Fluctuation-dissipation relation for nonlinear Langevin equations

V. Kumaran

Phys. Rev. E **83**, 041126 — Published 25 April 2011
DOI: 10.1103/PhysRevE.83.041126

Fluctuation-dissipation relation for non-linear Langevin equations.

V. Kumaran
Department of Chemical Engineering, Indian Institute of Science, Bangalore 560 012, India.

It is shown that the fluctuation-dissipation theorem is satisfied by the solutions of a general set of non-linear Langevin equations with a quadratic free-energy functional (constant susceptibility) and field-dependent kinetic coefficients, provided the kinetic coefficients satisfy the Onsager reciprocal relations for the irreversible terms and the antisymmetry relations for the reversible terms. The analysis employs a perturbation expansion of the non-linear terms, and a functional integral calculation of the correlation and response functions, and it is shown that the fluctuations-dissipation relation is satisfied at each order in the expansion.

PACS numbers: 05.20.Jj, 05.70.Ln

I. INTRODUCTION

Non-linear Langevin equations are encountered in applications such as the mode-coupling theories for supercooled liquids[1, 2], complex and polymeric fluids[3, 4], dynamical critical phenomena[5] and in numerous other applications. Suppressing the position and time (or frequency and wave vector) dependence of the fields, the non-linear Langevin equations have the general form,

$$\frac{\partial \psi_i}{\partial t} = -\Theta_{ij}(\{\psi\})\frac{\delta F}{\delta \psi_j} - \Gamma_{ij}(\{\psi\})\frac{\delta F}{\delta \psi_j} + \xi_i, \tag{1}$$

where ψ_i $(i = 1, N)$ are the fields which are space- and time-dependent, F is variously referred to as the free energy functional or the entropy function, and is usually written in the form (suppressing position and time dependence again)

$$F = \frac{1}{2}\sum_{i,j} \psi_i \chi_{ij}^{-1}(\{\psi\})\psi_j, \tag{2}$$

and the inverse of the susceptibility matrix χ_{ij}^{-1} is symmetric. In equation 1, the first term on the right is the 'reversible' part where the matrix Θ_{ij} is antisymmetric (Hermitian in the case of complex functions), and the second term is the 'irreversible' part where the matrix Γ_{ij} is symmetric. The last term on the right side of equation 1 is the noise, which is assumed to be a random process with Gaussian distribution whose correlation is a delta function in time. The variance of the noise distribution is related to the coefficient Γ_{ij} in the irreversible part of the evolution equation via the fluctuation-dissipation theorem.

The Langevin equation 1 is linear if the coefficients Θ_{ij} and Γ_{ij}, and the susceptibility matrix χ_{ij}, are independent of the fields ψ_i. In this case, the fluctuation-dissipation theorem is satisfied if the noise correlation is related to the kinetic coefficient Γ_{ij} in the irreversible part of equation 1,

$$\langle \xi_i(t)\xi_j(t')\rangle = 2k_B T\Gamma_{ij}\delta(t-t'). \tag{3}$$

where k_B is the Boltzmann constant and T is the absolute temperature. In non-linear Langevin equations, one source of non-linearity is the dependence of the coefficients Θ_{ij} and Γ_{ij} and the random noise correlations on the ψ fields. The second source is the dependence of the susceptibility χ_{ij} on the field variables. These are correctly formulated only if, at equilibrium, the fluctuation-dissipation relation between the kinetic coefficients and the noise correlations are satisfied in these equations.

One way to derive the macroscopic Langevin equations is to use 'coarse-graining' of the microscopic equations for all the particles in the system, using the projection-operator technique, for example. Since the microscopic equations satisfy all conservation laws, it is expected that the macroscopic field equations will also satisfy the fluctuation-dissipation theorem. However, strong approximations are usually made in the coarse-graining procedure in order to arrive at a tractable set of field equations, and it is not clear whether the macroscopic field equations also satisfy the fluctuation-dissipation relations at equilibrium. Due to this, it is important to have a framework to independently demonstrate the relation between the correlation function and the time derivative of the response function.

The non-linear Langevin equations are difficult to solve, in general, analytically or numerically. The evaluation of correlation and response functions in perturbation expansions of the non-linear terms was facilitated by the development of the Martin-Siggia-Rose[6] (MSR) formalism. This permits us to evaluate renormalisations of the correlation and response functions in a perturbative manner. In practical calculations, approximate solutions are obtained by truncating the perturbation expansions at some order in the expansion (usually one-loop order). In these calculations, it is important to demonstrate that the fluctuation-dissipation theorem is valid upto the order of truncation in the expansion, since one is not concerned about terms that are neglected. However, more fundamental questions are whether the fluctuation-dissipation relation is valid upto all orders in the expansion, and whether the relation between the response function and the time derivative of the correlation function is valid at each order in the expansion. Here, we will examine these issues using a functional integral formalism.

The MSR formalism was first used by Deker and Haake[7], to prove the fluctuation-dissipation theorem is valid at all orders in the perturbation expansion for certain restricted classes of non-linear Langevin equations. It was shown that the relations between the MSR correlation and response functions are satisfied to all orders in perturbation theory for three specific classes of problems, 'class A' where the Θ_{ij} is zero and Γ_{ij} is field-independent while χ_{ij} could be field-dependent, 'class B' where Θ_{ij} is non-zero and both Γ_{ij} and χ_{ij} are field-independent, and 'class C' for Hamiltonian systems without an irreversible part. There have been many subsequent studies showing the validity of the FDT relations[8–11] for a specific non-linear Langevin equation, particularly those for interacting Brownian particles[12, 13].

The MSR formalism was used by Miyazaki and Reichman[14] to show that the FDT relation is valid to one-loop order when χ_{ij}^{-1} in equation 2 is field-independent, and the coefficient Γ_{ij} has a contribution linear in the fields ψ_i. The authors realised the difficulty in extending this to field-dependent χ_{ij}, and observed that fluctuation-dissipation relations may not be valid at each order in the loop expansion in this case. Andreanov et al[15] showed that it is important to satisfy time-reversal symmetries to preserve the fluctuation-dissipation theorem. The authors considered the particular case of the fluctuating non-linear hydrodynamics equations[1], and the equations for interacting Brownian particles[12, 13]. In all of these cases, the focus has been on examining whether a particular set of non-linear Langevin equation, derived on the basis of physical considerations, satisfies the fluctuation-dissipation relations.

It is also of interest to ask the complementary question, that is, what is the form of non-linearities in coupled non-linear Langevin equations which will ensure that fluctuation-dissipation relations are preserved at equilibrium. In the case of conservative non-linearities proportional to Θ_{ij} in equation 1, the derivation is usually on the basis of the Poisson-bracket relations (see, for example, the model H equations[5]), and consequently these can be shown to satisfy fluctuation-dissipation relations quite easily. It is more difficult to solve equations where the kinetic coefficients Γ_{ij} and Θ_{ij} and the susceptibility χ_{ij} are field-dependent. Moreover, the noise correlations in equation 2 are also field dependent in cases where the kinetic coefficients Γ_{ij} are field-dependent, and this introduces more complications as discussed below.

Here, we use the functional integral formalism[16] in order to examine relations between correlation and response functions in general non-linear Langevin equations. This approach turns out to be simpler and more natural for analysing diagrammatic perturbation expansions in comparison to the classical MSR approach[6, 17, 18]. In this approach, conjugate 'hatted' fields are defined in a manner very similar to that in the MSR approach, but there are small differences in the physical interpretation of unhatted and hatted fields. The MSR approach was used by Miyazaki and Reichmann[14] to build upon the earlier work of Deker and Haake[7], and they showed that the fluctuation-dissipation relation is satisfied at one-loop order for the case of the linear dependence of the kinetic coefficients on the fields. In an earlier work[3], we examined the relations between the correlation functions for the hatted and unhatted fields using this formalism, and it was shown that relations between the functional integral correlation and response functions are satisfied to all orders in perturbations theory for dissipative non-linearities. Though not subsequently cited in this context, this work[3] preceded, and was more general than that of Andreanov et al[15] and Miyazaki and Reichmann[14], because the relation between the functional integral correlation and response functions were proved for a general non-linear Langevin equation with multiple fields and with no restriction on the exponents in the non-linear terms. In contrast, the proofs of Miyazaki and Reichmann[14] were restricted to one-loop expansions. Though the proof Andreanov et al[15] was valid at all orders in the perturbation expansion, it was restricted to a non-linearity in the form of a three-leg vertex for a non-linear Langevin equation containing only one field; this enabled the authors to prove the correlation-response relations at increasing orders in the perturbation expansion using induction.

Here, we build upon these analyses and prove the fluctuation-dissipation relations for the case of reversible and dissipative non-linearities. It is shown that the fluctuation-dissipation relations are identically satisfied, at each order in the perturbation expansion, when the kinetic coefficients Γ_{ij} and Θ_{ij} are field-dependent and the susceptibility χ_{ij} is field-independent. In the opposite case, where Γ_{ij} and Θ_{ij} are field-independent and χ_{ij} are field-dependent, it is quite an easy exercise to show that the fluctuation-dissipation relations are satisfied. The more complicated case is where Γ_{ij}, Θ_{ij} and χ_{ij} are field-dependent; it is almost certain that the fluctuation-dissipation relations are not valid at each order in the perturbation expansion in this case.

In equations where the kinetic coefficients Γ_{ij} are dependent on the field variables, there is the 'Ito-Stratonovich' paradox[19, 20] in the interpretation of the noise correlations. If the random noise in equation 1 is a delta function in time, the field variable ψ_i is a step function. Therefore, there is ambiguity whether the value of the function ψ_i to be used in the kinetic coefficient Γ_{ij} is before the step change (Ito calculus), after the step change, or the average of the two (Stratonovich calculus). The Stratonovich calculus is useful for writing down the Fokker-Planck analogue of the Langevin equation, since it is possible to use the Novikov theorem. Since we use the functional integral formalism[16] to relate the correlation and response functions, it is more convenient to use the Ito formulation because the Jacobian in the functional integrals are constants. It should also be noted that the form of equation 1 changes when the interpretation of noise correlations is changed, and we will use a form which is consistent with the Ito formulation which is slightly different from equation 1. This form is determined from the condition that the averages of the fields ψ_i are zero at equilibrium, in order to remove any gauge ambiguities.

II. NON-LINEAR LANGEVIN EQUATIONS

We use a quadratic free-energy functional,

$$F(\{\psi\}) = \sum_{i,j} \int_{\mathbf{k}} \int_{\mathbf{k}'} \psi_i(-\mathbf{k}) \chi_{ij}^{-1}(\mathbf{k}, \mathbf{k}') \psi_j(-\mathbf{k}'), \tag{4}$$

The equilibrium probability distribution is given by,

$$P_E(\{\psi\}) = Z_E^{-1} \exp\left(-\beta F(\{\psi\})\right), \tag{5}$$

where the equilibrium partition function Z_E is,

$$Z_E = \int_{\psi} \exp\left(-\beta F(\{\psi\})\right), \tag{6}$$

where $\int_{\psi} \equiv \prod_i \int D[\psi_i]$ is the functional integral over the ψ fields. The fields $\{\psi\}$ are defined to have zero equilibrium averages,

$$\begin{aligned} \langle \psi_i \rangle^{\text{eq}} &= Z_E^{-1} \int_{\psi} \psi_i \exp\left(-\beta F\right) \\ &= 0, \end{aligned} \tag{7}$$

where the notation $\langle \cdot \rangle^{\text{eq}}$ is used for equilibrium averages as defined above, to distinguish them from dynamical averages defined a little later. The equilibrium correlation function is given by,

$$\langle \psi_i(\mathbf{k}) \psi_j(\mathbf{k}') \rangle^{\text{eq}} = Z_E^{-1} \int_{\psi} \psi_i \psi_j \exp\left(-\beta F\right) = (\chi(\mathbf{k}))_{ij}^{-1} \delta(\mathbf{k} + \mathbf{k}'). \tag{8}$$

In the reminder of the analysis, we will set $\beta = (k_B T)^{-1} = 1$ without loss of generality, since the susceptibility can always be scaled by $k_B T$. Here k_B is the Boltzmann constant and T is the absolute temperature.

