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Stochastic Bifurcations in a Bistable Duffing-Van der Pol Oscillator with Colored Noise

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Abstract:

This paper aims to investigate Gaussian colored noise-induced stochastic bifurcations, and the dynamical influence of correlation time and noise intensity in a bistable Duffing-van der Pol oscillator. By using the stochastic averaging method, one can obtain the stationary probability density function of amplitude for the Duffing-van der Pol oscillator theoretically and reveal interesting dynamics under the influence of Gaussian colored noise. Stochastic bifurcations are discussed through a qualitative change of the stationary probability distribution, which indicates that system parameters, noise intensity and noise correlation time can be treated as bifurcation parameters respectively. They also imply that the effects of multiplicative noise are rather different from that of additive noise. The results of direct numerical simulation verify the effectiveness of the theoretical analysis. Moreover, the largest Lyapunov exponent calculations indicate that the P-bifurcation and D-bifurcation have no direct connection.

Key words: Gaussian colored noise, bistable Duffing-van der Pol oscillator, stochastic averaging method, stochastic bifurcations, and correlation time

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I. Introduction

Various physical, chemical, and biological processes can be modeled as nonlinear dynamical systems in which oscillatory motions are influenced by internal

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or external noise [1-3]. The investigation of the influence of random forces on dynamical behaviors, especially bifurcation phenomena, is one of the intensively developing research subjects [4-14]. However, the theory of stochastic bifurcations is still in its infancy [15]. There are few rigorous general theorems and criteria to detect stochastic bifurcations, which are often only verified by computer simulations, or for some particular models. In fact, it is much harder to deal with stochastic bifurcation problems than deterministic bifurcation problems. The definition of deterministic bifurcation is based upon the sudden change of topological properties of the phase portraits, while stochastic bifurcations may be characterized with a qualitative change of the stationary probability distribution, e.g., a transition from unimodal to bimodal distribution. At present, there are mainly two definitions for stochastic bifurcations. One is based on the sudden change of shape of the stationary probability density function—the so-called phenomenological (P)-bifurcation; and the other is based on the sudden change of sign of the largest Lyapunov exponent—the so-called dynamical (D)-bifurcation [15]. D-bifurcation is a dynamical concept, which is similar in nature to deterministic bifurcations, while P-bifurcation is a static concept. Unfortunately these two definitions do not agree well, and this means a new definition of stochastic bifurcation may be explored.

As we know, random noise may induce shift of the bifurcations with respect to different control parameter values compared to their deterministic counterparts. New types of dynamics can be found in the presence of random excitations, generally referred to the noise-induced effects. The Gaussian white noise in most theoretical studies is employed as the random driving force due to its mathematical simplicity, while realistic models of physical systems require considering colored noise. There has been a growing interest in the theoretical study of nonlinear dynamical systems subject to colored noise with finite correlation time scale [16-17]. It has been realized that colored noise gives rise to new intriguing effects such as the reentrant phenomenon in a noise-induced transition [18] and a resonant activation in bistable systems [19].

The Duffing-van der Pol oscillator is a prototypical system in modeling certain physical phenomena and its “simple” nonlinear structure has given rise to thorough studies of its dynamical behaviors [20,21]. The Gaussian white noise was firstly reported to create a purely noise-induced D-bifurcation with a single attractor in the Duffing-van der Pol system [12]. Stochastic bifurcation has been recently discussed for a self-sustained bistable Duffing-Van der Pol oscillator subject to additive Gaussian white noise in Ref. [13]. It is desirable to understand stochastic bifurcations in the bistable Duffing-Van der Pol oscillator driven by Gaussian colored noise.

In this paper, we explore effects of additive and multiplicative Gaussian colored noises on a bistable Duffing-Van der Pol oscillator. Furthermore, one can find the relation of stochastic bifurcation and noise correlation time on the dynamical properties. Based on the stochastic averaging method to separate fast and slow variables of the oscillator, the bifurcation analysis will be presented, taking the system parameters and statistical characteristics of noise (e.g., noise intensity and noise correlation time) as bifurcation parameters. Two types of qualitative changes are observed and bifurcation diagrams of the system in different parameter planes are presented. We find that the effects of multiplicative noise and that of additive noise are quite different (or not directly related).

