in the form of shock and solitary waves are presented. The effects of ion correlations and electron degeneracy on the speed, width and amplitude of both shocks and solitary waves are examined.

II. MODEL NONLINEAR EQUATIONS

We consider one-dimensional nonlinear propagation of electrostatic IMs associated with inertialess degenerate electron fluids and strongly coupled non-degenerate inertial ions in an unmagnetized dense plasma. The dynamics of nonlinear IMs in our plasma is governed by the momentum equation for inertialess degenerate electron fluids, given by,

$$0 = en_e\frac{\partial\phi}{\partial x} - \frac{\partial P_e}{\partial x}, \tag{4}$$

and the generalized viscoelastic ion hydrodynamic equations (similar to those in Refs. [24–26]) composed of the ion continuity and ion momentum equations

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0, \tag{5}$$

$$D_\tau\left[m_i n_i D_t u_i + Z_i e n_i\frac{\partial\phi}{\partial x} + k_B T_{ef}\frac{\partial n_i}{\partial x}\right] = \eta_l\frac{\partial^2 u_i}{\partial x^2}. \tag{6}$$

The equations are closed by Poisson's equation

$$\frac{\partial^2\phi}{\partial x^2} = 4\pi e\left(n_e - Z_i n_i\right), \tag{7}$$

where ϕ is the electrostatic potential, u_i the component of the ion fluid velocity along the x-axis in a Cartesian coordinate system, t and x are time and space variables, respectively, m_i the ion mass, and $T_{ef} = (\mu_i T_i + T_\star)$ the effective ion-temperature. The latter consists of two parts: one ($T_\star$) arising from the electrostatic interaction among strongly correlated positive ions, and the other ($\mu_i T_i$) arising from the ion-thermal pressure. We have denoted $D_\tau = 1 + \tau_m\left(\partial/\partial t + u_i\partial/\partial x\right)$, $D_t = \partial/\partial t + u_i\partial/\partial x$, τ_m is the viscoelastic ion relaxation time, μ_i the ion compressibility, and η_l represents the longitudinal ion viscosity coefficient. There are various approaches for calculating the ion transport coefficients, similar to those for one-component strongly coupled plasmas [25, 26, 29, 30]. The parameter $T_\star$ (which arises from the electrostatic interactions among strongly correlated positive ions), viscoelastic ion relaxation time τ_m, and the ion compressibility μ_i, for our purposes, are [25, 26, 30]

$$T_\star = \frac{N_{nn}}{3}\frac{Z_i^2 e^2}{a_i k_B}\left(1+\kappa\right)e^{-\kappa}, \tag{8}$$

$$\tau_m = \frac{\eta_l}{n_{i0}k_B T_i}\left[1 - \mu_i + \frac{4}{15}u(\Gamma)\right]^{-1}, \tag{9}$$

$$\mu_i = \frac{1}{k_B T_i}\frac{\partial P_i}{\partial n_i} = 1 + \frac{1}{3}u(\Gamma) + \frac{\Gamma_i}{9}\frac{\partial u(\Gamma)}{\partial \Gamma_i}, \tag{10}$$

where N_{nn} is determined by the ion structure, and corresponds to the number of nearest neighbors (viz. in crystalline state $N_{nn} = 8$ for bcc lattice, $N_{nn} = 12$ for fcc lattice, etc.), $\kappa = a_i/\lambda_D$, and λ_D is the Thomas-Fermi screening length, $u(\Gamma_i)$ is a measure of the excess internal energy of the system, and is calculated for weakly coupled plasmas ($\Gamma_i < 1$) as [26] $u(\Gamma_i) \simeq -(\sqrt{3}/2)\Gamma_i^{3/2}$. To express $u(\Gamma_i)$ in terms of Γ_i for a range of $1 < \Gamma_i < 100$, Slattery *et al.* [29] have analytically derived a relation

$$u(\Gamma_i) \simeq -0.89\Gamma_I + 0.95\Gamma_i^{1/4} + 0.19\Gamma_i^{-1/4} - 0.81, \tag{11}$$

where a small correction term due to finite number of particles has been neglected. The dependence of the other transport coefficient η_l on Γ_i is somewhat more complex, and cannot be expressed in such a closed analytical form. However, tabulated/graphical results of their functional behavior derived from molecular dynamic simulations, and a variety of statistical schemes are available in the literature [25].