It should be noted that in the free energy functional equation 4, all terms should necessarily have even time parity. That is, all terms in the equation should remain unchanged under time reversal (when the direction of time is reversed), even though the fields ψ_i could, in general, have either odd time parity (sign of field changes when direction of time is reversed) or even time parity (sign of field remains unchanged when direction of time is reversed). One important implication of the even time parity of the free energy functional is that all terms in the quadratic approximation, equation 4, should contain products of fields with the same time parity.

The general expression for the non-linear Langevin equation for the variables ψ_i is,

$$\begin{aligned} \frac{\partial \psi_i(\mathbf{x})}{\partial t} &= -\int_{\mathbf{x}'} \sum_j \left(\Gamma_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t) \frac{\delta F}{\delta \psi_j(\mathbf{x}')} - \frac{\delta \Gamma_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t)}{\delta \psi_j(\mathbf{x}', t)} \right) \\ &\quad - \int_{\mathbf{x}'} \sum_j \left(\Theta_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t) \frac{\delta F}{\delta \psi_j(\mathbf{x}')} \right) + G_i(\{\psi\}, \mathbf{x}) \theta(t). \end{aligned} \tag{9}$$

In the above equations, the kinetic coefficients $\Gamma_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t)$ and $\Theta_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t)$ are functions of the field variables $\psi_i(\mathbf{x}, t)$. These coefficients also depend on time through the time dependence of the field variables, as indicated in equation 9. The second term in the first integral on the right side of equation 9 is required, in the non-linear Langevin equation, to ensure that the relaxation rate, averaged over the equilibrium realisations of the field variables, is zero when the average values of the field variables are zero. In the absence of this term, there will be a non-zero relaxation rate even at equilibrium when the field variables have zero average. The Onsager reciprocal relations require that,

$$\Gamma_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t) = \Gamma_{ji}(\{\psi\}, \mathbf{x}', \mathbf{x}, t). \tag{10}$$

The transport coefficients are local if the value of Γ_{ij} at the position $\mathbf{x}$ depends only on the field variables at $\mathbf{x}$. However, we also account for the possibility that the transport coefficients are non-local, so that the value of the coefficient at $\mathbf{x}$ depends on the fields at other locations. The case of local transport coefficients is a special case of the the more general formulation considered here.

The 'reversible' nonlinearities on the right side of equation 9, proportional to $\Theta_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t)$, arise from the Poisson bracket relations in the microscopic equations. The convective term in the convection-diffusion equation, $(\nabla.(\mathbf{v}c))$ (where $\mathbf{v}$ is the velocity and c is the concentration), as well as the reciprocal terms in the model H equations[5] for the concentration field, are examples of reversible non-linearities. These are antisymmetric,

$$\Theta_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t) = -\Theta_{ji}(\{\psi\}, \mathbf{x}', \mathbf{x}, t). \tag{11}$$

In addition, these terms have opposite time parity to the field ψ_i, that is, the term $\Theta_{ij}(\delta F/\delta\psi_j)$ reverses sign on time reversal if ψ_i is invariant under time reversal, and vice versa. The antisymmetry in equation 11, as well as the time reversal symmetry, will be important later in the diagrammatic expansion.

The fluctuating force $G_i(\{\psi\}, \mathbf{x}, t)\theta(t)$ is modeled as Gaussian white noise with zero average. The term θ represents the rapidly fluctuating component in time,

$$\langle\theta(t)\rangle = 0, \tag{12}$$

$$\langle\theta(t)\theta(t')\rangle = \delta(t - t'), \tag{13}$$

where the average is over all possible realisations of the Gaussian noise distribution. $G_i(\{\psi\}, \mathbf{x}, t)$ is the noise amplitude which is also a function of time, through the time dependence of the field variables ψ_i. This is related to the transport coefficients,

$$G_i(\{\psi\}, \mathbf{x}, t)G_j(\{\psi\}, \mathbf{x}', t) = 2\Gamma_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t). \tag{14}$$

There is subjectivity in the value of the field variables ψ to be used in the above expression for the noise amplitude. Since the noise is a delta function in time, equation 9 indicates that the field variables ψ are step functions. Due to this, the value of the variable ψ to be used in equation 14 could be either just before the step change (Ito formulation), just after the step change, or the average of the two (Stratonovich formulation). Here, we use the Ito formulation where the value of ψ before the step change is used, since the Jacobian is field-independent.

The transport coefficient $\Gamma_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t)$ is expanded in a series in the fields ψ as follows,

$$\begin{aligned}
&\Gamma_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t) \\
&= \bar{\Gamma}(\mathbf{x} - \mathbf{x}') + \sum_m \int_{\mathbf{x}_m} \Gamma^{(1)}(\mathbf{x}, \mathbf{x}', \mathbf{x}_m)\psi_m(\mathbf{x}_m, t) + \ldots \\
&\quad + \sum_{m,n,\ldots,z} \int_{\mathbf{x}_l, \mathbf{x}_m, \cdots, \mathbf{x}_z} \Gamma^{(n)}_{ijm\ldots z}(\mathbf{x}, \mathbf{x}', \mathbf{x}_l, \mathbf{x}_m, ..., \mathbf{x}_z, t)\psi_l(\mathbf{x}_l, t)\psi_m(\mathbf{x}_m, t) \cdots \psi_z(\mathbf{x}_z, t) \\
&\quad + \ldots,
\end{aligned} \tag{15}$$

where the coefficients $\Gamma^{(n)}_{ij\cdots z}$ are now independent of time, and depend only on positions. We assume the Onsager reciprocal relations are valid at each order in the expansion, that is,

$$\Gamma^{(n)}_{ijm\ldots z}(\mathbf{x}, \mathbf{x}', \mathbf{x}_l, \mathbf{x}_m, ..., \mathbf{x}_z) = \Gamma^{(n)}_{jim\ldots z}(\mathbf{x}', \mathbf{x}, \mathbf{x}_l, \mathbf{x}_m, ..., \mathbf{x}_z). \tag{16}$$

The transport $\Theta_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t)$ of the reversible term in equation 9 is also expanded in a series similar to equation 15.

$$\begin{aligned}
&\Theta_{ij}(\{\psi\}, \mathbf{x}, \mathbf{x}', t) \\
&= \sum_m \int_{\mathbf{x}_m} \Theta^{(1)}_{ijm}(\mathbf{x}, \mathbf{x}', \mathbf{x}_m)\psi_m(\mathbf{x}_m, t) + \ldots \\
&\quad + \sum_{m,n,\ldots,z} \int_{\mathbf{x}_l, \mathbf{x}_m, \cdots, \mathbf{x}_z} \Theta^{(n)}_{ijm\ldots z}(\mathbf{x}, \mathbf{x}', \mathbf{x}_l, \mathbf{x}_m, ..., \mathbf{x}_z, t)\psi_l(\mathbf{x}_l, t)\psi_m(\mathbf{x}_m, t) \cdots \psi_z(\mathbf{x}_z, t) \\
&\quad + \ldots,
\end{aligned} \tag{17}$$

where the coefficients $\Theta^{(n)}_{ij\cdots z}$ are independent of time, and depend only on positions. We assume each of the coefficients $\Theta^{(n)}_{ijm...n}$ is antisymmetric under the interchange of i and j,

$$\Theta^{(n)}_{ijm...z}(\mathbf{x},\mathbf{x}',\mathbf{x}_l,\mathbf{x}_m,...,\mathbf{x}_z) = -\Theta^{(n)}_{jim...z}(\mathbf{x}',\mathbf{x},\mathbf{x}_l,\mathbf{x}_m,...,\mathbf{x}_z). \tag{18}$$

It is important to note that there is no field-independent contribution to $\Theta_{ij}(\{\psi\},\mathbf{x},\mathbf{x}',t)$, analogous to the term $\bar{\Gamma}(\mathbf{x},\mathbf{x}',t)$ in equation 15. The reason is as follows. As noted in the discussion of the time parity of the free energy functional, the quadratic approximation for the free energy equation 4 contains products of fields with the same time parity. Therefore, the contribution from the quadratic free energy in the third term on the right side of equation 9 would have the same parity as the field ψ_i. However, this violates the requirement that the reversible term proportional to $\Theta_{ij}(\{\psi\},\mathbf{x},\mathbf{x}',t)$ has to have time parity opposite to the field ψ_i. Therefore, the field-independent contribution to Θ_{ij} in equation 17 has to be zero.

The Fourier transforms of the field variables are defined as,

$$\psi(\mathbf{k},t) = \int_{\mathbf{x}} \exp(\imath\mathbf{k}.\mathbf{x})\psi(\mathbf{x},t), \tag{19}$$

where $\int_{\mathbf{x}} \equiv \int d\mathbf{x}$. The inverse Fourier transform of this is,

$$\psi(\mathbf{x},t) = \int_{\mathbf{k}} \exp(-\imath\mathbf{k}.\mathbf{x})\psi(\mathbf{k},t), \tag{20}$$

where $\int_{\mathbf{k}} \equiv (2\pi)^{-3}\int d\mathbf{k}$. The terms in equation 9 can be expressed in terms of Fourier components as follows. First, we define $\tilde{\psi}_i(\mathbf{x},t)$, the chemical potential, as the functional derivative of the free energy functional with respect to the variable ψ_i,

$$\begin{aligned}\tilde{\psi}_i(\mathbf{x},t) &= \frac{\delta F}{\delta\psi_i(\mathbf{x},t)}\\ &= \int_{\mathbf{x}'} (\chi(\mathbf{x}-\mathbf{x}'))^{-1}_{ij}\psi_j(\mathbf{x}',t),\end{aligned} \tag{21}$$

and the associated Fourier transform,

$$\begin{aligned}\tilde{\psi}_i(\mathbf{k},t) &= \frac{\delta F}{\delta\psi_i(-\mathbf{k},t)}\\ &= (\chi(\mathbf{k}))^{-1}_{ij}\psi_j(\mathbf{k},t).\end{aligned} \tag{22}$$

The Fourier transform of the first term on the right of equation 16 is,

$$\begin{aligned}&\int_{\mathbf{x}} \exp(\imath\mathbf{k}\cdot\mathbf{x}) \int_{\mathbf{x}',\mathbf{x}_l,\cdots,\mathbf{x}_z} \Gamma^{(n)}_{ijl\cdots z}(\mathbf{x},\mathbf{x}',\mathbf{x}_l,\cdots,\mathbf{x}_z)\tilde{\psi}_j(\mathbf{x}',t)\psi_l(\mathbf{x}_l,t)\cdots\psi_z(\mathbf{x}_z,t)\\ &= \int_{\mathbf{k}',\mathbf{k}_l,\cdots,\mathbf{k}_z} \Gamma^{(n)}_{ijl\cdots z}(\mathbf{k},\mathbf{k}',\mathbf{k}_l,\cdots\mathbf{k}_z)\tilde{\psi}_j(-\mathbf{k}',t)\psi_l(-\mathbf{k}_l,t)\cdots\psi_z(-\mathbf{k}_z,t).\end{aligned} \tag{23}$$