This paper is organized as follows. In Section II, the stochastic averaging method is carried out to obtain the stationary probability density function of amplitude for the noisy Duffing-van der Pol oscillator theoretically. Then the stochastic P-bifurcations will be discussed in Section III. Here we analyze the influence of the noise correlation time and noise intensity on stochastic P-bifurcations in two cases of additive noise and multiplicative noises. Finally, Section IV is devoted to concluding remarks and discussions.

II. Stationary probability distribution of a bistable oscillator with Gaussian colored noise

In this section, we consider a bistable Duffing–Van der Pol oscillator with colored Gaussian noise:

$$\ddot{x}-(\varepsilon+\beta_1 x^2-\beta_2 x^4)\dot{x}+x+\beta_0 x^3=\eta(t)+x\xi(t), \quad \beta_i \geq 0, \tag{1}$$

where ε, β_0, β_1 and β_2 are real parameters(β_0 is a small parameter), while $\eta(t)$ and $\xi(t)$ are Gaussian colored noises with zero mean and correlation

$$\begin{aligned}
&\langle \eta(t)\eta(s)\rangle=\frac{D_1}{\tau_1}\exp\left[-\frac{|t-s|}{\tau_1}\right],\\
&\langle \xi(t)\xi(s)\rangle=\frac{D_2}{\tau_2}\exp\left[-\frac{|t-s|}{\tau_2}\right],\\
&\langle \eta(t)\xi(s)\rangle=0.
\end{aligned}\tag{2}$$

Here τ_1, τ_2 and D_1, D_2 denote the correlation time and intensity of the colored noises $\eta(t)$ and $\xi(t)$, respectively.

In the deterministic case ($D_1 = D_2 = 0$), when $-\frac{\beta_1}{8\beta_2} < \varepsilon < 0$, the system Eq.(1) is characterized with a bistable behavior: two attractors are in the phase plane: a stable focus at the origin and a stable limit cycle, as Fig.1 shows. Thus, the bistability region is restricted to a saddle-node bifurcation of cycles at $\varepsilon = -\beta_1 / 8\beta_2$, and a subcritical Andronov-Hopf bifurcation at $\varepsilon = 0$. Furthermore, the parameter β_0 defines the anisochronicity of oscillations: for $\beta_0 = 0$ the nonisochronicity of system Eq.(1) is quite small.

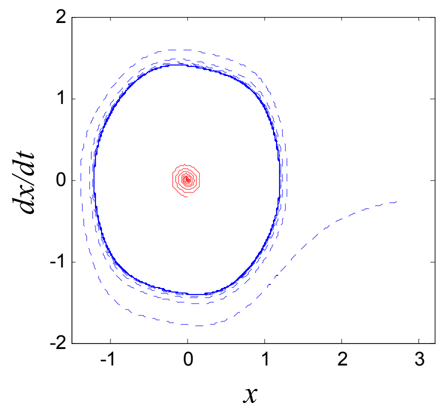

Fig.1. (Color online) Two attractors of system Eq.(1) for *D=0* when $\varepsilon = -0.11, \beta_1 = \beta_2 = 1.0$.

When $D_1 \neq 0$ and/or $D_2 \neq 0$, we assume that the noise intensity is small and introduce a change of variables,

$$x(t) = a\cos\theta, \dot{x}(t) = -a\sin\theta, \ \ \theta = t + \varphi. \tag{3}$$

Substituting Eq.(3) into Eq.(1), we can obtain

$$\begin{cases} \dot{a} = a\sin^2\theta(\varepsilon + \beta_1 a^2\cos^2\theta - \beta_2 a^4\cos^4\theta) + \beta_0 a^3\cos^3\theta\sin\theta - \sin\theta\eta(t) - a\sin\theta\cos\theta\xi(t) \\ \dot{\varphi} = \sin\theta\cos\theta(\varepsilon + \beta_1 a^2\cos^2\theta - \beta_2 a^4\cos^4\theta) + \beta_0 a^2\cos^4\theta - \frac{\cos\theta}{a}\eta(t) - \cos^2\theta\xi(t) \end{cases} \tag{4}$$

