Anisotropy and feedthrough in magneto-Rayleigh-Taylor instability

Y. Y. Lau, J. C. Zier, I. M. Rittersdorf, M. R. Weis, and R. M. Gilgenbach

Phys. Rev. E **83**, 066405 — Published 14 June 2011

DOI: 10.1103/PhysRevE.83.066405

April 15, 2011

Anisotropy and Feedthrough in Magneto-Rayleigh-Taylor Instability

Y. Y. Lau, J. C. Zier[1], I. M. Rittersdorf, M. R. Weis, and R. M. Gilgenbach

Department of Nuclear Engineering and Radiological Sciences, University of Michigan,

Ann Arbor, MI 48109-2104

ABSTRACT

The magneto-Rayleigh-Taylor instability (MRT) of a finite slab is studied analytically using the ideal MHD model. The slab may be accelerated by an arbitrary combination of magnetic pressure and fluid pressure, thus allowing an arbitrary degree of anisotropy intrinsic to the acceleration mechanism. The effect of feedthrough in the finite slab is also analyzed. The classical feedthrough solution obtained by Taylor in the limit of zero magnetic field, the single interface MRT solution of Chandrasekhar in the limit of infinite slab thickness, and Harris' stability condition on purely magnetic driven MRT, are all readily recovered in the analytic theory as limiting cases. In general, we find that MRT retains robust growth if it exists. However, feedthrough may be substantially reduced if there are magnetic fields on both sides of the slab, and if the MRT mode invokes bending of the magnetic field lines.

[1]Present Address: Naval Research Laboratory, Washington, DC 20375

I. INTRODUCTION

When a fluid slab of a finite width is accelerated in vacuum, one interface is subjected to the Rayleigh-Taylor instability (RT) while the other interface is stable, intuitively. How the perturbation on the unstable interface "feeds through" to the stable interface is an important issue. While this RT feedthrough was solved by Taylor in hydrodynamics [1] and used in laser fusion [2], it was hardly addressed when the slab is driven by a magnetic pressure, in which case the instability is known as the magneto-Rayleigh-Taylor instability (MRT) [3-8]. MRT is important to peta-watt pulsed-power system development, wire-array z-pinches and magnetized target fusion [8,9], and equation-of-state studies using flyer plates [10] or isentropic compression [11]. It is also important to the study of the crab nebulae [12]. MRT, because of the presence of magnetic fields, is necessarily anisotropic on the interface. This anisotropy is markedly different from the conventional RT, defined here to be free of magnetic field. Surprisingly, the fundamental question of MRT anisotropy on a *finite* fluid slab was also rarely studied analytically. In this paper, we treat MRT anisotropy and feedthrough with an ideal magnetohydrodyamic (MHD) model.

For a study of MRT feedthrough and anisotropy, one needs to go back to Harris' 1962 paper [3], which remains a key reference in recent literature [8]. This pioneering work of Harris, unfortunately, obscures the distinction between MRT and RT as far as anisotropy and feedthrough are concerned. One reason is that, early on, Harris eliminated the magnetic field in favor of gravity through the equilibrium condition. Thus, his stabilization condition for MRT, stated in his abstract, becomes independent of the magnetic field. This stabilization condition, at first sight, bears little resemblance to those developed by Kruskal and Schwartzschild [4] and by Chandrasekhar [5]. These prior works on MRT were not cited by Harris. When Harris considered the feedthrough factor, he concentrated only on the MHD mode which does not bend the magnetic field line. Not surprisingly, then, the feedthrough factor that he obtained is the same as that obtained for RT by Taylor [1], further obscuring the crucial distinction between RT and MRT. Here we vastly extend Harris' slab model. In the lab frame, the magnetized fluid slab may be accelerated by an arbitrary combination of fluid pressure or magnetic pressure. The classical results of Taylor [1], Chandrasekhar [5], and Harris [3] are all readily recovered as limiting cases. MRT anisotropy and feedthrough is analyzed in detail for a specific case that is relevant to our ongoing MRT experiment [13].

II. EQUILIBRIUM AND STABILITY

The model under study is shown in Fig. 1. It consists of three regions, I, II, and III. In the accelerated frame, the three regions are stationary. In this *rest frame of the interfaces*, we use the ideal MHD model. In each region, we assume that the fluid is incompressible, and is perfectly conducting. Thus we solve the equations: $\rho(\partial/\partial t + \mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \mathbf{J}\times\mathbf{B} - \rho g\mathbf{x}$, $\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{v}) = 0$, $\nabla\cdot\mathbf{v} = 0$, $\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{v}\times\mathbf{B})$, and $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$. Here, ρ is the mass density, **v** is the fluid velocity, p is the fluid pressure which is assumed to be isotropic, **J** is the current density, **B** is the magnetic field, g is the gravity in the negative x-direction [Fig. 1], **x** is the unit vector, and μ_0 is the free space permeability.