It is obvious that (5)-(7) are coupled with the electron number density n_e, which can be deduced from (1) and (4) as

$$\begin{aligned} n_e &= n_{e0}\left[1 + \frac{(\gamma-1)e\phi}{\gamma K n_{e0}^{\gamma-1}}\right]^{\frac{1}{\gamma-1}} \\ &\simeq n_{e0}\left[1 + \frac{C_1}{1!}e\phi - \frac{C_2}{2!}(e\phi)^2 + \frac{C_3}{3!}(e\phi)^3 + \cdot\cdot\right], \end{aligned} \tag{12}$$

where $n_{e0} = Z_i n_{i0}$, and

$$C_1 = \frac{1}{\gamma K n_{e0}^{\gamma-1}},$$

$$C_2 = \frac{\gamma-2}{\gamma^2 K^2 n_{e0}^{2(\gamma-1)}},$$

$$C_3 = \frac{(\gamma-2)(2\gamma-3)}{\gamma^3 K^3 n_{e0}^{3(\gamma-1)}}.$$

Hence, n_e for the weakly and ultra-relativistic electron fluids can be obtained by substituting (2) and (3) into (12).

III. IM SHOCK

To derive a dynamical equation for the shock waves from Eqs. (5)−(7) and (12), we use the reductive perturbation technique [27], and the stretched coordinates [28]

$$\left.\begin{array}{l}\xi = \epsilon(x - V_p t),\\ \tau = \epsilon^2 t,\end{array}\right\} \tag{13}$$

where ϵ is a smallness parameter measuring the weakness of the dispersion, and V_p is the phase speed of the electrostatic IMs. We can expand the perturbed quantities n_i, u_i, and ϕ about their equilibrium values in power series of ϵ as

$$\left.\begin{array}{l}n_i = n_{i0} + \epsilon n_i^{(1)} + \epsilon^2 n_i^{(2)} + \cdots,\\ u_i = \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \cdots,\\ \phi = \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \cdots.\end{array}\right\} \tag{14}$$

We now use (12)−(14) into (5)−(7), and develop equations in various powers of ϵ. To the lowest order in ϵ, i.e. taking the coefficients of ϵ^2 from both sides of (5) and (6), and the coefficients of ϵ from both sides of (7) and (12), one obtains the first order continuity equation, momentum equation, and Poisson's equation which, in turn, give

$$u_i^{(1)} = \frac{Z_i e V_p \phi^{(1)}}{m_i V_p^2 - k_B T_{ef}}, \tag{15}$$

$$n_i^{(1)} = \frac{Z_i e n_{i0} \phi^{(1)}}{m_i V_p^2 - k_B T_{ef}}, \tag{16}$$

$$V_p = C_\hbar \sqrt{1 + \beta_T}, \tag{17}$$

where $\beta_T = k_B T_{ef}/Z_i \gamma K n_{e0}^{\gamma-1}$, and $C_\hbar = (Z_i \gamma K n_{e0}^{\gamma-1}/m_i)^{1/2}$ is the speed of the IMs in our dense plasma. We note that $\beta_T = a_e^2 k_B T_{ef}/Z_i L_c \hbar c$ and $\beta_T = a_e K_B T_{ef}/Z_i \hbar c$ for the weakly- and ultra-relativistic degenerate electron fluids, respectively. Equation (17) represents the linear dispersion relation for the IMs in which the restoring force comes from the pressure of degenerate electrons and the ion mass provides the inertia. It is obvious from (17) that the phase speed (V_p) is increased by T_{ef} (i.e. by the effective ion-temperature). It is also clear that for $\beta_T \ll 1$ (which is valid for both the weakly and ultra-relativistic limits), we have $V_p \sim C_\hbar$, which along with γ, K, β_h are provided in Table I. The expression for C_h, given in Table I, dictates that unlike the usual ion-acoustic speed in the usual non-degenerate electron-ion plasma, C_h (or V_p for $\beta_T \ll 1$) is independent of the electron-temperature, but it depends on the unperturbed electron number density, and it is directly proportional to $n_{e0}^{1/3}$, i.e. inversely proportional to inter-electron distance ($a_e = (3/4\pi n_{e0})^{1/3}$). It is also directly proportional to $\sqrt{Z_i}$, and inversely proportional to $\sqrt{m_i}$. To the next higher order in ϵ, i.e. taking the coefficients of ϵ^3 from both sides of (5) and (6), and the coefficients of ϵ^2 from both sides of (7) and (12), one obtains another set of coupled equations for $n_i^{(2)}$, $u_i^{(2)}$, and $\phi^{(2)}$, which along with the first set of coupled linear equations for $n_i^{(1)}$, $u_i^{(1)}$, and $\phi^{(1)}$, reduce to a nonlinear dynamical equation