Apply the stochastic averaging method [22-23], we can obtain the following pair of stochastic equations for the amplitude $a(t)$ and phase $\varphi(t)$:

$$\begin{cases} da = \left[\dfrac{\varepsilon a}{2} + \dfrac{\beta_1 a^3}{8} - \dfrac{\beta_2 a^5}{16} + \dfrac{3D_2 a}{8(1+4\tau_2^{\,2})} + \dfrac{D_1}{2a(1+\tau_1^{\,2})} \right] dt + \sqrt{\dfrac{D_1}{1+\tau_1^{\,2}} + \dfrac{D_2 a^2}{4(1+4\tau_2^{\,2})}}\, dW_1(t), \\ d\varphi = \left(\dfrac{3\beta_0 a^2}{8} - \dfrac{D_2 \tau_2}{2(1+4\tau_2^{\,2})} \right) dt + \sqrt{\dfrac{D_1}{(1+\tau_1^{\,2})a^2} + \dfrac{2D_2\tau_2^{\,2}}{1+4\tau_2^{\,2}} + \dfrac{3D_2}{4(1+4\tau_2^{\,2})}}\, dW_2(t), \end{cases} \tag{5}$$

where $W_1(t)$ and $W_2(t)$ represents independent normalized Wiener processes. Clearly, da does not depend on φ, thus we can develop a probability density for a, rather than a joint density for a and φ.

The probability density function $p(a,t|a_0,t_0)$ for amplitude is governed by the Fokker-Planck-Kolmogorov equation,

$$\begin{aligned} \frac{\partial p}{\partial t} = &-\frac{\partial}{\partial a}\left[\left(\frac{\varepsilon a}{2} + \frac{\beta_1 a^3}{8} - \frac{\beta_2 a^5}{16} + \frac{3D_2 a}{8(1+4\tau_2^{\,2})} + \frac{D_1}{2a(1+\tau_1^{\,2})} \right) p \right] \\ &+ \frac{1}{2}\left(\frac{D_1}{(1+\tau_1^{\,2})} + \frac{D_2 a^2}{4(1+4\tau_2^{\,2})} \right) \frac{\partial^2 p}{\partial a^2}. \end{aligned} \tag{6}$$

Let $\dfrac{\partial P(a,t)}{\partial t} = 0$, according to Zhu [22], the stationary solution of Eq.(6) is

$$p(a) = \frac{N}{B(a)} \exp\left[2\int \frac{A(a)}{B(a)} da \right], \tag{7}$$

where

$$\begin{aligned} A(a) &= \left(\frac{\varepsilon a}{2} + \frac{\beta_1 a^3}{8} - \frac{\beta_2 a^5}{16} + \frac{3D_2 a}{8(1+4\tau_2^{\,2})} + \frac{D_1}{2a(1+\tau_1^{\,2})} \right), \\ B(a) &= \left(\frac{D_1}{(1+\tau_1^{\,2})} + \frac{D_2 a^2}{4(1+4\tau_2^{\,2})} \right), \end{aligned} \tag{8}$$

where N is a normalization constant.

III. Stochastic bifurcations

This section is devoted to discussing stochastic bifurcations through qualitative changes of the stationary probability density $p(a)$. The exact probability densities are presented in the case of additive noise, and the case of combined multiplicative noise and additive noise, respectively. The number and the extrema of the stationary densities have been carefully examined.

A. The case of additive colored noise

We first consider the system (1) with only additive colored noise with $D_2 = 0, D_1 \neq 0$. By Eq.(7) and Eq.(8), we get

$$p(a) = N(1+\tau_1^{\ 2})a\exp\left[\frac{1+\tau_1^{\ 2}}{48D_1}(24\varepsilon a^2 + 3\beta_1 a^4 - \beta_2 a^6)\right], \tag{9}$$

where N is a normalization constant.

In the limit $\tau_1 \to 0$, the colored noise $\eta(t)$ tends to a white noise, and this case was discussed in [13]. From Eq.(9), we find that the shape of amplitude in Eq.(5) does not depend on the phase φ and system parameter β_0.