We assume that in equilibrium, *within each region* of Fig. 1,

$$\rho_0 = \text{constant},\ \mathbf{v}_0 = 0,\ \mathbf{B}_0 = \mathbf{z}B_0 = \text{constant},\ \mathbf{J_0} = 0\ , \tag{1}$$

and $p_0(x)$ is a linear function of x. The fluid pressure p_0, as well as the magnetic pressure $B_0^2/2\mu_0$, may suffer a discontinuity at the interfaces, x = α and x = β [Fig. 1]. The total pressure, $p_0 + B_0^2/2\mu_0$, is continuous at both interfaces. In addition, the weight in region II of thickness Δ = α - β, is supported by the pressure difference between the two interfaces, α and β,

$$g\rho_{02}\Delta = \left[p_I + \frac{B_{01}^2}{2\mu_0}\right] - \left[p_{III} + \frac{B_{03}^2}{2\mu_0}\right], \tag{2}$$

where p_I is the equilibrium fluid pressure at the top of region I, and p_{III} is the equilibrium fluid pressure at the bottom of region III. In the lab frame, the acceleration, which equals –g, may therefore be driven by an arbitrary mix of fluid pressure (p_I, p_{III}) or magnetic pressure (B_{01}, B_{03}), as long as the above equilibrium conditions are satisfied. If the magnetic field is discontinuous at an interface, there is a surface current, K_0, at that interface. At the interfaces α and β, the surface current is given by $K_{0\alpha} = (B_{02} - B_{03})/\mu_0$, $K_{0\beta} = (B_{01} - B_{02})/\mu_0$. When the interface α is very far away from the interface β, Δ becomes large and this reduces to the single interface problem treated by Kruskal and Schwartzschild [4] and by Chandrasekhar [5]; the equilibrium condition, Eq. (2), is to be interpreted accordingly. In our analysis, we assume that the gravity, g, is an independent, pre-assigned constant. The only requirement is that the equilibrium conditions are satisfied.

We next consider a small signal perturbation on the equilibrium of the 3-region geometry shown in Fig. 1. Within each region, I, II, or III, all perturbation quantities are assumed to be of the form $u_1(x)e^{i\omega t-ik_y y-ik_z z}$. We next follow Chandrasekhar [5] to solve the linearized MHD equations for each region, and match the various conditions at the lower interface, and then at the upper interface. This leads to a dispersion relation of the form, $\omega^4 - R\omega^2 + S = 0$, where R and S are functions of B_{01}, B_{02}, B_{03}, ρ_{01}, ρ_{02}, ρ_{03}, g, Δ, k_y, and k_z. The details will be given elsewhere. Here, we note that there are four modes of ω. They appear in positive and negative pairs, $\omega = \pm\omega_1$, and $\omega = \pm\omega_3$. We shall henceforth denote ω_1 the most unstable mode,

$$\omega_1^2 = \frac{R}{2} - \sqrt{\frac{R^2}{4} - S}\ . \tag{3}$$

We note that ω_3^2 is also given by Eq. (3) except that the minus sign in front of the square root is replaced by the plus sign. From the energy principle for ideal MHD, ω^2 is real [14], and therefore, $R^2/4 - S > 0$, always.

III. FEEDTHROUGH AND ANISOTROPY

We follow Taylor's approach [1] to examine the feedthrough factor by considering the temporal evolution of sinusoidal ripples at the lower and upper interfaces, respectively denoted as, $\xi_\beta(t)$ and $\xi_\alpha(t)$ [Fig. 1]. We assume that, at t = 0, $\xi_\beta(0) = \xi_{\beta 0}$, $\dot{\xi}_\beta(0) = 0$, $\xi_\alpha(0) = 0$, $\dot{\xi}_\alpha(0) = 0$, i.e., initially, the sinusoidal ripple at the interface at $x = \beta$ has an initial amplitude $\xi_{\beta 0}$, but zero initial velocity; whereas the interface at $x = \alpha$ is undisturbed. We find the following solution,

$$\xi_\beta(t) = p_1\cos(\omega_1 t) + p_3\cos(\omega_3 t)\,, \tag{4}$$

$$\xi_\alpha(t) = p_1 F(\omega_1)\cos(\omega_1 t) + p_3 F(\omega_3)\cos(\omega_3 t)\,, \tag{5}$$

where

$$p_1 = \frac{\xi_{\beta 0} F(\omega_3)}{F(\omega_3) - F(\omega_1)}, \qquad p_3 = -\frac{\xi_{\beta 0} F(\omega_1)}{F(\omega_3) - F(\omega_1)}. \tag{6}$$