TABLE I: The expressions for K, $C_h = V_p(T_{ef} = 0)$, and corresponding $\Phi_m^{(1)}(T_{ef} = 0)$ for the weakly-relativistic ($\gamma = 5/3$) and ultra-relativistic ($\gamma = 4/3$) limits:

γ	K	β_T	$C_\hbar$	$\Phi_m^{(1)}(T_{ef} = 0)$
$\frac{5}{3}$	$\frac{5}{3}L_c\hbar c$	$\frac{a_e^2 T_{ef}}{Z_i L_c \hbar c}$	$\left(\frac{Z_i L_c \hbar c}{a_e^2 m_i}\right)^{\frac{1}{2}}$	$\frac{3 m_i U_0 C_\hbar^{nr}}{4 Z_i e}$
$\frac{4}{3}$	$\frac{3}{4}\hbar c$	$\frac{a_e T_{ef}}{Z_i \hbar c}$	$\left(\frac{Z_i \hbar c}{a_e m_i}\right)^{\frac{1}{2}}$	$\frac{6 m_i U_0 C_\hbar^{ur}}{7 Z_i e}$

$$\frac{\partial\phi^{(1)}}{\partial\tau} + A\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi} = C\frac{\partial^2\phi^{(1)}}{\partial\xi^2}, \tag{18}$$

where the nonlinear coefficient A and the dissipation coefficient C are, respectively,

$$A = \frac{Z_i e}{2 m_i C_\hbar \sqrt{1 + \beta_T}}\left[1 + \gamma + (\gamma - 2)\beta_T\right], \tag{19}$$

$$C = \frac{\eta_l}{2 n_{i0} m_i}. \tag{20}$$

Equation (18) is the well-known Burgers equation describing the nonlinear propagation of the IMs in our dense plasma. It is obvious from (18) and (20) that the dissipative term, i.e. the right-hand side of (18) is due to the strong correlation among positive ions.

We now look for a stationary shock wave solution of (18), by introducing $\zeta = \xi - U_0\tau'$ and $\tau' = \tau$, where U_0 is the shock speed (in the reference frame). This leads us to write (18), under the steady state condition ($\partial/\partial\tau' = 0$), as

$$-U_0\frac{\partial\phi^{(1)}}{\partial\zeta} + A\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\zeta} = C\frac{\partial^2\phi^{(1)}}{\partial\zeta^2}. \tag{21}$$

It can be easily shown [31] that (21) describes the IM shock whose speed U_0 is related to the extreme values $\phi^{(1)}(-\infty)$ and $\phi^{(1)}(\infty)$ by $\phi^{(1)}(\infty) - \phi^{(1)}(-\infty) = 2U_0/A$. Thus, under the condition that $\phi^{(1)}$ is bounded at $\zeta = \pm\infty$, the IM shock solution of (21) is [31]

$$\phi^{(1)} = \frac{\Phi_m}{2}[1 - \tanh(\zeta/\Delta)], \tag{22}$$

where Φ_m and Δ are the height and width of the IM shock, respectively. We have

$$\Phi_m = \frac{2U_0}{A}, \tag{23}$$

$$\Delta = \frac{2C}{U_0}. \tag{24}$$

The IM shock arises due to the balance between nonlinearities associated with the generation of harmonics on account of IM couplings and strong correlations between the positive ions in our dense plasma. It is seen from (19), (20), and (23) that the IM shock height is independent of the longitudinal viscosity coefficient (η_l), but the shock thickness is directly proportional to the longitudinal viscosity coefficient η_l, (i.e. the IM shock thickness increases with the increase of the longitudinal ion viscosity coefficient), and is inversely proportional to the ion mass density ($\rho_{i0} = n_{i0}m_i$) and to U_0. It is also seen from (19) and (22) that for $\gamma < 2$ [which is satisfied for both the weakly-relativistic ($\gamma = 5/3$) and ultra-relativistic ($\gamma = 4/3$) limits], the nonlinear coefficient A decreases, i.e. the IM shock height increases (non-linearly and slowly) with the increase of T_{ef}. On the other hand, for $\beta_T \ll 1$, the expression for the shock height [$\phi_m^{(1)}$] for non-relativistic ($\gamma = 5/3$) and ultra-relativistic ($\gamma = 4/3$) limits are provided in Table I.