A similar transform can be used for the Θ non-linearities. The equivalent of the Onsager reciprocal relations 16 and the antisymmetry relation 18 in Fourier space are,

$$\Gamma^{(n)}_{ijl...z}(\mathbf{k},\mathbf{k}',\mathbf{k}_l,\cdots,\mathbf{k}_z) = \Gamma^{(n)}_{jil...z}(\mathbf{k}',\mathbf{k},\mathbf{k}_l,\cdots,\mathbf{k}_z). \tag{24}$$

$$\Theta^{(n)}_{ijl...z}(\mathbf{k},\mathbf{k}',\mathbf{k}_l,\cdots,\mathbf{k}_z) = -\Theta^{(n)}_{jil...z}(\mathbf{k}',\mathbf{k},\mathbf{k}_l,\cdots,\mathbf{k}_z). \tag{25}$$

In Fourier space, the non-linear Langevin equation 9 is,

$$\begin{aligned}\partial_t\psi_i(\mathbf{k},t) &= \int_{\mathbf{k}'}\sum_j \Big[-\Gamma_{ij}(\{\psi\},\mathbf{k},\mathbf{k}',t)\tilde{\psi}_j(\mathbf{k}',t) + \Gamma'_{ij}(\{\psi\},\mathbf{k},\mathbf{k}',t)\\ &\quad - \int_{\mathbf{k}'}\sum_j \Theta_{ij}(\{\psi\},\mathbf{k},\mathbf{k}',t)\tilde{\psi}_j(\mathbf{k}',t)\Big] + G_i(\{\psi\},\mathbf{k},t)\theta(t),\end{aligned} \tag{26}$$

where $\Gamma'_{ij}(\{\psi\},\mathbf{k},\mathbf{k}',t) = (\delta\Gamma_{ij}(\{\psi\},\mathbf{k},\mathbf{k}',t)/\delta\psi_j(-\mathbf{k}',t))$ is the second term on the right side of equation 9 required to ensure zero relaxation rate at equilibrium. It is convenient to use the temporal Fourier transform of the field,

$$\psi_i(\mathbf{q}) = \int_{-T/2}^{T/2} dt \exp(\imath\omega t)\psi(\mathbf{k},t), \tag{27}$$

where $\mathbf{q} = (\mathbf{k},\omega)$, and T is the averaging time interval which is much longer than the longest relaxation time in the system. The temporal Fourier transform of the irreversible non-linear term equation 23 is,

$$\begin{aligned}
&\int_t \exp(\imath\omega t)\int_{\mathbf{k}',\mathbf{k}_l,\cdots,\mathbf{k}_z} \Gamma^{(n)}_{ijl\cdots z}(\mathbf{k},\mathbf{k}',\mathbf{k}_l,\cdots\mathbf{k}_z)\tilde{\psi}_j(-\mathbf{k}',t)]\psi_l(-\mathbf{k}_l,t)\cdots\psi_z(-\mathbf{k}_z,t) \\
&= \int_{\mathbf{q}',\mathbf{q}_l,\cdots,\mathbf{q}_z} \Gamma^{(n)}_{ijl\cdots z}(\mathbf{k},\mathbf{k}',\mathbf{k}_l,\cdots\mathbf{k}_z)\tilde{\psi}_j(-\mathbf{q}')\psi_l(-\mathbf{q}_l)\cdots\psi_z(-\mathbf{q}_z) \\
&\quad \times\delta(\omega+\omega'+\omega_l+\cdots+\omega_z) \\
&= \int_{\mathbf{q}',\mathbf{q}_l,\cdots,\mathbf{q}_z} \Gamma^{(n)}_{ijl\cdots z}(\mathbf{q},\mathbf{q}',\mathbf{q}_l,\cdots\mathbf{q}_z)\tilde{\psi}_j(-\mathbf{q}')\psi_l(-\mathbf{q}_l)\cdots\psi_z(-\mathbf{q}_z),
\end{aligned} \tag{28}$$

where $\int_t = \int_{-T/2}^{T/2} dt$, and $\int_{\mathbf{q}} = (2\pi)^{-4}\int_{\mathbf{k}}\int_{-\infty}^{\infty} d\omega$, and we have used the notation $\Gamma^{(n)}_{ijl\cdots z}(\mathbf{q},\mathbf{q}',\mathbf{q}_l,\cdots,\mathbf{q}_z) \equiv \Gamma^{(n)}_{ijl\cdots z}(\mathbf{k},\mathbf{k}',\cdots,\mathbf{k}_z)\ \delta(\omega+\omega'+\omega_l+\cdots+\omega_z)$. With this, the Fourier transform of the non-linear term in equation 26 is,

$$\begin{aligned}
&\int_{\mathbf{x},t} \exp(\imath(\mathbf{k}\cdot\mathbf{x}+\omega t))\Gamma^{(n)}_{ij}(\{\psi\},\mathbf{x},\mathbf{x}')\tilde{\psi}_j(\mathbf{x}',t) \\
&= \bar{\Gamma}_{ij}(\mathbf{k})\delta(\mathbf{q}+\mathbf{q}')\tilde{\psi}_j(-\mathbf{q}') + \sum_l \int_{\mathbf{q}',\mathbf{q}_l} \Gamma^{(1)}_{ijl}(\mathbf{q},\mathbf{q}',\mathbf{q}_l)\tilde{\psi}_j(-\mathbf{q}')\psi_l(-\mathbf{q}_l) \\
&\quad +\ldots \\
&\quad + \sum_{l,m,\ldots,z}\int_{\mathbf{q}',\mathbf{q}_l,\mathbf{q}_m,\cdots,\mathbf{q}_z} \Gamma^{(n)}_{ijlm\ldots z}(\mathbf{q},\mathbf{q}',\mathbf{q}_l,\mathbf{q}_m,\ldots,\mathbf{q}_z)\tilde{\psi}_j(-\mathbf{q}')\psi_l(-\mathbf{q}_l)\psi_m(-\mathbf{q}_m)\cdots\psi_z(-\mathbf{q}_z) \\
&\quad +\ldots
\end{aligned} \tag{29}$$

While discussing the diagrammatic expansions a little later, we will use the notation,

$$\begin{aligned}
&\Gamma_{ij}(\{\psi\},\mathbf{q},\mathbf{q}') \\
&= \bar{\Gamma}_{ij}(\mathbf{k})\delta(\mathbf{q}+\mathbf{q}') + \sum_l \int_{\mathbf{q}_l} \Gamma^{(1)}(\mathbf{q},\mathbf{q}',\mathbf{k}_l)\psi_l(-\mathbf{q}_l) \\
&\quad +\ldots \\
&\quad + \sum_{l,m,\ldots,z}\int_{\mathbf{q}_l,\mathbf{q}_m,\cdots,\mathbf{q}_z} \Gamma^{(n)}_{ijlm\ldots z}(\mathbf{q},\mathbf{q}',\mathbf{q}_l,\mathbf{q}_m,\ldots,\mathbf{q}_z)\psi_l(-\mathbf{q}_l)\psi_m(-\mathbf{q}_m)\cdots\psi_z(-\mathbf{q}_z) \\
&\quad +\ldots
\end{aligned} \tag{30}$$

Equations equivalent to 28, 29 and 30 can also be derived for the Θ non-linearities, with $\Gamma^{(n)}_{ijlm\ldots z}(\mathbf{q},\mathbf{q}',\mathbf{q}_l,\ldots,\mathbf{q}_z)$ replaced by $\Theta^{(n)}_{ijlm\ldots z}(\mathbf{q},\mathbf{q}',\mathbf{q}_l,\ldots,\mathbf{q}_z)$. Using the notation in equation 30, equation 26 becomes,

$$\begin{aligned}
\partial_t\psi_i(\mathbf{q}) &= \int_{\mathbf{q}'}\sum_j \Big[-\Gamma_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\tilde{\psi}_j(-\mathbf{q}') + \Gamma'_{ij}(\{\psi\},\mathbf{q},\mathbf{q}') \\
&\quad - \Theta_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\tilde{\psi}_j(-\mathbf{q}')\Big] + G_i(\{\psi\},\mathbf{q})\theta(\omega),
\end{aligned} \tag{31}$$

with the noise correlation

$$\langle\theta(\omega)\theta(\omega')\rangle = \delta(\omega+\omega'). \tag{32}$$

In the special case where transport is local, so that $\Gamma^{(n)}_{ijl\cdots z}$ is independent of position, the first term on the right side of equation 9 becomes,

$$\Gamma_{ijl\cdots z}\tilde{\psi}_j(\mathbf{x},t)\psi_l(\mathbf{x},t)\cdots\psi_z(\mathbf{x},t). \tag{33}$$

The Fourier transform of this is,

$$\int_{\mathbf{q}',\mathbf{q}_l,\cdots,\mathbf{q}_z} \Gamma_{ijl\cdots z}\tilde{\psi}_j(-\mathbf{q}')\psi_l(-\mathbf{q}_l)\cdots\psi_z(-\mathbf{q}_z)\delta(\mathbf{q}+\mathbf{q}'+\cdots+\mathbf{q}_z). \tag{34}$$

The generating functional for the dynamics of the system is defined as,

$$\begin{aligned}\mathcal{Z} \ = \ & \int_{\psi}\delta\left(-\imath\omega\psi_i(\mathbf{q})+\sum_j\left[\int_{\mathbf{q}'}\Gamma_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\tilde{\psi}_j(-\mathbf{q}')-\Gamma'_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\right.\right. \\ & \left.\left.+\ \Theta_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\tilde{\psi}_j(-\mathbf{q}')\right]-G_i(\{\psi\},\mathbf{k})\theta(\omega)\right). \end{aligned} \tag{35}$$

where $\int_\psi \equiv \prod_i \int D[\psi_i]$ is the functional integral. A functional Fourier transform is used to express the generating functional as,

$$\mathcal{Z} = c\int_{\psi,\hat{\psi}} \exp\left(-\mathcal{L}\right), \tag{36}$$

where $\int_{\psi,\hat{\psi}} = \prod_i \int D[\psi_i]\int D[\hat{\psi}_i]$, and

$$\begin{aligned}\mathcal{L} \ = \ & \int_{\mathbf{q}}\hat{\psi}_i(-\mathbf{q})\left[-\imath\omega\psi_i(\mathbf{q})+\int_{\mathbf{q}'}\sum_j\Big(\Gamma_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\tilde{\psi}_j(-\mathbf{q}')-\Gamma'_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\right. \\ & \left.+\Theta_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\tilde{\psi}_j(-\mathbf{q}')\Big)-G_i(\{\psi\},\mathbf{q})\theta(\omega)\right]. \end{aligned} \tag{37}$$

The Jacobian c in equation 36 is a constant in the Ito formulation, and $\hat{\psi}_i$ are the auxiliary fields in the functional Fourier transforms.

The generating functional $\mathcal{Z}$ can be explicitly averaged over the noise realisations to obtain the averaged generating functional,

$$Z = \langle\mathcal{Z}\rangle_{\text{noise}} = c\int_{\psi,\hat{\psi}} \exp\left(-L\right), \tag{38}$$

where $\langle\mathcal{Z}\rangle_{\text{noise}}$ is the average of the generating functional over the Gaussian noise realisations, and

$$\begin{aligned}L \ = \ & \int_{\mathbf{q}}\hat{\psi}_i(-\mathbf{q})\left[-\imath\omega\psi_i(\mathbf{q})+\int_{\mathbf{q}'}\sum_j\Big(\Gamma_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\tilde{\psi}_j(-\mathbf{q}')-\Gamma'_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\right. \\ & \left.+\ \Theta_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\tilde{\psi}_j(-\mathbf{q}')-\Gamma_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\hat{\psi}_j(-\mathbf{q}')\Big)\right]. \end{aligned} \tag{39}$$

III. CORRELATION AND RESPONSE FUNCTIONS:

The correlation functions of the hatted and unhatted fields, C_{ij} and $C_{i\hat{j}}$, are different from those used in the fluctuation-dissipation theorem, equation 44 and equation 49, discussed a little later. The functional integral correlation and response functions are,

$$C_{ij}(\mathbf{x},t) = \langle\psi_i(\mathbf{x}+\mathbf{x}',t+t')\psi_j(\mathbf{x}',t')\rangle, \tag{40}$$

$$\hat{C}_{i\hat{j}}(\mathbf{x},t) = \langle\psi_i(\mathbf{x}+\mathbf{x}',t+t')\hat{\psi}_j(\mathbf{x}',t')\rangle, \tag{41}$$

where the averages are defined over the Lagrangian function 39,

$$\langle\bullet\rangle = c\int_{\psi,\hat{\psi}} \bullet\exp\left(-L\right). \tag{42}$$

Note that L is the Lagrangian averaged over different noise realisations in equation 37.