Equations (4) and (5) clearly satisfy the initial conditions. In Eq. (5), $F(\omega_1)$ is the feedthrough factor from the β interface to the α interface for the modes $\omega = \pm\omega_1$, and $F(\omega_3)$ is the feedthrough factor from the β interface to the α interface for the modes $\omega = \pm\omega_3$. Below, we shall recover the familiar feedthrough factor $F(\omega_1) = e^{-k\Delta}$ for the unstable mode, $\omega_1 = -i(kg)^{1/2}$, that was obtained by Taylor [1] and Harris [3] in the appropriate limits.

We shall henceforth consider the special case where regions I and III are a vacuum, and region II is magnetic field free, i.e., $\rho_{01} = \rho_{03} = 0$, and $B_{02} = 0$ in Fig. 1. This case mimics the MRT experiments currently being conducted at the University of Michigan [13], and is also a generalization of Harris' model to nonzero B_{03}. For this case, we find,

$$R = (k_z^2 V_\ell^2 + k_z^2 V_u^2)\coth(k\Delta), \quad S = (k_z^2 V_\ell^2 - kg)(k_z^2 V_u^2 + kg), \tag{7}$$

$$F(\omega) = \cosh(k\Delta) + \frac{kg - k_z^2 V_\ell^2}{\omega^2}\sinh(k\Delta), \tag{8}$$

where $V_\ell = \sqrt{B_{01}^2 / \mu_0 \rho_{02}}$, $V_u = \sqrt{B_{03}^2 / \mu_0 \rho_{02}}$ and $k = (k_y^2 + k_z^2)^{1/2}$. Since $R \geq 0$ in (7), Eq. (3) shows that MRT exists if and only if $S < 0$, i.e., if and only if

$$kg > k_z^2 V_\ell^2. \tag{9}$$

[If $g < 0$, Eq. (7) shows that MRT exists if and only if $-kg > k_z^2 V_u^2$. This is expected because in this case, the direction of g is reversed (Fig. 1), so is the role of the upper and lower interface.] Note that the instability condition (9), as written, is independent of the slab width Δ, and therefore must be the same as the $\Delta \to \infty$ limit, which becomes the single interface MRT problem treated by Kruskal and Schwarzschild [4] and by Chandrasekhar [5].

As $\Delta \to \infty$, $\coth(k\Delta) = 1$. Substitution of (7) into Eq. (3) yields, $\omega_1^2 = -kg + k_z^2 V_\ell^2$, as the square root in Eq. (3) can now be completed. This is the classical, single interface MRT dispersion relation [4,5] for a *non*magnetized (but perfectly conducting) fluid of density ρ_{02}, supported against gravity $\mathbf{g} = -\mathbf{x}g$ by an external magnetic field B_{01} in vacuum from below ($\rho_{01} = 0$); and Eq. (9) is the condition for MRT excitation. Since $k = (k_y^2 + k_z^2)^{1/2}$, the magnetic field will have no effect on the MRT growth rate $(kg)^{1/2}$ if $k_z = 0$. This, of course, simply states the well-known fact that the magnetic field has an effect on MRT only when the unstable mode bends the magnetic field line, i.e, only when $\mathbf{k} \cdot \mathbf{B_0} \neq 0$, and this effect of magnetic tension is always stabilizing [6,14].

Taylor's classical feedthrough solution [1] can also be recovered in the conventional RT limit: $V_\ell = V_u = 0$. Then, we have R = 0, $S = -(kg)^2$ from Eq. (7). Equation (3) gives $\omega_1^2 = -kg$, and $\omega_3^2 = kg$, yielding $\omega_1 = -i(kg)^{1/2}$, and $\omega_3 = (kg)^{1/2}$. Then, $F(\omega_1) = e^{-k\Delta}$ and $F(\omega_3) = e^{k\Delta}$ from Eq. (8); and the feedthrough solutions (4), (5) are identical to Eqs. (27) and (28) of Taylor [1].