Table I shows that for $\beta_T \ll 1$, the IM shock height is i) directly proportional to U_0 and $\sqrt{m_i}$, but inversely proportional to $\sqrt{Z_i}$ in both the weakly- and ultra-relativistic limits; ii) directly proportional to $n_{e0}^{1/3}$ in the weakly-relativistic limit, but to $n_{e0}^{1/6}$ in the ultra-relativistic limit.

IV. SOLITARY IMS

To derive a dynamical equation for the solitary IMs from Eqs. (5)-(7) and (12), we again use the reductive perturbation technique [27], with another set of stretched coordinates

$$\left.\begin{aligned} \xi &= \epsilon^{1/2}(x - V_p t), \\ \tau &= \epsilon^{3/2} t. \end{aligned}\right\} \tag{25}$$

We now use (12),(14), and (25) in (5)−(7), and develop equations in various powers of ϵ. To the lowest order in ϵ, i.e. taking the coefficients of $\epsilon^{3/2}$ from both sides of (5) and (6), and the coefficients of ϵ from both sides of (7) and (12), we obtain the first order continuity equation, momentum equation, and Poisson's equation which, in turn, give a set of equations that are completely identical to the set of equations given by (17).

To the next higher order in ϵ, i.e. taking the coefficients of $\epsilon^{5/2}$ from both sides of (5) and (6), and the coefficients of ϵ^2 from both sides of (7) and (12), we obtain another set of coupled equations for $n_i^{(2)}$, $u_i^{(2)}$, and $\phi^{(2)}$, which, along with (17), reduce to a nonlinear dynamical equation

$$\frac{\partial \phi^{(1)}}{\partial \tau} + A\phi^{(1)}\frac{\partial \phi^{(1)}}{\partial \xi} + B\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = 0, \tag{26}$$

where the nonlinear coefficient A is given by (19), and the dispersion coefficient B is

$$B = \frac{C_\hbar^3(1+\beta_T)^{\frac{3}{2}}}{2\omega_{pi}^2} = \left(\frac{Z_i}{m_i}\right)^{\frac{1}{2}} \frac{\gamma^{\frac{3}{2}} K^{\frac{3}{2}} n_{e0}^{\frac{1}{2}(3\gamma-1)}}{8\pi e^2}. \tag{27}$$

Equation (26) is the well-known K-dV equation describing the nonlinear propagation of IMs in our dense plasma system.

The stationary solitary IM solution of the K-dV equation (26) is obtained by transforming the independent variables to $\zeta = \xi - U_0\tau'$ and $\tau' = \tau$, where U_0 is the speed of the solitary IM, and imposing the appropriate boundary conditions, viz. $\phi^{(1)} \to 0$, $d\phi^{(1)}/d\zeta \to 0$, $d^2\phi^{(1)}/d\zeta^2 \to 0$ at $\zeta \to \pm\infty$. Accordingly, the stationary solitary IM solution of the K-dV equation (26) is

$$\phi^{(1)} = \phi_m \mathrm{sech}^2(\zeta/\delta), \tag{28}$$

where ϕ_m and δ are the amplitude and the width of the solitary IM, respectively. We have

$$\phi_m = \frac{3U_0}{A}, \tag{29}$$

$$\delta = \sqrt{\frac{4B}{U_0}}. \tag{30}$$

It is obvious from (23) and (29) that $\Phi_m = 2U_0/A$ for the IM shock and $\phi_m = 3U_0/A$ for the solitary IM, i.e. $\phi_m/\Phi_m = 3/2$. This means that the solitary IM amplitude differs by an amount of 3/2 from the IM shock height, and that the variation of the solitary IM amplitude with Z_i, m_i, n_{e0}, and T_{ef} is exactly the same as that of the IM shock height.

It is easy to show from (2), (3), (27), and (30) that

$$\delta^{nr} = \left[\frac{Z_i L_c^3 \hbar^3 c^3}{m_i U_0^2 \pi^2 e^4}(1+\beta_T)^3\right]^{\frac{1}{4}} \tag{31}$$

for the weakly-relativistic limit ($\gamma = 5/3$), and

$$\delta^{ur} = \left[\frac{Z_i a_e^3 \hbar^3 c^3}{m_i U_0^2 \pi^2 e^4}(1+\beta_T)^3\right]^{\frac{1}{4}} \tag{32}$$

for the ultra-relativistic limit ($\gamma = 4/3$).