Moreover, let $\frac{\partial p(a)}{\partial a} = 0$, the extrema of the distribution Eq.(9) are the roots of

$$a_m^{\ 6} - 2\frac{\beta_1}{\beta_2}a_m^{\ 4} - \frac{8\varepsilon}{\beta_2}a_m^{\ 2} - \frac{8D_1}{\beta_2(1+\tau_1^{\ 2})} = 0, \tag{10}$$

where a_m is the amplitude corresponding to the extremum of distribution Eq. (9) and m is the index number of the extremum. The number of real roots of Eq.(10) is either one or three for different parameters, which represents the unimodal distribution and the bimodal distribution of the amplitude, respectively. This effect means a type of stochastic bifurcation will take place. It is necessary to note that the transitions between the unimodal and the bimodal stationary probability density are also referred to as the noise-induced transitions, and stochastic bifurcation discussed here is closely connected to the noise induced transition [24].

In the parameter plane of D_1 and τ_1, Fig.2 (a) displays the bifurcation graph of the system (5) from the analysis of Eq.(10) with parameters $\varepsilon = -0.14$, and $\beta_1 = \beta_2 = 1.0$, The stationary amplitude distribution is bimodal in the tinted region and unimodal in the colourless region. The lines l_1 and l_2 represent the appearance and disappearance of one of the maxima of $p(a)$ which bounds the region corresponding to stochastic P-bifurcation. Increasing τ_1, the bimodality region will shift to larger values of D_1 and will become wider. The numerical solutions of the oscillator Eq.(1) could be obtained by order-2 stochastic Runge-Kutta algorithm [25] with the initial conditions $t_0 = 0, x(0) = 0.2, \dot{x}(0) = 0.1$, take the parameter $\beta_0 = 0.1$ and the time step $\Delta t = 0.01$ in numerical calculations. As $a(t) = \sqrt{x(t)^2 + \dot{x}(t)^2}$, then the stationary probability density function $p(a)$ can be obtained by Monte-Carlo simulation method with simulation data length $N = 10^7$. With parameters $\tau_1 = 0.5, \varepsilon = -0.14, \beta_1 = \beta_2 = 1.0$ we demonstrate the figure of stationary probability density for amplitude versus different noise intensity D_1 in Fig.2 (b). One can observe that the amplitude distribution has only one maximum situated in the vicinity of zero when the noise intensity is small. As $D_1 \approx 0.0182$ (see a point A in Fig.2 (a)), a transition from a unimodal to a bimodal distribution occurs, and for $D_1 \approx 0.03$ (the point B in Fig.2 (a)) the second stochastic bifurcation will appear. The amplitude distribution becomes unimodal again, but its maximum is shifted toward larger amplitude values, as curve 3 in Fig.2 (b) depicts.

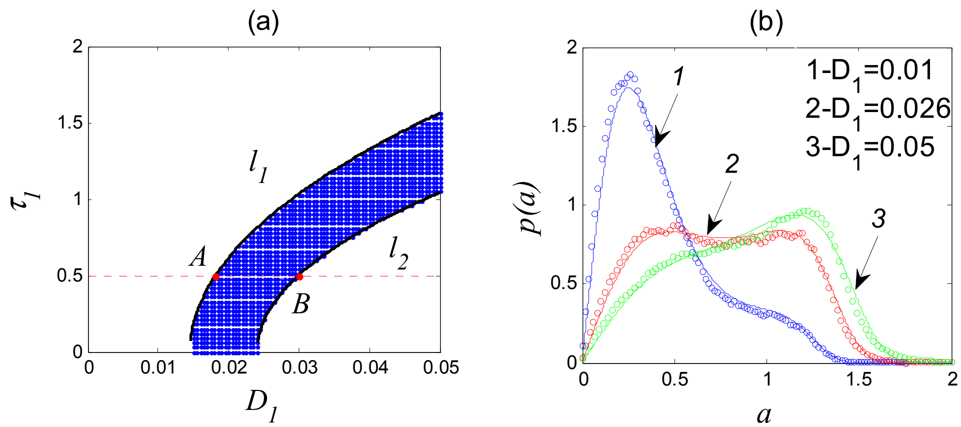

Fig.2. (Color online) Stochastic *P*-bifurcations in the Duffing–Van der Pol oscillator [Eq.(1) and Eq.(5)]. (a). Bifurcation diagram of the system (5) in the parameter plane (D_1, τ_1) for $\varepsilon = -0.14, \beta_1 = \beta_2 = 1.0$. Point A and point B are intersection points of the horizontal dash line $\tau_1 = 0.5$ and l_1, l_2, respectively. (b). Stationary probability density of amplitude for $\tau_1 = 0.5, \varepsilon = -0.14$, $\beta_1 = \beta_2 = 1.0$ and different values of noise intensity. The solid lines denote the algebraic calculations using formula Eq.(9), whereas the normalization constant N is defined numerically; The circles represent the numerical solutions for the oscillator Eq.(1).