For Harris' model [3], we set $B_{03} = 0$ and $V_u = 0$. Harris assumes that the weight of region II is supported *solely* by the magnetic field B_{01} in the vacuum region I, i.e., $g\Delta = V_\ell^2/2$ [cf. Eq. (2)]. The condition (9) for MRT excitation then becomes $k - k_z^2(2\Delta) > 0$, which is the main result of Harris [3], stated in his Abstract, and displayed in his Eq. (36). The independence of g and B_{01}, and the explicit dependence on Δ, in Harris' condition is in sharp contrast to the condition of Eq. (9) that is derived in Refs. [4] and [5], which shows explicit dependence on g and B_{01}, but independence of Δ. Using $V_u = 0$ and $V_\ell^2 = 2g\Delta$ in Eq. (7), the eigenvalue equation, $\omega^4 - R\omega^2 + S = 0$, becomes Eq. (26) of Harris.

Having recovered the classical results of Taylor, Chandrasekhar, and Harris, we show in Figs. 2-4 the normalized MRT growth rate, $\bar{\gamma} = -\mathrm{Im}(\omega_1)/\sqrt{kg}$, and the feedthrough factor $F(\omega_1)$ as a function of kΔ at various combination of magnetic field effects according to Eqs. (3), (7) and (8). The magnetic field effects enter through the normalized magnetic tension, b_ℓ^2 and b_u^2, in the lower and upper region. They are defined as,

$$b_\ell^2 = k_z^2 V_\ell^2 / kg, \quad b_u^2 = k_z^2 V_u^2 / kg\,. \tag{10}$$

Once more, MRT exists if and only if $b_\ell^2 < 1$ [cf. Eq. (9)]. The dashed lines in Figs. 2-4 show the asymptotic dependence of the normalized growth rate and the feedthrough factors,

$$\bar{\gamma} \cong \sqrt{k\Delta} \times \sqrt{\frac{(1-b_\ell^2)(1+b_u^2)}{b_\ell^2 + b_u^2}}, \qquad k\Delta << (b_\ell^2 + b_u^2)\,, \tag{11a}$$

$$\bar{\gamma} \cong \sqrt{1-b_\ell^2}\,, \qquad k\Delta >> 1\,, \tag{11b}$$

$$F(\omega_1) \cong \frac{1-b_\ell^2}{1+b_u^2}, \qquad k\Delta \to 0\,, \tag{12a}$$

$$F(\omega_1) \cong e^{-k\Delta}\,\frac{2(1-b_\ell^2)}{2-b_\ell^2+b_u^2}, \qquad k\Delta >> 1\,. \tag{12b}$$

In Fig. 2, we fixed $b_u = 0$. We may take this case to be $B_{03} = 0$, which becomes the case studied by Harris [3]. The normalized growth rate as a function of kΔ is shown by the solid curves in Fig. (2a) for various values b_ℓ. For $b_\ell = 0$, (e.g., $k_z = 0$ with a nonzero B_{01}), the normalized growth rate is unity, i.e., $\gamma = (kg)^{1/2}$, and the feedthrough factor for this $b_\ell = 0$ case is simply $e^{-k\Delta}$, as already discussed and shown in Fig. (2b). As b_ℓ increases, the MRT growth rates decreases. When b_ℓ is as large as 0.99 (i.e., the magnetic tension reaches 99 percent of the gravity force as measured by kg), the MRT growth rates still exceed 10 percent of the growth rate at much smaller values of b_ℓ [Fig. (2a)]. However, the feedthrough factor is reduced tremendously as b_ℓ approaches unity [Fig. (2b)]. It is interesting to note that the asymptotic formulas for large kΔ are already very accurate when kΔ > 2.

In Fig. 3, we fixed $b_u = 0.5$. To be consistent with g pointing downward so that the lower interface is MRT unstable (Fig. 1), as we are now considering, we require $b_\ell \geq b_u$. Figure 3 shows the data for $b_\ell = 0.5$, 0.7, 0.9, and 0.99. In all cases, the MRT growth rates are essentially the same as the $b_u = 0$ cases, as Eq. (11b) shows that $\bar{\gamma}$ is independent of b_u. The feedthrough factor is somewhat reduced if $b_u > 0$. The underlying reason is that with a nonzero b_u, the upper interface is less likely to form a ripple because of the magnetic tension there. Thus, the MRT growth at the lower interface is less likely to be transmitted to the upper interface. The asymptotic formulas, Eqs. (11) and (12), readily provide a quantitative evaluation of the growth rates and feedthrough factors for both small and large kΔ.

In Fig. 4, we fixed $b_u = 0.9$. Thus, the magnetic tension in the upper region III is about 90 percent of the gravity force. In keeping with $b_\ell \geq b_u$, we show the data for $b_\ell = 0.9$, 0.922, 0.945, and 0.99. In all cases, the MRT growth rate remains significant, in some cases exceeding 40 per cent of $(kg)^{1/2}$ [Fig. (4a)]. However, the feedthrough factor is minimal, with value always less than 0.11 [Fig. (4b)].