Equations (31) and (32) reveal that the width (δ) of the solitary IMs increases with the effective ion-temperature T_{ef}. Furthermore, we see from (31) and (32) that for $\beta_T \ll 11$, i) the width (δ) is directly proportional to $Z_i^{1/4}$, but inversely proportional to $m_i^{1/4}$ and $\sqrt{U_0}$, and ii) it is independent of the plasma number density in the weakly-relativistic limit, but is inversely proportional to $n_{e0}^{3/4}$ in the ultra-relativistic limit.

V. DISCUSSION AND CONCLUSION

In this paper, we have investigated the nonlinear propagation of electrostatic IMs in a dense plasma composed of inertialess degenerate electron fluids and strongly coupled non-degenerate strongly ion-fluids. By using the appropriate electron density response and the generalized hydrodynamic equations for ions [25, 26], together with Poisson's equation, we have derived the governing equations for nonlinear IMs. When the longitudinal ion viscosity, resulting from strong ion correlations, prevails over the dispersion arising from the charge separation, one finds that the dynamics of nonlinear IMs is governed by the Burgers equation. On the other hand, when the charge separation effects overwhelms the strong ion-coupling effects, the nonlinear IM dynamics is governed by the K-dV equation. Here the electron degeneracy plays a crucial role. Our main results are summarized as follows:

1. The phase speed of the nonlinear IMs, V_p (which is, in fact, the critical phase speed for which the solitary or shock IMs are formed) increases due to the effect of the effective ion-temperature T_{ef}.

2. Unlike the usual nonlinear ion-acoustic waves in a non-degenerate electron-ion plasma, C_h (which is, in fact, the phase speed for $\beta_T \ll 1$, i.e. $C_\hbar = V_p(T_{ef} = 0)$) is independent of the electron-temperature, but it depends on the electron number density, and is directly proportional to $n_{e0}^{1/3}$, i.e. inversely proportional to the inter-electron distance (a_e). However, like the usual nonlinear ion-acoustic waves in a non-degenerate electron-ion plasma, it is also directly proportional to $\sqrt{Z_i}$ and inversely proportional to $\sqrt{m_i}$.

3. The IM shock, which is due to the balance between nonlinearities and dissipation, exists with a positive potential only. The strong correlation among positive ions is the source of dissipation, and is responsible for the formation of the IM shock.

4. The IM shock height is independent of the longitudinal viscosity coefficient (η_l), but the IM shock thickness is directly proportional to the longitudinal viscosity coefficient η_l, and is inversely proportional to the ion mass density ($\rho_{i0} = n_{i0}m_i$).

5. The IM shock height is directly proportional to $\sqrt{m_i}$, but inversely proportional to $\sqrt{Z_i}$ in both the weakly- and ultra-relativistic limits. However, it is directly proportional to $n_{e0}^{1/3}$ (i.e inversely proportional to inter-electron distance a_e) in the weakly-relativistic limit, but to $n_{e0}^{1/6}$ (i.e inversely proportional to the inter-electron distance $\sqrt{a_e}$) in the ultra-relativistic limit.

6. The solitary IM, which is due to the balance between nonlinearities and dispersion, exists with a positive potential only. The solitary IM amplitude differs by an amount of 3/2 from the IM shock height, and that the variation of the solitary IM amplitude with Z_i, m_i, n_{e0}, and T_{ef} is exactly the same as that of the IM shock height.

7. The width (δ) of the solitary IM increases with the increase of T_{ef}. On the other hand, for $\beta_T \ll 1$, the width (δ) is directly proportional to $Z_i^{1/4}$, but inversely proportional to $m_i^{1/4}$ in both the weakly and ultra-relativistic limits. However, it is independent of the plasma number density in the weakly-relativistic limit, but is inversely proportional to $n_{e0}^{3/4}$ in the ultra-relativistic limit.