The fluctuation-dissipation theorem relates the time derivative of the correlation function to the response function. In order to derive the Fourier transform of the time derivative of the correlation function, it is necessary to return to the original formulation in equation 9 which incorporates the Gaussian noise.

$$\begin{aligned}\frac{dC_{ij}}{dt} &= \langle \frac{\partial \psi_i(\mathbf{x}+\mathbf{x}', t+t')}{\partial t} \psi_j(\mathbf{x}', t') \rangle \\ &= -\langle \psi_i(\mathbf{x}+\mathbf{x}', t+t') \frac{\partial \psi_j(\mathbf{x}', t')}{\partial t'} \rangle \\ &= -\langle c \int_{\psi, \hat{\psi}} \exp(-\mathcal{L}) \psi_i(\mathbf{x}+\mathbf{x}', t+t') \qquad (43) \\ &\times \int_{\mathbf{x}''} \Big(\sum_k \Big(-\Gamma_{jk}(\{\psi\}, \mathbf{x}', \mathbf{x}'', t') \tilde{\psi}_k(\mathbf{x}'') + \Gamma'_{jk}(\{\psi\}, \mathbf{x}', \mathbf{x}'', t') \\ &-\Theta_{ij}(\{\psi\}, \mathbf{x}', \mathbf{x}'', t') \tilde{\psi}_k(\mathbf{x}'', t') \Big) + G_j(\mathbf{x}') \theta(t') \Big) \rangle_{\text{noise}}. \end{aligned} \tag{44}$$

where $\langle \cdot \rangle_{\text{noise}}$ is the average over noise realisations. Note that $\mathcal{L}$ in the above equation is the Lagrangian in equation 37 which has not yet been averaged over the noise realisations. When we take the average of this over the noise realisations, the first, second and third terms in the spatial integral on the right are unchanged because they do not depend on the noise, while the last term is linear in the noise. As shown in Appendix A, we obtain after averaging,

$$\begin{aligned}\frac{dC_{ij}}{dt} &= -\int_{\psi, \hat{\psi}} \exp(-L) \psi_i(\mathbf{x}+\mathbf{x}', t+t') \\ &\times \int_{\mathbf{x}''} \sum_k \Big(-\Gamma_{jk}(\{\psi\}, \mathbf{x}', \mathbf{x}'', t') \tilde{\psi}_k(\mathbf{x}'', t') + \Gamma'_{jk}(\{\psi\}, \mathbf{x}', \mathbf{x}'', t') \\ &-\Theta_{jk}(\{\psi\}, \mathbf{x}', \mathbf{x}'', t') \tilde{\psi}_k(\mathbf{x}'', t') + 2\Gamma_{jk}(\{\psi\}, \mathbf{x}', \mathbf{x}'', t') \hat{\psi}_k(\mathbf{x}'', t')) \Big) \\ &= -\langle \psi_i(\mathbf{x}+\mathbf{x}', t+t') \int_{\mathbf{x}''} (-\Gamma_{jk}(\{\psi\}, \mathbf{x}', \mathbf{x}'', t') \tilde{\psi}_k(\mathbf{x}'', t') + \Gamma'_{jk}(\{\psi\}, \mathbf{x}', \mathbf{x}'', t') \\ &-\Theta_{jk}(\{\psi\}, \mathbf{x}, \mathbf{x}'', t') \tilde{\psi}_k(\mathbf{x}'', t') + 2\Gamma_{jk}(\{\psi\}, \mathbf{x}', \mathbf{x}'', t') \hat{\psi}_k(\mathbf{x}'', t')) \rangle, \end{aligned} \tag{45}$$

where L is the Lagrangian defined in equation 39. The Fourier transform of the time derivative of the correlation function is,

$$\begin{aligned}\imath\omega S_{ij}(\mathbf{q}) &= -\frac{1}{TV} \langle \psi_i(\mathbf{q}) \int_{\mathbf{q}'} \sum_k \Big(-\Gamma_{jk}(\{\psi\}, -\mathbf{q}, \mathbf{q}') \tilde{\psi}_k(-\mathbf{q}') + \Gamma'_{jk}(\{\psi\}, -\mathbf{q}, \mathbf{q}') \\ &-\Theta_{jk}(\{\psi\}, -\mathbf{q}, \mathbf{q}') + 2\Gamma_{jk}(\{\psi\}, -\mathbf{q}, \mathbf{q}') \hat{\psi}_k(-\mathbf{q}') \Big) \rangle, \end{aligned} \tag{46}$$

where the average $\langle \cdot \rangle$ is defined for the Lagrangian L, V is the total volume and T is the time period of averaging.

The response function $R_{ij}(\mathbf{q})$ is the value of ψ_i due to a force f_j conjugate to the variable ψ_j is added to the free energy functional. In the presence of the conjugate force, the generating functional is modified as,

$$\begin{aligned} Z &= c \int_{\psi, \hat{\psi}} \exp(-L) \exp(-\int_{\mathbf{x}', \mathbf{x}''} \int_{t'} \hat{\psi}_k(\mathbf{x}'', t') \Gamma_{kj}(\{\psi\}, \mathbf{x}'', \mathbf{x}', t') f_j(\mathbf{x}', t') \\ &= c \int_{\psi, \hat{\psi}} \exp(-L)(1 - \int_{\mathbf{x}', \mathbf{x}''} \int_{t'} \hat{\psi}_k(\mathbf{x}'', t') \Gamma_{kj}(\{\psi\}, \mathbf{x}'', \mathbf{x}', t') f_j(\mathbf{x}', t')), \end{aligned} \tag{47}$$

where the linearisation approximation has been used in the final step for small force. The change in ψ_i at $(\mathbf{x}+\mathbf{x}', t+t')$

due to this applied force is,

$$\begin{aligned}\langle\Delta\psi_i(\mathbf{x},t)\rangle_f &= -c\int_{\psi,\hat{\psi}}\exp\left(-L\right)\int_{\mathbf{x}',\mathbf{x}''}\int_{t'}(\psi_i(\mathbf{x},t)\hat{\psi}_k(\mathbf{x}'',t') \\ &\quad\times\Gamma_{kj}(\{\psi\},\mathbf{x}'',\mathbf{x}',t'))f_j(\mathbf{x}',t') \\ &= -\int_{\mathbf{x}',\mathbf{x}''}\int_{t'}\langle(\psi_i(\mathbf{x},t)\hat{\psi}_k(\mathbf{x}'',t')\Gamma_{kj}(\{\psi\},\mathbf{x}'',\mathbf{x}',t')\rangle f_j(\mathbf{x}',t'),\end{aligned} \tag{48}$$

where $\langle\cdot\rangle_f$ is the average value of the variable $\cdot$ in the presence of a force. Therefore, the response function due to the force f_j is,

$$R_{ij}(\mathbf{x}-\mathbf{x}',t-t') = \frac{\delta\langle\Delta\psi(\mathbf{x},t)\rangle_f}{\delta f_j(\mathbf{x}',t')}. \tag{49}$$

Using spatial inhomogeneity and time-translation invariance, the above equation can be recast as,

$$R_{ij}(\mathbf{x},t) = -\int_{\mathbf{x}''}\langle\psi_i(\mathbf{x}+\mathbf{x}',t+t')\hat{\psi}_k(\mathbf{x}'',t')\Gamma_{kj}(\{\psi\},\mathbf{x}'',\mathbf{x}',t')\rangle \tag{50}$$

The Fourier transform of the response function is,

$$\begin{aligned}R_{ij}(\mathbf{q}) &= -\int_{\mathbf{x}}\int_t\exp\left(\imath(\mathbf{k}\cdot\mathbf{x}+\omega t)\right)R_{ij}(\mathbf{x},t) \\ &= -\frac{1}{TV}\int_{\mathbf{q}''}\langle\psi_i(\mathbf{q})\Gamma_{kj}(\{\psi\},\mathbf{q}'',-\mathbf{q})\hat{\psi}_k(-\mathbf{q}'')\rangle.\end{aligned} \tag{51}$$

Due to the symmetry of the kinetic coefficients Γ_{kj}, the above response functions can also be written as,

$$R_{ij}(\mathbf{q}) = \frac{1}{TV}\int_{\mathbf{q}''}\langle\psi_i(\mathbf{q})\Gamma_{jk}(\{\psi\},-\mathbf{q},\mathbf{q}'')\hat{\psi}_k(-\mathbf{q}'')\rangle. \tag{52}$$

IV. BARE CORRELATION FUNCTIONS:

For a linear Langevin equation 9, where $\Gamma^{(n)}_{ijl\cdots z} = 0$ for $n > 0$, the Lagrangian L_0 is quadratic,

$$L_0 = \sum_{ij}\int_{\mathbf{q}}\hat{\psi}_i(-\mathbf{q})\left(-\imath\omega\psi_i(\mathbf{q}) + \bar{\mathbf{\Gamma}}_{ij}(\mathbf{k})\tilde{\psi}_j(\mathbf{q}) - \bar{\mathbf{\Gamma}}_{ij}(\mathbf{k})\hat{\psi}_j(\mathbf{q})\right). \tag{53}$$

Here, we have used the property $\bar{\mathbf{\Gamma}}_{ij}(\mathbf{x},\mathbf{x}') = \bar{\mathbf{\Gamma}}_{ij}(\mathbf{x}-\mathbf{x}')$ in a spatially homogeneous system. This Lagrangian can be symmetrised and written in matrix form as,

$$L_0 = \frac{1}{2}\int_{\mathbf{q}}\begin{pmatrix}\Psi^{*T} & \hat{\Psi}^{*T}\end{pmatrix}\begin{pmatrix} & \\ & \bar{\mathbf{M}}^{-1} & \\ & \end{pmatrix}\begin{pmatrix}\Psi \\ \hat{\Psi}\end{pmatrix}, \tag{54}$$

where Ψ and $\hat{\Psi}$ are column vectors whose elements are $\psi_i(\mathbf{q})$ and $\hat{\psi}_i(\mathbf{q})$, while Ψ^* and $\hat{\Psi}^*$ are column vectors whose elements are the complex conjugates $\psi_i(-\mathbf{q})$ and $\hat{\psi}_i(-\mathbf{q})$ respectively, and the superscript ^{T} is the transpose. The matrix $\bar{\mathbf{M}}(\mathbf{q})$ is a block-diagonal matrix, whose inverse is given by,

$$\bar{\mathbf{M}}^{-1}(\mathbf{q}) = \begin{pmatrix}\mathbf{0} & \imath\omega\mathbf{I} + (\chi)^{-1}\cdot\bar{\mathbf{\Gamma}} \\ -\imath\omega\mathbf{I} + \bar{\mathbf{\Gamma}}\cdot(\chi)^{-1} & 2\bar{\mathbf{\Gamma}}\end{pmatrix}, \tag{55}$$

where $\mathbf{I}$ is the identity matrix, and $\bar{\mathbf{\Gamma}}$ and $(\chi)^{-1}$ are a square matrices whose elements are $\bar{\Gamma}_{ij}(\mathbf{k})$ and $(\chi(\mathbf{k}))^{-1}_{ij}$, and $\mathbf{0}$ is a null square matrix. The the product $\bar{\mathbf{\Gamma}}\cdot(\chi)^{-1}$ represents the matrix multiplication $\bar{\Gamma}_{ik}(\chi)^{-1}_{kj}$. In equation 55, the block $2\bar{\mathbf{\Gamma}}$ is real and symmetric. In the off-diagonal blocks $(-\imath\omega\mathbf{I}+\bar{\mathbf{\Gamma}}\cdot\chi^{-1})$ and $(\imath\omega\mathbf{I}+\chi^{-1}\cdot\bar{\mathbf{\Gamma}})$, both the matrices $\bar{\mathbf{\Gamma}}$ and $(\chi)^{-1}$ are symmetric. Moreover, the elements of the matrix $(\chi)^{-1}$ and $\bar{\mathbf{\Gamma}}$ are real, since the free energy is a real function and the transport coefficients represent irreversible processes. Therefore, one of the off-diagonal blocks is obtained by taking the transpose of the complex conjugate of the other, and the square matrix in L_0 is Hermetian.