Additionally, fix $D_1 = 0.05$, $\beta_1 = \beta_2 = 1.0$, we consider the influence of noise correlation time on stochastic P-bifurcation. The bifurcation diagram of the system (5) in the parameter plane (ε, τ_1) is given in Fig.3 (a). Similar discussions as Fig.2 (a) in previous words, the stationary amplitude distribution is bimodal in the tinted region and unimodal in the gap region. The lines l_3 and l_4 are boundaries of the tinted region, which mean stochastic P-bifurcations. Decreasing the value of ε, the bimodal region shifts to smaller values of τ_1 and becomes narrow. If ε is further decreasing (e.g., for $\varepsilon < \varepsilon_1 \approx -0.155$), then the bimodality region does not exist any more and P-bifurcation can not be observed for any correlation time. For the fixed noise intensity $D_1 = 0.05$ and parameter $\varepsilon = -0.11$, Fig.3 (b) shows the Stationary probability density of amplitude with different values of correlation time τ_1 . In this case, there are two attractors in the deterministic system, see Fig.1. For small correlation time (below the point C in Fig.3(a)), the amplitude distribution has only one peak, as curve 1 in Fig.3(b) shows, and for $\tau_c \approx 1.72$, a transition from a unimodal to a bimodal distribution occurs which can be found in curve 2 of Fig.3(b). Additionally, it could be worth noticing that the Stationary probability density $p(a)$ keeps bimodal as τ_1 increases ($\tau_1 > \tau_c$). However, value of the peak correspond to lager amplitude become very small if the correlation time is large (e.g., for $\tau_1 > 6.0$), as curve 3 in Fig.3(b) shows. The phase trajectory visits more and more frequently the regions close to the origin, and the nonlinearity of the system becomes weak.

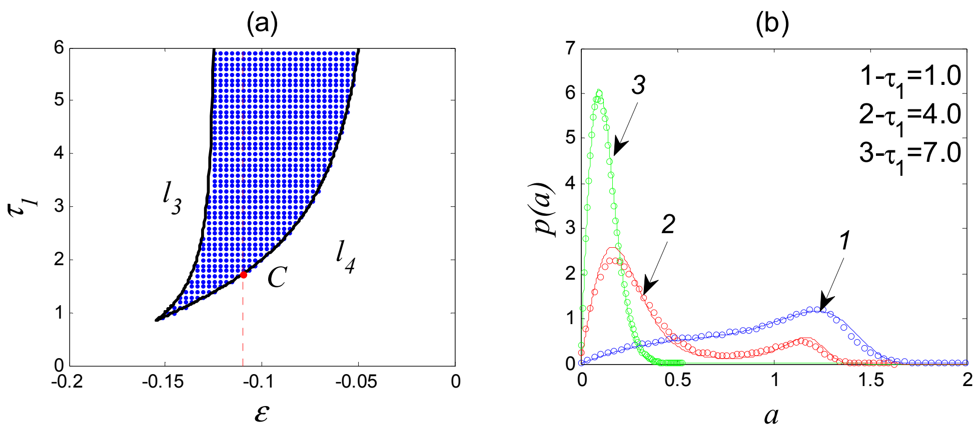

Fig.3. (Color online) Stochastic *P*-bifurcations in the Duffing–Van der Pol oscillator [Eq.(1) and Eq.(5)] for $D_1 = 0.05, \beta_1 = \beta_2 = 1.0$. (a). Bifurcation diagram of the system (5) in the parameter plane (ε, τ_1) . Point *C* is the intersection point of the vertical dash line $\varepsilon = -0.11$ and l_4 . (b). Stationary probability density of amplitude for $\varepsilon = -0.11, D_1 = 0.05$ and different values of correlation time. The solid lines and the circles have the same meanings as Fig.2 (b).