IV. CONCLUDING REMARKS

Finally, for the general case shown in Fig. 1, our formulation confirms that there is no instability if g = 0, regardless of the values of B_{01}, B_{02}, B_{03}, ρ_{01}, ρ_{02}, ρ_{03}, Δ, k_y, and k_z according to Eq. (3) [subject, of course, to the equilibrium condition (2), and pressure balance across the interface at $x = \alpha$ and $x = \beta$ in equilibrium]. Thus, regardless of the surface current $K_{0\alpha}$ and $K_{0\beta}$ on the interfaces, there is no kink or sausage instability of a current-carrying Cartesian slab if g = 0, in sharp contrast to a current-carrying cylinder. This is also a well-known result in ideal MHD theory [14,15].

V. ACKNOWLEDGMENTS

We acknowledge fruitful discussions with Sasha Velikovich, Denis Colombant, Mike Cuneo, Steve Slutz, and Bruce Remington. This work was supported by DoE award number DE-SC0002590, NSF award number PHY 0903340, and by DoE through Sandia National Laboratories award numbers 240985 and 76822 to the University of Michigan. J. C. Z. was supported by an NPSC fellowship through Sandia.

References

1. G. I. Taylor, *Proc. Roy. Soc.* (*London*), **A201**, 192 (1950).
2. S. Atzeni and J. Meyer-Ter-Vehn, *The Physics of Inertial Fusion*, Oxford University Press, (NY, 2004), p. 253.
3. E.G. Harris, *Phys. Fluids* **5**, 1057 (1962).
4. M. Kruskal and M. Schwarzschild , *Proc. Roy. Soc.* (*London*), **A223**, 348 (1954).
5. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford U. Press (London, UK, 1961), p. 429.
6. D. D. Ryutov, M. S. Derzon, and M. K. Matzen, *Rev. Mod. Phys*. **72**, 167 (2000).
7. D. L. Peterson *et al*., *Phys. Plasmas* **6**, 2178 (1999).
8. D. B. Sinars *et al*., *Phys. Rev. Lett*. **105,** 185001 (2010).
9. M. K. Matzen *et al*., *Plasma Phys. Control. Fusion* **41**, A175 (1999); S. A. Slutz *et al*., *Phys. Plasmas* **17**, 056303 (2010).
10. R. W. Lemke *et al*., *Phys. Plasmas* **10**, 1867, and 1092 (2003).
11. J. Eggert, in 14th APS Topical Conference on Shock Compression of Condensed Matter (Baltimore, MD, August 2005) ; T.D. Pointon, et al., *2003 IEEE International Pulsed Power Conference*, Dallas, Texas (IEEE Cat. No. 03CH37272).
12. J. J. Hester *et al*., *Astrophys. J*. **456**, 225 (1996).
13. J. C. Zier *et al*., Bull. Am. Phys. Soc. vol. 55, no. 15, p. 87 (November, 2010).
14. See, e.g., P. M. Bellan, *Fundamentals of Plasma Physics*, Cambridge U. Press (Cambridge, UK, 2006), p. 349; also, Ref. 5, p. 599.
15. H. P. Furth, J. Killian, and M. N. Rosenbluth, *Phys. Fluids* **6**, 459 (1963).