It is convenient to define the bare averages as,

$$\langle \bullet \rangle_0 = c \int_{\hat{\psi},\psi} \int_{\mathbf{q}} \bullet \exp\left(-L_0\right). \tag{56}$$

The matrix $\bar{\mathbf{M}}$, which is the inverse of the matrix $\bar{\mathbf{M}}^{-1}$ (equation 55), is

$$\bar{\mathbf{M}}(\mathbf{q}) = \begin{pmatrix} (-\imath\omega\mathbf{I}+\bar{\mathbf{\Gamma}}\cdot\chi^{-1})^{-1}(2\bar{\mathbf{\Gamma}})(\imath\omega\mathbf{I}+\chi^{-1}\cdot\bar{\mathbf{\Gamma}})^{-1} & (-\imath\omega\mathbf{I}+\bar{\mathbf{\Gamma}}\cdot(\chi)^{-1})^{-1} \\ (\imath\omega\mathbf{I}+(\chi)^{-1}\cdot\bar{\mathbf{\Gamma}})^{-1} & \mathbf{0} \end{pmatrix}. \tag{57}$$

The bare correlation and response functions, evaluated as shown in the Appendix B, are,

$$\langle \psi_i(\mathbf{q})\hat{\psi}_j(\mathbf{q}')\rangle_0 \;=\; (-\imath\omega\mathbf{I}+\bar{\mathbf{\Gamma}}(\mathbf{k})\cdot(\chi(\mathbf{k}))^{-1})^{-1}_{ij}\delta(\mathbf{q}+\mathbf{q}'), \tag{58}$$

$$\begin{aligned} &\langle \psi_i(\mathbf{q})\psi_j(\mathbf{q}')\rangle_0 \\ &= ((-\imath\omega\mathbf{I}+\bar{\mathbf{\Gamma}}(\mathbf{k})\cdot(\chi(\mathbf{k}))^{-1})^{-1}\cdot(2\bar{\mathbf{\Gamma}}(\mathbf{k}))\cdot(\imath\omega\mathbf{I}+(\chi(-\mathbf{k}))^{-1}\cdot\bar{\mathbf{\Gamma}}(-\mathbf{k}))^{-1})_{ij}\delta(\mathbf{q}+\mathbf{q}'), \end{aligned} \tag{59}$$

$$\langle \hat{\psi}_i(\mathbf{q})\hat{\psi}_j(\mathbf{q}')\rangle_0 = 0. \tag{60}$$

The final result above is a consequence of the causal discretisation scheme used, where averages involving $\hat{\psi}_i(t)$ vanish if t is the latest time. In addition, we can show the following relations between the bare correlation functions. Since the matrix $\bar{\mathbf{M}}$ is Hermetian, the correlations between the hatted and unhatted fields satisfy,

$$\langle \psi_i(\mathbf{q})\hat{\psi}_j(\mathbf{q}')\rangle_0 = \langle \psi_j(-\mathbf{q})\hat{\psi}_i(-\mathbf{q}')\rangle_0 \tag{61}$$

and

$$(\chi(\mathbf{k}))^{-1}_{ik}\langle \psi_k(\mathbf{q})\hat{\psi}_j(-\mathbf{q})\rangle_0 + \langle \hat{\psi}_i(\mathbf{q})\psi_k(-\mathbf{q})\rangle_0(\chi(\mathbf{k}))^{-1}_{kj} = (\chi(\mathbf{k}))^{-1}_{ik}\langle \psi_k(\mathbf{q})\psi_l(-\mathbf{q})\rangle_0(\chi(\mathbf{k}))^{-1}_{lj}. \tag{62}$$

From this, by pre- and post-multiplying by χ, we obtain the reciprocal relation

$$\langle \psi_i(\mathbf{q})\hat{\psi}_k(-\mathbf{q})\rangle_0\chi_{kj}(\mathbf{k}) + \chi_{ik}(\mathbf{k})\langle \hat{\psi}_k(\mathbf{q})\psi_j(-\mathbf{q})\rangle_0 = \langle \psi_i(\mathbf{q})\psi_j(-\mathbf{q})\rangle_0. \tag{63}$$

The bare time correlation functions an be obtained by taking the inverse Fourier transforms of the structure factors,

$$\langle \psi_i(\mathbf{k},t'+t)\hat{\psi}_j(\mathbf{k}',t')\rangle_0 \;=\; \int_\omega\int_{\omega'} \exp\left(-\imath\omega(t+t')-\imath\omega' t'\right)\langle \psi_i(\mathbf{q})\hat{\psi}_j(\mathbf{q}')\rangle_0. \tag{64}$$

Since the correlation function depends only on the time difference t, stationarity can be used to reformulate the correlation function as,

$$\begin{aligned} \langle \psi_i(\mathbf{k},t+t')\hat{\psi}_j(\mathbf{k}',t')\rangle_0 \;&=\; \frac{1}{T}\int_{-T/2}^{T/2} dt \langle \psi_i(\mathbf{k},t+t')\hat{\psi}_j(\mathbf{k}',t')\rangle_0 \\ &= (\exp\left(-t(\bar{\mathbf{\Gamma}}(\mathbf{k})\cdot(\chi(\mathbf{k}))^{-1})\right)_{ij}\delta(\mathbf{k}+\mathbf{k}') \;\text{ for }\; t>0 \\ &= 0 \;\text{ for }\; t<0. \end{aligned} \tag{65}$$

The equal-time response function is interpreted as if the time argument of the hatted field is displaced by an infinitesimal interval after the unhatted field, in which case the equal time response function is zero.

$$\langle \psi_i(\mathbf{k},t)\hat{\psi}_j(-\mathbf{k},t)\rangle_0 \;=\; 0. \tag{66}$$

The inverse Fourier transform of the correlation function $\langle\psi_i(\mathbf{k},t+t')\psi_j(\mathbf{k}',t')\rangle_0$ is given by,

$$\begin{aligned}
&\langle\psi_i(\mathbf{k},t+t')\psi_j(\mathbf{k}',t')\rangle_0 \\
&= T^{-1}\int_\omega \exp(-\imath\omega t)\langle\psi_i(\mathbf{k},\omega)\psi_j(\mathbf{k}',-\omega)\rangle_0 \\
&= (\exp(-|t|\bar{\mathbf{\Gamma}}(\mathbf{k})\cdot\chi(\mathbf{k})^{-1})\cdot\chi(\mathbf{k}))_{ij}\delta(\mathbf{k}+\mathbf{k}'), \\
&= (\chi(\mathbf{k})\cdot\exp(-|t|\chi(\mathbf{k})^{-1}\cdot\bar{\mathbf{\Gamma}}(\mathbf{k})))_{ij}\delta(\mathbf{k}+\mathbf{k}'). \qquad (67)
\end{aligned}$$

The equal time bare correlation function is given by the equilibrium correlation function,

$$\langle\psi_i(\mathbf{k},t)\psi_j(\mathbf{k}',t)\rangle_0 = \chi_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}'). \qquad (68)$$

From equations 65 and 67, the Fourier transforms of the correlation functions satisfy the relations,

$$\begin{aligned}
\langle\psi_i(\mathbf{k},t+t')\hat{\psi}_j(\mathbf{k}',t')\rangle_0 &= \langle\psi_i(\mathbf{k},t+t')\psi_k(\mathbf{k}',t')\rangle_0(\chi(\mathbf{k}))^{-1}_{kj} \text{ for } t>0 \\
&= \langle\psi_i(\mathbf{k},t+t')((\chi(\mathbf{k}))^{-1}_{jk}\psi_k(\mathbf{k}',t'))\rangle_0 \text{ for } t>0 \\
&= \langle\psi_i(\mathbf{k},t+t')\tilde{\psi}_j(\mathbf{k}',t')\rangle_0 \text{ for } t>0. \qquad (69)
\end{aligned}$$

The following relation is valid for both positive and negative t,

$$\langle\psi_i(\mathbf{k},t'+t)\hat{\psi}_j(\mathbf{k}',t')\rangle_0 + \langle\psi_i(\mathbf{k},t'-t)\hat{\psi}_j(\mathbf{k}',t')\rangle_0 = \langle\psi_i(\mathbf{k},t+t')\tilde{\psi}_j(\mathbf{k}',t')\rangle_0. \qquad (70)$$

The correlation between the hatted and tilde fields also satisfy reciprocal relations,

$$\begin{aligned}
\langle\tilde{\psi}_i(\mathbf{k},t+t')\hat{\psi}_j(\mathbf{k}',t')\rangle_0 &= (\chi(\mathbf{k}))^{-1}_{ik}\langle\psi_k(\mathbf{k},t+t')\hat{\psi}_j(\mathbf{k}',t')\rangle_0\delta(\mathbf{k}+\mathbf{k}') \\
&= (\chi(\mathbf{k}))^{-1}_{ik}(\exp(-t\bar{\mathbf{\Gamma}}(\mathbf{k})\cdot(\chi(\mathbf{k}))^{-1}))_{kj}\delta(\mathbf{k}+\mathbf{k}') \\
&= (\exp(-t(\chi(\mathbf{k}))^{-1}\cdot\bar{\mathbf{\Gamma}}(\mathbf{k})))_{ik}(\chi(\mathbf{k}))^{-1}_{kj})\delta(\mathbf{k}+\mathbf{k}') \\
&= \langle\hat{\psi}_i(\mathbf{k}',t')\psi_k(\mathbf{k},t+t')\rangle_0(\chi(\mathbf{k}))^{-1}_{kj} \\
&= \langle\hat{\psi}_i(\mathbf{k}',t)\tilde{\psi}_j(\mathbf{k},t+t')\rangle_0. \qquad (71)
\end{aligned}$$

We shall use the above reciprocal relations to show that the correlation-response relations are also valid for the renormalised correlation and response functions.