B. The case of multiplicative and additive colored noises

With $D_1 = 0, D_2 \neq 0$ The random noisy oscillator will be reduced to a Duffing–Van der Pol system excited by the multiplicative noise, and the stationary probability density function due to Eq.(7) for amplitude can be obtained as

$$p(a) = Na^{1+\frac{\varepsilon}{L}} \exp\left[\frac{4\beta_1 a^2 - \beta_2 a^4}{32L}\right], \tag{11}$$

where $L = \dfrac{D_2}{4(1+4\tau_2^{\ 2})}$, and N is a normalization constant.

Let $\dot{p}(a) = 0$, the extrema of the distribution Eq.(11) are the roots of

$$a_m^{\ 4} - \frac{2\beta_1}{\beta_2} a_m^{\ 2} - \frac{8(L+\varepsilon)}{\beta_2} = 0. \tag{12}$$

The real positive root of Eq.(12) is $\sqrt{\dfrac{\beta_1}{\beta_2} + \sqrt{\dfrac{\beta_1^2}{\beta_2^{\ 2}} + \dfrac{8(L+\varepsilon)}{\beta_2}}}$ for $\varepsilon > -L$, and then the probability density function in Eq.(11) has a maximum [curve 2 in Fig.4(a)]. With $\varepsilon \le -L$ there are two real positive roots of Eq.(12), and the probability density $p(a)$ has the minimum and maximum respectively, whose shape is similar to a crater, as curve 1 in Fig.4(a). Thus, a transition from a crater-like density to a unimodal density are observed, which can be defined as a type of P-bifurcations and this is completely different from the case of additive noise. It should be noticed that Eq.(11) is an singular integral which is singular at $a = 0$ for $\varepsilon < -L$. However, on the basis of the convergence criterion of singular integral, we can find Eq.(11) is integrable in the condition of $\varepsilon > -2L$. For $\varepsilon = -0.01$ the bifurcation diagram of the system (5) in the parameter plane (τ_2, D_2) is given in Fig.4 (b), where in region *II*, the stationary amplitude is crater-like distribution, and region *I* represents the unimodal distribution. The line l_5 denoted the boundary of region *I* and region *II* corresponding to stochastic P-bifurcations.

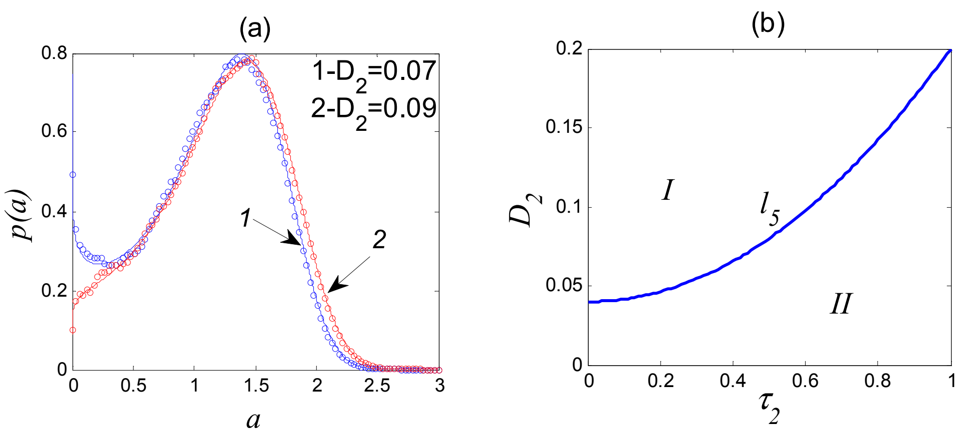

Fig.4. (Color online) Stochastic *P*-bifurcations in the Duffing–Van der Pol oscillator excite by multiplicative noise for $\varepsilon = -0.01$. (a). Stationary probability density for $\tau_2 = 0.5, \beta_1 = \beta_2 = 0.1$

and different D_2. (b). Bifurcation diagram of the system (5) in the parameter plane (τ_2, D_2). The line l_5 is the boundary of region *I* and region *II*.

If $D_1 \neq 0, D_2 \neq 0$, system (1) will be driven by a combination of multiplicative and additive colored noises.