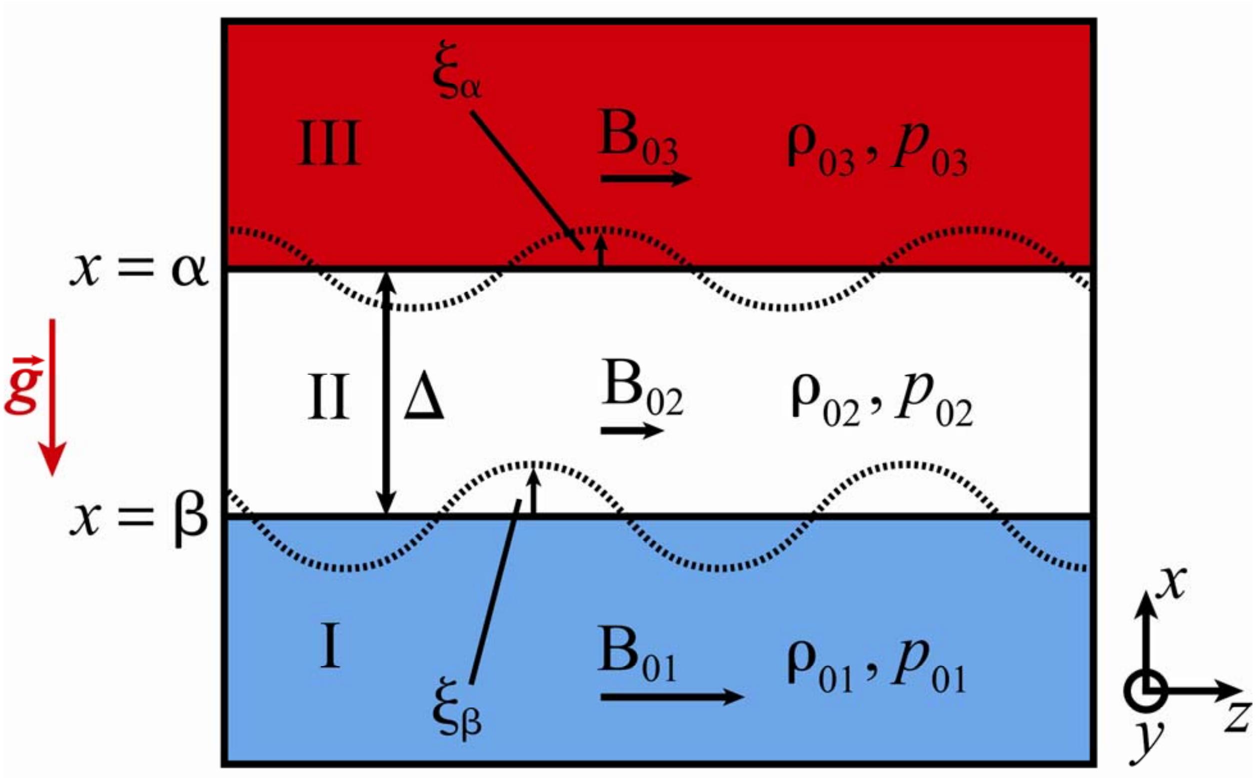
Figure Captions

Fig. 1. (Color online) MRT model with two interfaces at $x = \alpha$ and at $x = \beta$. This paper concentrates on the case $\rho_{01} = \rho_{03} = 0$, and $B_{02} = 0$. Other parameters, B_{01}, B_{03}, g, and Δ are arbitrary, allowing recovery of the classical results by Taylor, Chandrasekhar, and Harris as limiting cases.

Fig. 2. (Color online) (a) The normalized MRT growth rate and (b) the feedthrough factor for $b_u = 0$. This case is the same as Harris' model as $B_{03} = 0$. The dashed lines show the asymptotic formulas, Eqs. (11) and (12).

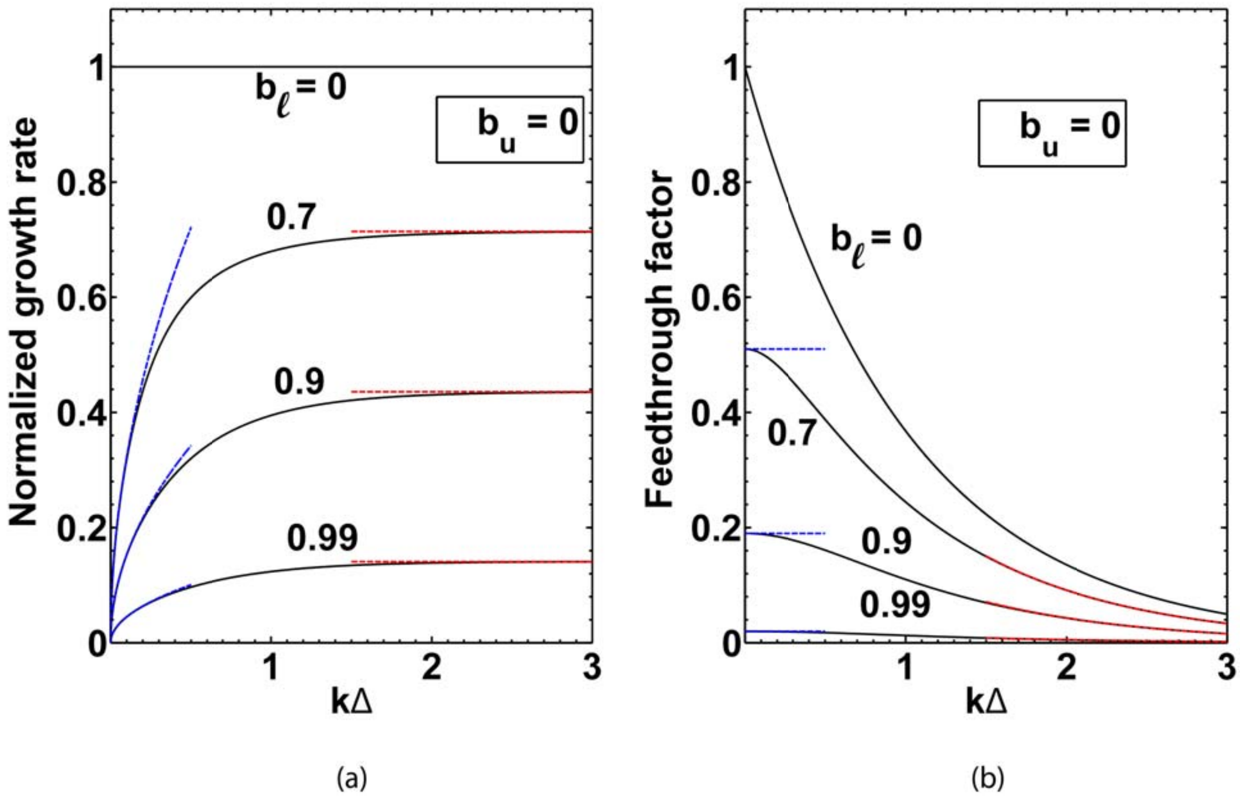
Fig. 3. (Color online) (a) The normalized MRT growth rate and (b) the feedthrough factor for $b_u = 0.5$. The dashed lines show the asymptotic formulas, Eqs. (11) and (12).