V. FIELD-DEPENDENT KINETIC COEFFICIENTS:

The nonlinearities in the Lagrangian, L', renormalise the bare propagators, through the self-energies $\mathbf{\Sigma}_{\hat{\psi}\psi}$ and $\mathbf{\Sigma}_{\psi\psi}$, resulting in the renormalisation $\bar{\mathbf{M}}$ matrix (equation 55),

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{0} & \imath\omega\mathbf{I} + (\chi(-\mathbf{k}))^{-1}\cdot\bar{\mathbf{\Gamma}}(-\mathbf{k}) - \mathbf{\Sigma}_{\psi\hat{\psi}}(-\mathbf{q}) \\ -\imath\omega\mathbf{I} + \bar{\mathbf{\Gamma}}(\mathbf{q})\cdot(\chi(\mathbf{k}))^{-1} - \mathbf{\Sigma}_{\hat{\psi}\psi}(\mathbf{q}) & -2\bar{\mathbf{\Gamma}}(\mathbf{q}) - \mathbf{\Sigma}_{\hat{\psi}\hat{\psi}}(\mathbf{q}) \end{pmatrix}. \qquad (72)$$

For a consistent functional-integral formulation, it is necessary to show that the self-energies $\mathbf{\Sigma}_{\psi\hat{\psi}}$, $\mathbf{\Sigma}_{\hat{\psi}\psi}$ and $\hat{\mathbf{\Sigma}}_{\hat{\psi}\hat{\psi}}$ satisfy the same relations as the bare correlation and response functions,

$$\begin{aligned}
(\chi(\mathbf{k}))^{-1}_{ik}\mathbf{\Sigma}_{\hat{\psi}_k\psi_j}(\mathbf{q}) &= \mathbf{\Sigma}_{\psi_i\hat{\psi}_k}(-\mathbf{q})(\chi(\mathbf{k}))^{-1}_{kj} \qquad (73) \\
\mathbf{\Sigma}_{\hat{\psi}_i\psi_k}(\mathbf{q})\chi_{kj}(\mathbf{k}) + \chi_{ik}(\mathbf{k})\mathbf{\Sigma}_{\psi_k\hat{\psi}_j}(-\mathbf{q}) &= -\mathbf{\Sigma}_{\psi_i\psi_j}(\mathbf{q}). \qquad (74)
\end{aligned}$$

The renormalised matrix $\mathbf{M}$ is also Hermetian if equation 73 is satisfied. Equation 74 ensures the diagonal and off-diagonal blocks of the renormalised matrix $\mathbf{M}^{-1}$ are related in manner identical to those for $\bar{\mathbf{M}}^{-1}$.

A diagrammatic expansion is used for obtaining the relationship between the self-energies in the renormalised matrix $\mathbf{M}$ (equation 72). In the expansion, solid lines are used for the ψ field, dashed lines are used for the $\hat{\psi}$ field, and dotted lines are used for the $\tilde{\psi}$ fields. The vertices due to the ψ dependence of the Onsager coefficient are represented

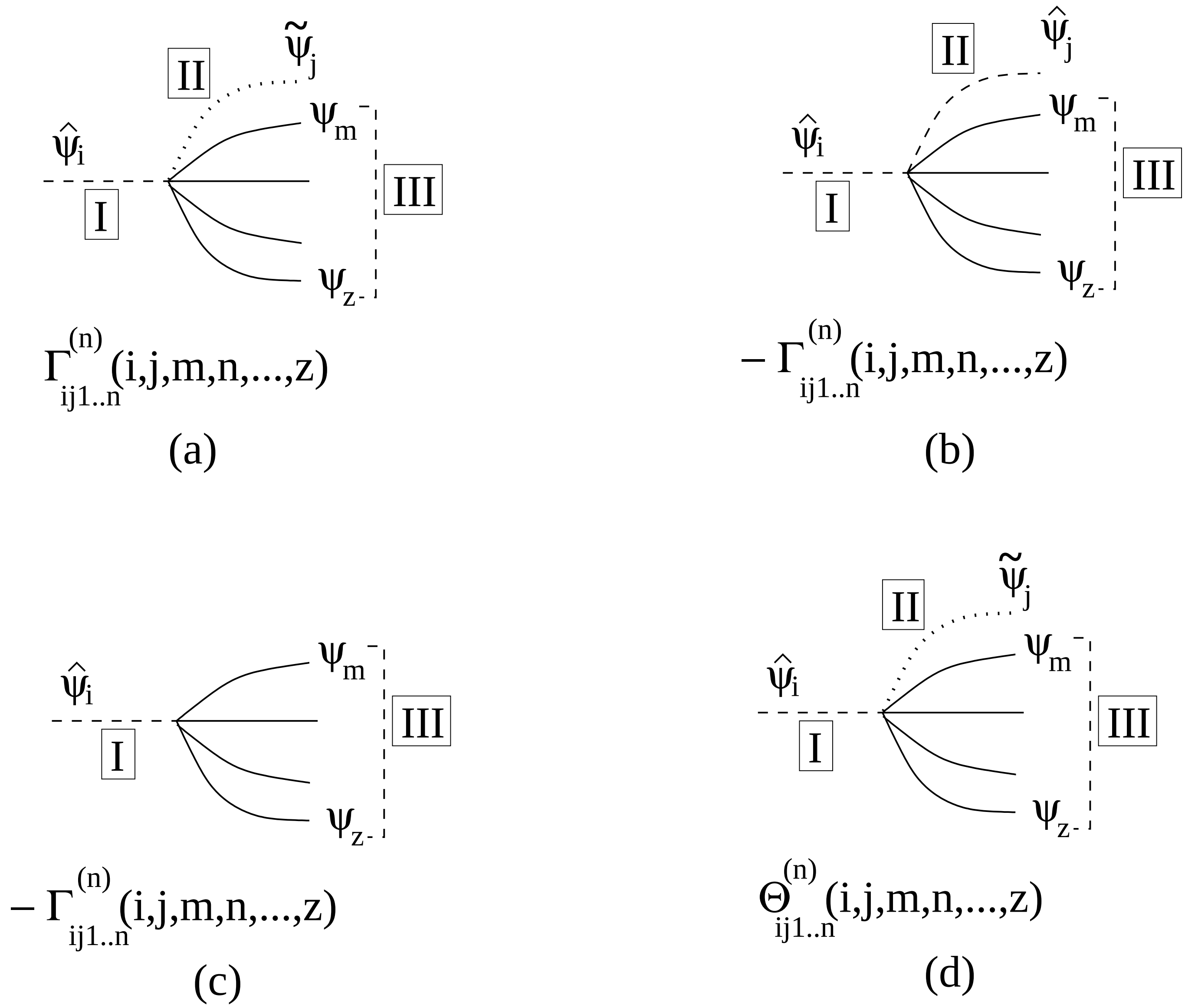

FIG. 1: Vertices due to the field dependence of the transport coefficients.

as shown in figure 1. For the non-linear terms proportional to $\Gamma^{(n)}_{ijmn..z}$ (second term on the right side of equation 39), the vertex is,

$$\hat{\psi}_i(-\mathbf{q}) \int_{\mathbf{q}',\mathbf{q}_l,\cdots,\mathbf{q}_z} \Gamma^{(n)}_{ijl\cdots z}(\mathbf{q},\mathbf{q}',\mathbf{q}_l,\cdots\mathbf{q}_z)\tilde{\psi}_j(-\mathbf{q}')\psi_l(-\mathbf{q}_l)\cdots\psi_z(-\mathbf{q}_z).$$

This vertex has $(n+2)$ legs, of which one is hatted ($\boxed{\text{I}}$), and the remaining are unhatted. Of the unhatted legs, one leg $\tilde{\psi}_k$ is designated as $\boxed{\text{II}}$, while all the others are $\boxed{\text{III}}$. This vertex is shown in figure 1 (a), and is called the $\boxed{\text{A}}$ vertex.

There is also a vertex due to noise correlations, the fourth term on the right side of equation 39, which can be derived in a manner similar to that above,

$$-\hat{\psi}_i(-\mathbf{q}) \int_{\mathbf{q}',\mathbf{q}_l,\cdots,\mathbf{q}_z} \Gamma^{(n)}_{ijl\cdots z}(\mathbf{q},\mathbf{q}',\mathbf{q}_l,\cdots\mathbf{q}_z)\hat{\psi}_j(-\mathbf{q}')\psi_l(-\mathbf{q}_l)\cdots\psi_z(-\mathbf{q}_z). \tag{75}$$

In this case, the $\boxed{\text{II}}$ leg is also hatted, while all the $\boxed{\text{III}}$ legs are unhatted, as shown in figure 1(b). This is referred to as the $\boxed{\text{B}}$ vertex.

There is a vertex due to the concentration dependence of the transport coefficient, the third term on the right side of equation 39,

$$-\hat{\psi}_i(-\mathbf{q}) \int_{\mathbf{q}',\mathbf{q}_l,\cdots,\mathbf{q}_z} \frac{\delta}{\delta\psi_j(\mathbf{q}')} \left(\Gamma_{ijl\cdots z}(n)(\mathbf{q},\mathbf{q}',\mathbf{q}_l,\cdots\mathbf{q}_z)\psi_l(-\mathbf{q}_l,t)\cdots\psi_z(-\mathbf{q}_z,t)\right). \tag{76}$$

This is easily simplified to provide,

$$\begin{aligned} -\hat{\psi}_i(-\mathbf{q},t) \; \textstyle\int_{\mathbf{q}',\mathbf{q}_l,\cdots,\mathbf{q}_z} \; & (\Gamma_{ijl\cdots z}(n)(\mathbf{q},\mathbf{q}',\mathbf{q}_l,\cdots\mathbf{q}_z)\psi_l(-\mathbf{q}_l,t)\cdots\psi_z(-\mathbf{q}_z,t)) \\ & \times(\delta_{jl}\delta(\mathbf{q}'-\mathbf{q}_l)+\cdots+\delta_{jz}\delta(\mathbf{q}'-\mathbf{q}_z)). \end{aligned} \tag{77}$$

This vertex has one hatted $\boxed{\text{I}}$ leg, no $\boxed{\text{II}}$ legs and n $\boxed{\text{III}}$ legs, as shown in figure 1, and is called a $\boxed{\text{C}}$ vertex.

Finally, there is the vertex due to the time-reversible part of the equation, proportional to Θ_{ij},

$$\hat{\psi}_i(-\mathbf{q}) \int_{\mathbf{q}',\mathbf{q}_l,\cdots,\mathbf{q}_z} \Theta^{(n)}_{ijl\cdots z}(\mathbf{q},\mathbf{q}',\mathbf{q}_l,\cdots\mathbf{q}_z)\tilde{\psi}_j(-\mathbf{q}')\psi_l(-\mathbf{q}_l)\cdots\psi_z(-\mathbf{q}_z). \tag{78}$$

As in the $\boxed{\text{A}}$ vertex, this vertex has $(n+2)$ legs, of which one is hatted ($\boxed{\text{I}}$), and the remaining are unhatted. Of the unhatted legs, one vertex $\tilde{\psi}_k$ is designated as $\boxed{\text{II}}$, while all the others are $\boxed{\text{III}}$. This vertex is shown in figure 1 (d), and is called the $\boxed{\text{D}}$ vertex.

Next, we derive some general rules that govern the diagrams for the self-energies $\Sigma_{\hat{\psi}\psi}$, $\Sigma_{\psi\hat{\psi}}$ and $\Sigma_{\hat{\psi}\hat{\psi}}$ in equation 72. The self-energy for the $\Sigma_{\hat{\psi}\psi}$ contains one terminal hatted and one terminal unhatted leg. The diagrams are time ordered, with time increasing monotonically from the unhatted to the hatted leg. In these diagrams, the hatted legs are always at earlier times than the unhatted legs. We derive general rules of two types, the first for the terminal vertices and the other for the internal vertices, which can be used to obtain a set of 'reduced' diagrams after cancellation. The former are discussed in detail, while the latter, which are small modifications of the former, and briefly enumerated.

1. A terminal $\boxed{\text{C}}$ vertex with a terminal $\boxed{\text{I}}$ leg (figure 2(a)) is exactly cancelled by a terminal $\boxed{\text{A}}$ vertex with a bubble involving the $\boxed{\text{II}}$ leg (figure 2(b)). Therefore, the reduced diagrams for the self-energies do not have either $\boxed{\text{C}}$ vertices with terminal $\boxed{\text{I}}$ leg, or $\boxed{\text{A}}$ vertices with terminal $\boxed{\text{I}}$ leg and a bubble involving the $\boxed{\text{II}}$ leg.