According to Eq.(7) and Eq.(8), we have

$$p(a) = Na(K + La^2)^Q \exp\left[(4\beta_1 La^2 + 2\beta_2 Ka^2 - \beta_2 La^4)/32L^2\right], \tag{13}$$

where $K = \dfrac{D_1}{1+\tau_1^2}, L = \dfrac{D_2}{4(1+4\tau_2^2)}, Q = (8\varepsilon L^2 - 2\beta_1 KL - \beta_2 K^2)/16L^3.$

Similarly, let $\dot{p}(a) = 0$, the extrema of the distribution Eq.(13) are the roots of the equation

$$a_m^6 - ha_m^4 - ma_m^2 - n = 0, \tag{14}$$

with $h = \dfrac{2\beta_1}{\beta_2}, m = \dfrac{16QL^3 + 8L^3 + 2\beta_1 KL + \beta_2 K^2}{\beta_2 L^2}, n = \dfrac{8K}{\beta_2}.$

Taking $\varepsilon = -0.01, \beta_1 = \beta_2 = 0.1, D_1 = 0.01, D_2 = 0.01, \tau_1 = 5.0, \tau_2 = 1.0$, the roots of Eq.(14) are 0.215, 0.67,1.23, which represent corresponding amplitude of maximum, minimum and maximum of the stationary probability density in Eq.(13), respectively, as curve 1 in Fig.5(a) shows. The curve 2 in Fig.5 (a) is unimodal for $D_2 = 0.1$ and other parameters are the same as curve 1. Thus, there will be a stochastic P-bifurcation when the multiplicative noise intensity D_2 increases from 0.01 to 0.1. Fix $\varepsilon = -0.01, \beta_1 = \beta_2 = 0.05, D_1 = 0.005, D_2 = 0.06, \tau_1 = 2.0$, the stationary amplitude distribution in Eq.(13) with different τ_2 are shown in Fig.5(b), where $p(a)$ is unimodal for $\tau_2 = 0.5$, see curve 1 in Fig.5(b). As τ_2 increases to 1.0, the stationary probability density function will be bimodal, it turns to unimodal if $\tau_2 = 1.5$, as shown in Fig.5(b) curve 2&3. In other words, twice stochastic P-bifurcations take place when multiplicative noise correlation time τ_2 increases. Therefore, the type of stochastic P-bifurcations induced by multiplicative noise will vary when the system is excited by additive noise as well. For instance, a transition between crater-like density and a unimodal density will be changed as a transition between unimodal density and a bimodal density. Moreover, according to the number of the real roots of Eq.(14), bifurcation diagrams in different planes can be obtained but we omit here.

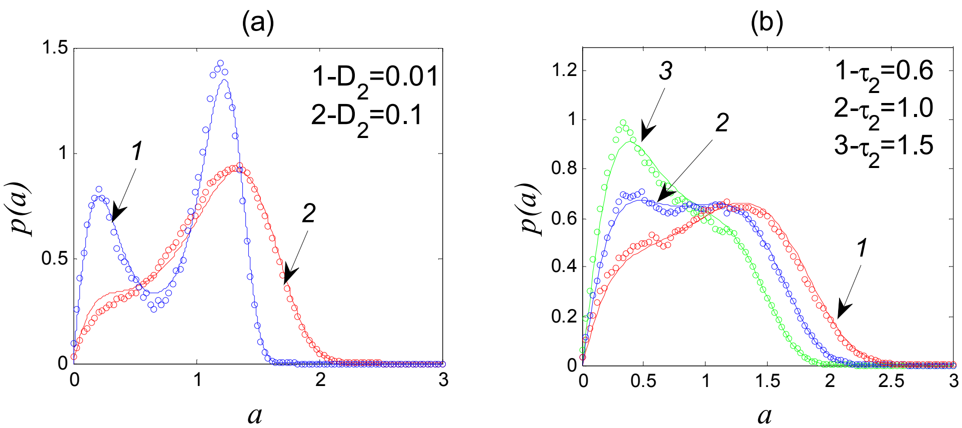

Fig.5.(Color online) Stationary probability density for $\varepsilon = -0.01$ and appropriate parameters. The solid lines denote the algebraic calculations using formula Eq.(13), whereas the normalization constant N is defined numerically; The circles represent the numerical solutions for the oscillator Eq.(1) by the same algorithm as Fig.1(b). (a) $\beta_1 = \beta_2 = 0.1, D_1 = 0.01, \tau_1 = 5.0, \tau_2 = 1.0$, different values of D_2;(b) $\beta_1 = \beta_2 = 0.05, D_1 = 0.005, D_2 = 0.06, \tau_1 = 2.0$, different values of τ_2.