Fig. 4. (Color online) (a) The normalized MRT growth rate and (b) the feedthrough factor for $b_u = 0.9$. The dashed lines show the asymptotic formulas, Eqs. (11) and (12).



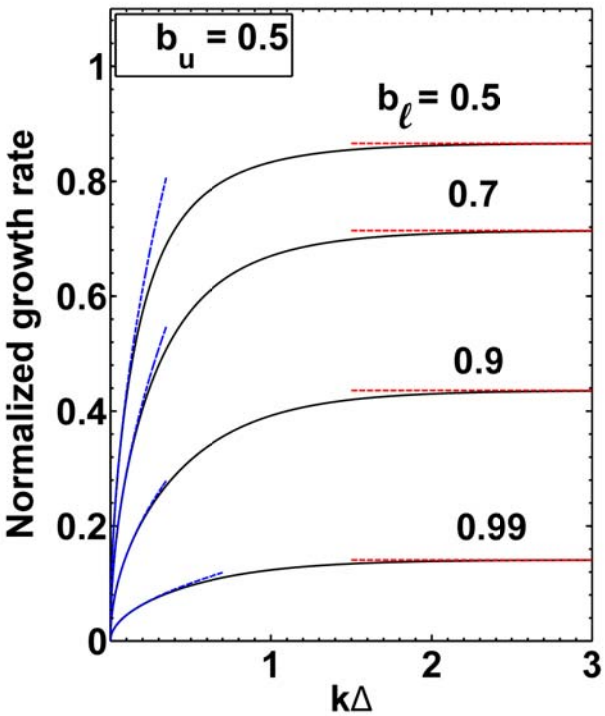
III
ξ_α
B_{03}
ρ_{03}, p_{03}
$x = \alpha$
$\vec{g}$
II
Δ
B_{02}
ρ_{02}, p_{02}
$x = \beta$
I
B_{01}
ρ_{01}, p_{01}
ξ_β
x
y
z

Fig. 1



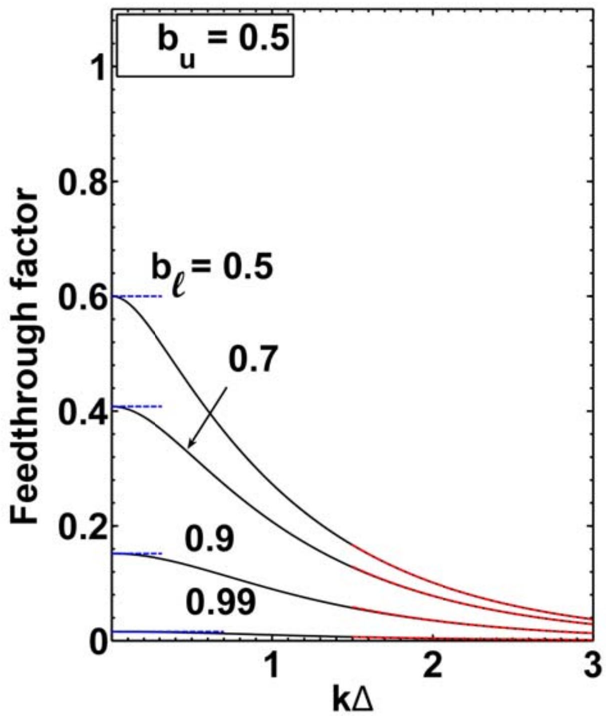
Normalized growth rate
kΔ
b_ℓ = 0
b_u = 0
0.7
0.9
0.99
Feedthrough factor
kΔ
b_ℓ = 0
b_u = 0
0.7
0.9
0.99
(a)
(b)