2. A terminal $\boxed{\text{C}}$ vertex with a terminal $\boxed{\text{III}}$ leg (figure 2(c)) is cancelled by a terminal $\boxed{\text{A}}$ vertex with a terminal $\boxed{\text{III}}$ leg, which has a bubble involving the $\boxed{\text{II}}$ leg, as shown in figure 2(d). From the above two rules, it is clear that there are no reduced diagrams with terminal $\boxed{\text{C}}$ vertices, and no reduced diagrams in which the terminal $\boxed{\text{A}}$ vertex has a bubble involving the $\boxed{\text{II}}$ leg.

3. Due to causality, there are no terminal $\boxed{\text{A}}$ or $\boxed{\text{B}}$ vertices with a bubble involving the hatted legs, as shown in figure 2(e), in the reduced diagrams. This is because such a bubble is,

$$\begin{aligned} (2\pi)^{-1}\int_{-\infty}^{\infty} d\omega\langle\hat{\psi}_i(-\mathbf{q})\psi_j(\mathbf{q})\rangle_0 &= \langle\hat{\psi}_i(-\mathbf{k},t)\psi_j(\mathbf{k},t)\rangle_0 \\ &= 0. \end{aligned} \tag{79}$$

   Here, the correlation $\langle\hat{\psi}_i(-\mathbf{k},t)\psi_j(\mathbf{k},t)\rangle_0$ is interpreted such that the hatted field is displaced by an infinitesimal time interval after the unhatted field.

4. An $\boxed{\text{A}}$ terminal vertex with a $\boxed{\text{III}}$ terminal leg, shown in figure 2(f), is exactly cancelled by a $\boxed{\text{B}}$ terminal vertex with the same $\boxed{\text{III}}$ terminal leg, shown in figure 2(g). This is because the correlation function due to the $\boxed{\text{II}}$ $\tilde{\psi}$ leg in figure 2 (f) is exactly equal to that involving the $\boxed{\text{II}}$ $\hat{\psi}$ leg in figure 2(g) from equation 69. Moreover, the coefficients of the $\boxed{\text{A}}$ and $\boxed{\text{B}}$ vertices are exactly equal in magnitude and opposite in sign from figure 1, and so these contributions cancel.

5. A $\boxed{\text{B}}$ terminal vertex with a $\boxed{\text{I}}$ or $\boxed{\text{II}}$ terminal leg, shown in figure 2(h), provides a non-zero contribution only if the vertex shown in figure 2(h) has the earliest time index in the diagram, and time increases both towards the left and right, that is, the vertex is the 'primordial vertex' in the diagram.

6. There are no terminal $\boxed{\text{D}}$ vertices with a terminal $\boxed{\text{III}}$ leg, and with the $\boxed{\text{I}}$ and $\boxed{\text{II}}$ legs directed inwards, as shown in figure 2(i) in the reduced diagrams. This is because the contribution due to the $\boxed{\text{D}}$ vertex in figure 2(i) is exactly cancelled by that due to the $\boxed{\text{D}}$ vertex in figure 2 (j). The vertex in figure 2(j) is obtained by interchanging the $\boxed{\text{I}}$ and $\boxed{\text{II}}$ legs of figure 2 (i), or by the transformation $\Theta^{(n)}_{ij\ldots z} \rightarrow \Theta^{(n)}_{ji\ldots z}$. The correlation functions involving the $\boxed{\text{I}}$ and $\boxed{\text{II}}$ legs are unchanged, due to the equality in equation 69. Due to the antisymmetry condition equation 25, the value of the diagram in figure 2(j) is exactly negative of that in figure 2(i), and therefore these two diagrams exactly cancel. Therefore, it is only possible to have terminal Θ vertices with terminal $\boxed{\text{I}}$ or $\boxed{\text{II}}$ legs.

Due to the above rules, there are only three possible terminal vertices in the reduced diagrams. The first is a $\boxed{\text{A}}$ type, with either a $\boxed{\text{I}}$ or $\boxed{\text{II}}$ leg as the terminal leg, with the additional restriction that the $\boxed{\text{II}}$ leg which is not a terminal leg cannot be part of a bubble. The second is a $\boxed{\text{B}}$ primordial vertex, with hatted legs in both directions of increasing time. The third is a $\boxed{\text{D}}$ terminal vertex, with terminal $\boxed{\text{I}}$ or $\boxed{\text{II}}$ legs.

For the internal vertices in correlation functions, the rules contain some minor modifications of the rules for terminal vertices above. The major modification is that the $\boxed{\text{III}}$ legs in all the vertices could be directed either forward or backward in time.

1. As in the case of the terminal vertices, $\boxed{\text{C}}$ vertices (figures 2 (a) and (c)) are cancelled by $\boxed{\text{A}}$ vertices with a bubble involving the $\boxed{\text{II}}$ legs (figure 2(b) and (d)).

2. There are no bubbles involving the $\boxed{\text{I}}$ or the hatted $\boxed{\text{II}}$ legs, (figure 2(e)) due to causality.

3. $\boxed{\text{A}}$ vertices with hatted $\boxed{\text{I}}$ and $\boxed{\text{II}}$ legs in the same direction (figure 2(f)) are cancelled by $\boxed{\text{B}}$ vertices with $\boxed{\text{I}}$ and $\boxed{\text{II}}$ legs in the same direction (figure 2(g)).

4. $\boxed{\text{B}}$ vertices with the hatted $\boxed{\text{I}}$ and $\boxed{\text{II}}$ legs in opposite directions (figure 2 (k)) are permitted only if time increases on both sides of the vertex, that is, the vertex has the earliest time index (primordial vertex). However, in this case, the $\boxed{\text{B}}$ vertex in figure 2(k) is exactly cancelled by a $\boxed{\text{A}}$ vertex shown in figure 2(l).

5. It is not possible to have $\boxed{\text{D}}$ vertices with both the $\boxed{\text{I}}$ and $\boxed{\text{II}}$ legs in the same direction, since these are exactly cancelled by equivalent $\boxed{\text{D}}$ vertices with $\boxed{\text{I}}$ and $\boxed{\text{II}}$ legs interchanged, as a consequence of the antisymmetry condition equation 25. The diagrams are identical to those for terminal vertices shown in figures 2(i) and 2(j), except that these are now internal vertices linked by $\boxed{\text{III}}$ legs on both sides.

Due to these rules, the only internal vertices in the reduced diagrams are $\boxed{\text{A}}$ or $\boxed{\text{D}}$ vertices in which the $\boxed{\text{I}}$ leg is directed towards increasing time, the $\boxed{\text{II}}$ leg is directed towards decreasing time and the $\boxed{\text{III}}$ legs could be directed towards increasing or decreasing time.

Using the above rules, the reciprocal relations for the correlations between hatted and unhatted fields, $\Sigma_{\psi\hat{\psi}}$, can be proved as follows. These diagrams contain a hatted leg at one end and an unhatted leg at the other end. Since there are no primordial internal vertices, all diagrams contain $\boxed{\text{A}}$ or $\boxed{\text{D}}$ vertices in which the $\boxed{\text{I}}$ hatted legs are directed towards increasing time, while the $\boxed{\text{II}}$ unhatted legs are directed towards decreasing time. A typical diagram for the self-energy $\langle\tilde{\psi}_i(\mathbf{q})\hat{\psi}_j(\mathbf{q})\rangle$ is shown in figure 3 (a). Note that this diagram represents the self-energy $(\chi(\mathbf{k}))^{-1}_{ik}\boldsymbol{\Sigma}_{\psi_k\hat{\psi}_j}(\mathbf{q})$, which is the left side of equation 73. From this diagram, we can obtain a diagram for the right side of equation 73, $\boldsymbol{\Sigma}_{\hat{\psi}_i\psi_k}(-\mathbf{q})\chi^{-1}_{kj}(\mathbf{k})$, which is the self-energy for $\langle\tilde{\psi}_j(-\mathbf{q})\hat{\psi}_i(\mathbf{q})\rangle$, as follows,

1. We interchange the all vertices

$$\Gamma^{(n)}_{ijl\cdots z}(\mathbf{k},\mathbf{k}',\mathbf{k}_l,\cdots,\mathbf{k}_z) \rightarrow \Gamma^{(n)}_{jil\cdots z}(\mathbf{k}',\mathbf{k},\mathbf{k}_l,\cdots,\mathbf{k}_z). \tag{80}$$

Due to the reciprocal relation 24, the value of the vertices remain unchanged. In addition, we also make the change,

$$\Theta^{(n)}_{ijl\cdots z}(\mathbf{k},\mathbf{k}',\mathbf{k}_l,\cdots,\mathbf{k}_z) \rightarrow -\Theta^{(n)}_{jil\cdots z}(\mathbf{k}',\mathbf{k},\mathbf{k}_l,\cdots,\mathbf{k}_z). \tag{81}$$

Therefore, all vertices due to the reversible term in the non-linear Langevin equations change sign.

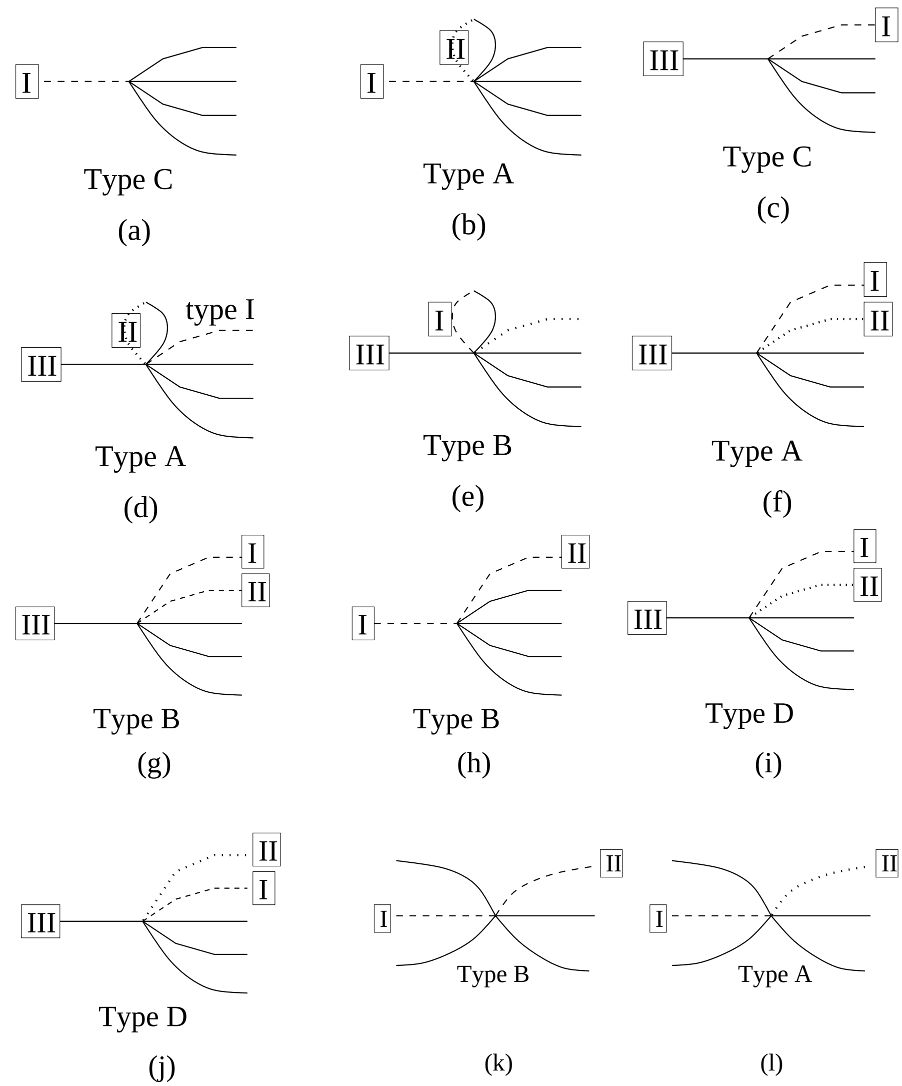

FIG. 2: Figures illustrating the rules for terminal vertices in the diagrams for the self-energies.