It is important to add here that we have followed Chandrasekhar [1, 2]) by assuming that the white dwarf core is pure He. However, recently Koester [32] assumes that the core may be pure carbon or pure oxygen. The important point to note here is that whatever the 'chemical' composition of the core (pure He, pure C, or pure O), for a given mass, the total number of nucleons (protons plus neutrons) would be the same, and since the number of electrons per nucleon would also be the same, namely 1/2 (He: 4 nucleons, 2 electrons; C: 12 nucleons, 6 electrons; O: 16 nucleons, 8 electrons), we would end up with the same value of $n_0 = n_{e0} = Z_i n_{i0}$ for all compositions. This means that our nonlinear theory for localized IMs is also valid for the recent assumption of Koester [32]. Our results have shown how the presence of the ions of heavier elements C or O (instead of He) can modify the basic features [viz. the speed, height, and thickness, which are expressed as function of m_i by (23), (24), (29), and (30)] of the IM shock and solitary waves that are formed in our dense plasma with degenerate inertialess electron fluids and strongly coupled non-degenerate inertial ion fluids.

It is also important to note here that when the dispersion (dissipation) effect is much more pronounced than the dissipation (dispersion) effect, and the dissipation (dispersion) effect is neglected, strongly coupled degenerate dense plasmas support solitary (shock) waves. To neglect the effect of the IM dispersion in comparison with that of the dissipation or vice-versa, one has to choose a suitable scaling (stretching of co-ordinates) that we used in our investigation of the IM shock and solitary waves. However, if it would be possible to keep both effects, i.e. possible to derive the K-dV-Burgers equation [31] by choosing appropriate stretched co-ordinates (which have not been found so far), one would have oscillatory IM shock in which the first few oscillations at the wavefront will be close to IM solitons [31].

The K-dV-Burgers equation has been derived by a number of authors, e. g. Shukla and Mamun [33], Pakzad and Javidan [34], etc., by using the stretched coordinates, defined by (25), along with an additional stretching of the viscosity coefficient, $\eta_l = \epsilon^{1/2}\eta_0$, in order to study the dust-acoustic [35] solitary and shock waves in a strongly coupled dusty plasma. However, the use of this additional stretching, $\eta_l = \epsilon^{1/2}\eta_0$, is not correct in general, at least from the mathematical points of view, since this additional stretching ($\eta_l = \epsilon^{1/2}\eta_0$) leads to the dissipation coefficient to contain the expansion parameter ϵ. So, to avoid this additional stretching ($\eta_l = \epsilon^{1/2}\eta_0$), in our present work, we used the stretched co-ordinates other than that used by Shukla and Mamun [33] or Pakzad and Javidan [34]. The Burgers equation has also been derived by a number of authors, e. g. Shukla [36], Rahman *et al.* [37], etc. by using the same additional stretching in order to study the dust-ion-acoustic [38] shock waves in a dusty plasma.

On the other hand, Pakzad [39] has derived the K-dV equation by using the stretched coordinates defined by (25), and the modified K-dV equation by using another set of stretched coordinates (viz. $\xi = \epsilon(x - V_p t)$, $\tau = \epsilon^3 t$), and has studied the ion-acoustic solitons [40] in a plasma containing warm ions (which are weakly-relativistic and weakly coupled) and electrons (which follow the Cairns distribution [40]), and positrons (which follow the Boltzmann distribution). It is, therefore, obvious that the plasma models (viz. dusty plasma [33–38] and nonthermal plasma [39, 40] models) as well as the scale and time lengths of the waves considered in all of these earlier investigations [33–40] are not valid for any degenerate plasma system, and are completely different from what we have considered in our present work (concerning waves in strongly coupled degenerate (quantum) plasma systems or compact objects like white dwarfs [1, 2]). We refer to the seminal works of Rao *et al.* [35], Shukla and Silin [38], and Cairns *et al.* [40] for the details of the dust-acoustic waves, dust-ion-acoustic waves, and nonthermal plasmas, respectively, which make clear how our present work (from view of its model, results, and possible applications) is completely different from the works of Pakzad and Javidan [34], Rahman *et al.* [37], and Pakzad [39], respectively.

To conclude, the results of the present investigation should be useful in understanding the salient features of localized IM excitation in dense (degenerate) plasmas, such as those in white dwarf stars. Specifically, from the phase speed of nonlinear IM structures, one may infer the composition and structural properties of massive stars in which the plasma particles are densely packed.

It may be noted here that the effects of nonplanar geometry and external magnetic field on the IM shock and solitary structures and their multidimensional instability are also problems of great importance, but beyond the scope of our present work.

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