Fig. 2



b_u = 0.5
b_ℓ = 0.5
0.7
0.9
0.99
Normalized growth rate
kΔ

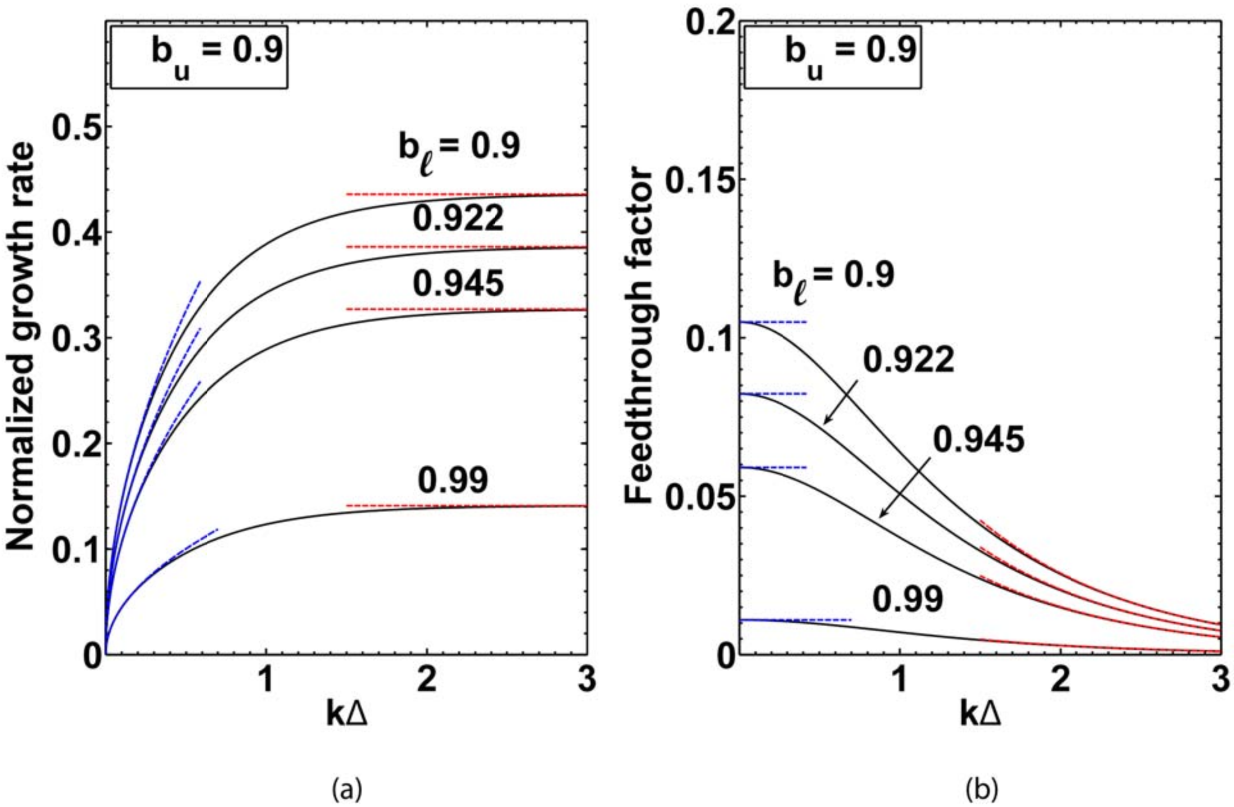
(a)



b_u = 0.5
b_ℓ = 0.5
0.7
0.9
0.99
Feedthrough factor
kΔ

(b)

Fig. 3



b_u = 0.9
b_ℓ = 0.9
0.922
0.945
0.99
Normalized growth rate
kΔ
(a)
b_u = 0.9
b_ℓ = 0.9
0.922
0.945
0.99
Feedthrough factor
kΔ
(b)

Fig. 4