It is worthy to note that the averaged model Eq.(5) does not reflect the properties of the original system completely for large values of noise intensity and correlation time. However, it can be found that the analytical solutions agree well with the numerically results from figures we presented in this paper. Additionally, the bifurcation diagrams are related not only to the averaged model, but also to the original system and do not depend on the parameter of anisochronicity $\beta_0 \geq 0$. Here we point out that one can find appropriate parameter ranges for $\varepsilon, \beta_1, \beta_2, D_1, D_2, \tau_1, \tau_2$ from expressions of the stationary probability density $p(a)$ (see Eq.(9), Eq.(11), Eq.(13)).

Now we apply the numerical algorithm in [26,27] to calculate the largest Lyapunov exponent of the initial system Eq.(1) and the averaged system Eq.(5) numerically for the additive noise case with $\varepsilon = -0.14, \beta_1 = \beta_2 = 1.0$, which are shown in Fig.6. The figures show the top Lyapunov exponent λ_1 remains negative for any $(D_1, \tau_1) \in (0 \sim 0.05, 0 \sim 4.0)$, so the stochastic bifurcation can not be found based on the sudden change of sign of the largest Lyapunov exponent. Additionally, P-bifurcation is nearly independent from β_0, while for D-bifurcation this parameter is crucial, i.e. P-bifurcation is not necessarily accompanied by D- bifurcation.

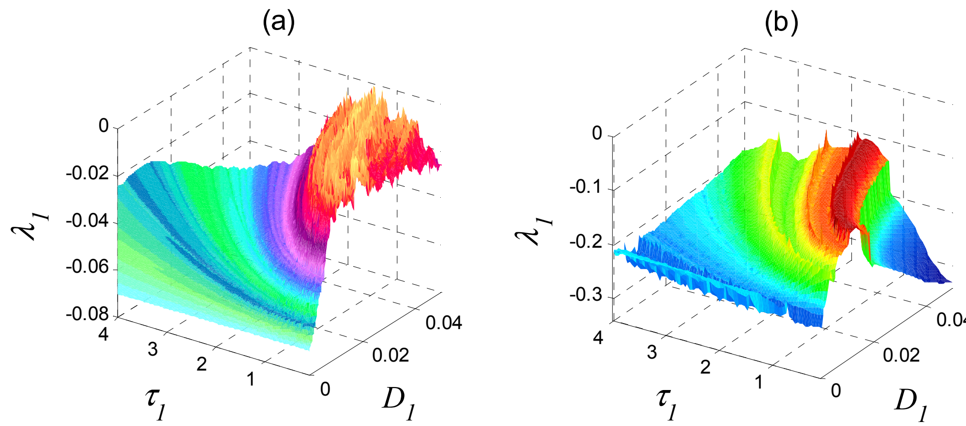

Fig.6. (Color online) The largest Lyapunov exponent λ_1 in the (D_1, τ_1) plane of the additive noise case for $\varepsilon = -0.14, \beta_1 = \beta_2 = 1.0$, respond to Fig.2(a). (a) The largest Lyapunov exponent of the original system Eq.(1) ; (b) The largest Lyapunov exponent of the averaged system Eq.(5).

IV. Concluding remarks

In this paper, we have presented results about stochastic bifurcations in a self-sustained bistable Duffing-Van der Pol oscillator with additive and/or colored noise. By applying a method of stochastic averaging based on a perturbation technique, we obtain the stationary probability density function of amplitude for the noisy oscillator. Two types of qualitative change are found, namely, a transition from unimodal density to a bimodal density and a transition from crater-like density to a unimodal density. The stochastic bifurcations based on the qualitative change of stationary measures are observed by discussing the extrema of the distribution. Bifurcation diagrams of the system in various parameter planes are obtained, and from which we point out that not only system parameters and noise intensity can be treated as the bifurcation parameter, but also the change of noise correlation time could induce stochastic bifurcations. Besides, the investigations show the effects of the multiplicative noise are different from that of additive noise. In addition, the D-bifurcation via the change of the largest Lyapunov exponent appears not to agree well with the results of P-bifurcation, and we remark that there is no direct connection between these two stochastic bifurcations.

Acknowledgments

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