2. All the $\boxed{\mathrm{I}}$ legs are interchanged to $\boxed{\mathrm{II}}$ legs, and vice versa, as shown in figure 3.

3. In the process, the direction of time, from left to right in figure 3(a), is reversed in figure 3 (b). Due to the time reversal, all irreversible terms (Γ vertices) in the Langevin equation with even time parity remain unchanged. All reversible terms (Θ vertices) with odd time parity change sign. However, note that the reversible terms have already changed sign once due to the antisymmetry in equation 81. Therefore, they recover the same sign as in figure 3(a).

4. Due to the above, all internal correlation functions involving $\boxed{\mathrm{II}}$ vertices are changed to correlation functions involving $\boxed{\mathrm{I}}$ vertices, and vice-versa. All $\boxed{\mathrm{III}}$ vertices remain unchanged. For example, taking just two correlation functions involving the terminal vertices,

$$\langle \psi_l(\mathbf{q}_n)\tilde{\psi}_a(\mathbf{q}')\rangle_0 \rightarrow \langle \psi_l(\mathbf{q}_n)\hat{\psi}_a(\mathbf{q}')\rangle_0. \tag{82}$$

It is clear that the values of the correlation function $\langle \psi_n(\mathbf{q}_n)\hat{\psi}_a(\mathbf{q}')\rangle_0$, with time ordered from left to right, is the complex conjugate of $\langle \psi_n(\mathbf{q}_n)\tilde{\psi}_a(\mathbf{q}')\rangle_0$ with time ordered from right to left, due to equation 69. In addition, we have also carried out the transformation,

$$\langle \hat{\psi}_b(\mathbf{q}_b)\tilde{\psi}_m(\mathbf{q}_m)\rangle_0 \rightarrow \langle \tilde{\psi}_b(\mathbf{q}_b)\hat{\psi}_m(\mathbf{q}_m)\rangle_0. \tag{83}$$

In this case, as well, the values of the right sides is the complex conjugate of the left side due to equation 71.

5. It can be easily verified that in this transformation process, all other correlation functions remain unchanged, since they involve only unhatted fields, and the correlations of these fields are all real.

Therefore, the term in self-energy in equation 3(a), which is $\chi_{ik}(\mathbf{k})\mathbf{\Sigma}_{\psi_k\hat{\psi}_j}(\mathbf{q})$, is the complex conjugate of 3(b), $\mathbf{\Sigma}_{\hat{\psi}_i\psi_k}(\mathbf{q})\chi_{kj}(\mathbf{q})$. Since every term in the equation for the self-energy of the type shown in figure 3(a) has an equivalent term of the type shown in figure 3(b), it is proved that equation 73 is valid term-by-term in the expansion.

Next, we come to the relation between the correlation functions for the hatted and unhatted fields, $\Sigma_{\hat{\psi}\psi}$ and $\Sigma_{\hat{\psi}\hat{\psi}}$. The diagrams for the self-energy for the correlation functions contain two terminal hatted legs, and these are obtained by modifications of the vertex with the terminal unhatted leg in figures 3 (a) and (b). The reasoning for relating the correlation functions for the hatted and unhatted fields is different for terminal Γ and Θ vertices, and so we discuss the two separately.

In the case of a Γ terminal vertex, time increases outward from a set of 'primordial' vertices somewhere in the diagram, which are at the earliest time in compared to vertices on either side. The primordial vertices are defined such that the vertices closest to these, on either side, have a later time index than the primordial vertices. On either side of the primordial vertices, the rules for the vertices are identical to those for the terminal and internal vertices for the response functions. Only the primordial vertices are different, because time increases outward on both sides. In these cases, the following rules are modified.

1. It is possible to have internal vertices with hatted legs directed in opposite directions at $\boxed{\mathrm{B}}$ primordial vertices, as shown in figure 2(k). This is because time is increasing outward on both sides of the primordial vertices.

2. For internal vertices, the diagrams due to $\boxed{\mathrm{B}}$ primordial vertices, as shown in figure 2(k), are exactly cancelled by $\boxed{\mathrm{A}}$ primordial vertices, as shown in figure 2(l). Therefore, the sum of all diagrams with internal primordial vertices is identically zero, and there are no internal primordial vertices in the reduced diagrams.

3. Terminal $\boxed{\mathrm{B}}$ primordial vertices for the correlation function can be of two types. The first is a $\boxed{\mathrm{B}}$ primoridal vertex with a terminal $\boxed{\mathrm{I}}$ leg, as shown in figure 4(a), while the second is a $\boxed{\mathrm{B}}$ primordial vertex with a terminal $\boxed{\mathrm{II}}$ leg, as shown in figure 4(b). However, the terminal $\boxed{\mathrm{B}}$ primordial vertex with a terminal $\boxed{\mathrm{I}}$ leg (figure 4(a)) is exactly cancelled by a terminal $\boxed{\mathrm{A}}$ primordial vertex, shown in figure 4(c). Therefore, there are non-zero contributions only from diagrams with terminal $\boxed{\mathrm{B}}$ vertices with a terminal $\boxed{\mathrm{II}}$ leg, as shown in figure 4(b).

The diagrams for the self energies $\mathbf{\Sigma}_{\hat{\psi}_i\hat{\psi}_j}(\mathbf{q})$ contain a primordial vertex at one of the two ends, as shown in figure 5. These diagrams are obtained by replacing the $\boxed{\mathrm{A}}$ terminal vertex in figure 3, which consist of a $\boxed{\mathrm{II}}$ unhatted terminal leg (on the right in figure 5(a) and on the left in figure 5(b)), with a $\boxed{\mathrm{B}}$ terminal vertex with a $\boxed{\mathrm{II}}$ terminal leg. Since this is the primordial vertex, the terminal leg has to be of type $\boxed{\mathrm{II}}$ (a diagram containing a vertex $\boxed{\mathrm{I}}$ terminal leg is cancelled by other equivalent diagrams due to rule 3 above).

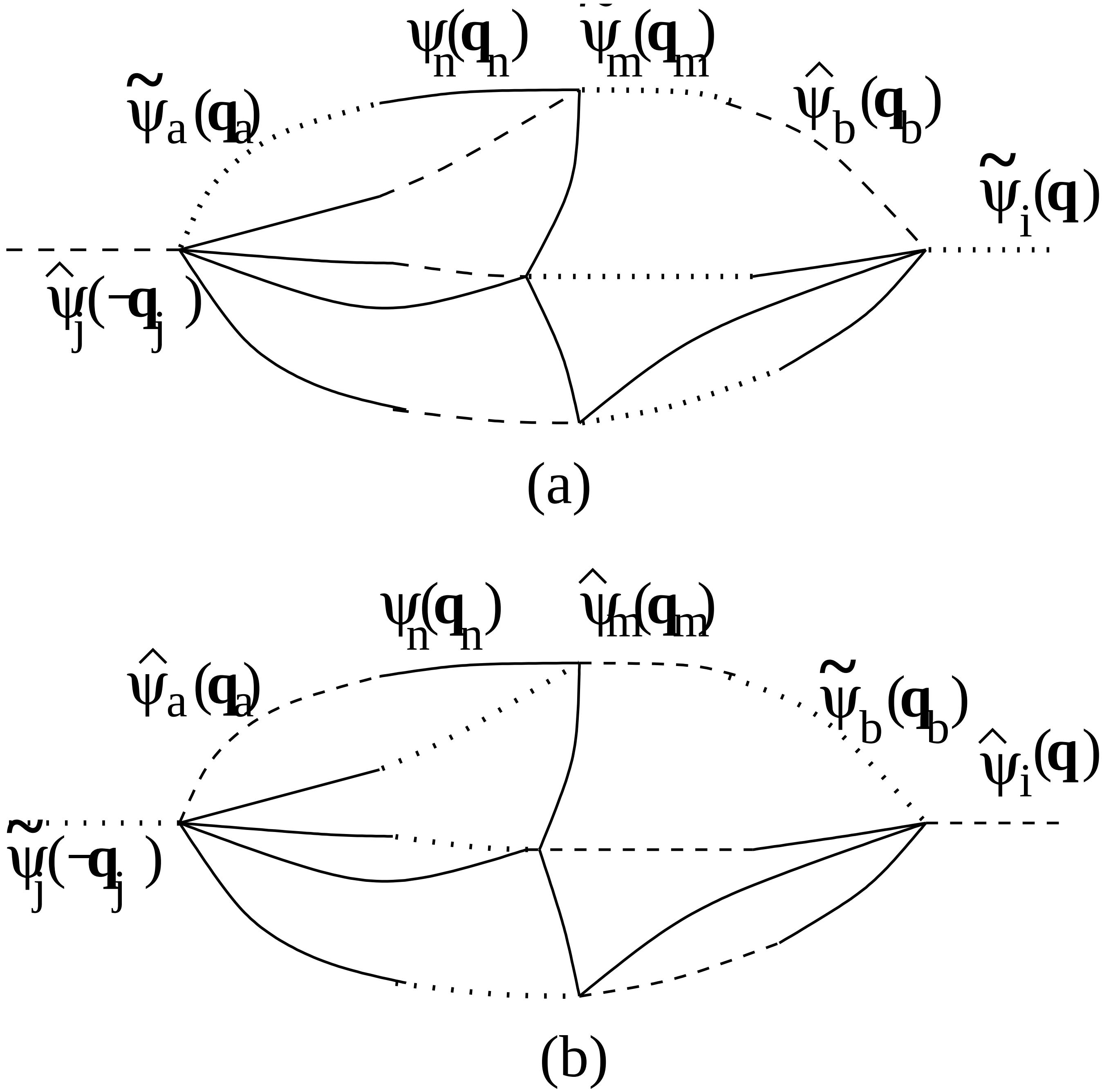

FIG. 3: Diagrams for $\chi_{ik}(\mathbf{q})\Sigma_{\psi_k\hat{\psi}_j}(\mathbf{q})$ and $\Sigma_{\hat{\psi}_i\psi_k}(-\mathbf{q})\chi_{kj}(-\mathbf{q})$.

Comparing figures 3(a) and 5(a), it is clear that all the internal vertices and correlation functions are identical. There is only a modification in the terminal vertex on the right, where $\tilde{\psi}_i(\mathbf{q}) = \chi^{-1}_{ik}\psi_k(\mathbf{q})$ is replaced by $\hat{\psi}_i(\mathbf{q})$. In this case, the coefficient of the $\boxed{\text{A}}$ vertex, $\Gamma^{(n)}_{bi\ldots z}$, had changed to that of the $\boxed{\text{B}}$ vertex, which is $-\Gamma^{(n)}_{ib\ldots z}$. Similarly, all the internal vertices in figures 3(b) and 5(b) are identical, but the terminal leg on the left, $\tilde{\psi}_j(-\mathbf{q}) = (\chi(\mathbf{k}))^{-1}_{jk}\psi_k(-\mathbf{q})$. The self energy $\mathbf{\Sigma}_{\hat{\psi}_i\hat{\psi}_j}(\mathbf{q})$ is just the sum of the two diagrams in figures 5(a) and 5(b). Therefore, we obtain equation 74 for the terminal Γ vertex.

In case of a terminal Θ ($\boxed{\text{D}}$) vertex, the equation 74 is obtained in a slightly different way. In this case, the equivalent of diagram 3(a), with a Θ vertex at the right, is the diagram 6(a). In this case, the vertex on the extreme right is a Θ vertex with a $\boxed{\text{I}}$ terminal hatted leg. The transformation from diagram 3(a) to 6(a) involves the

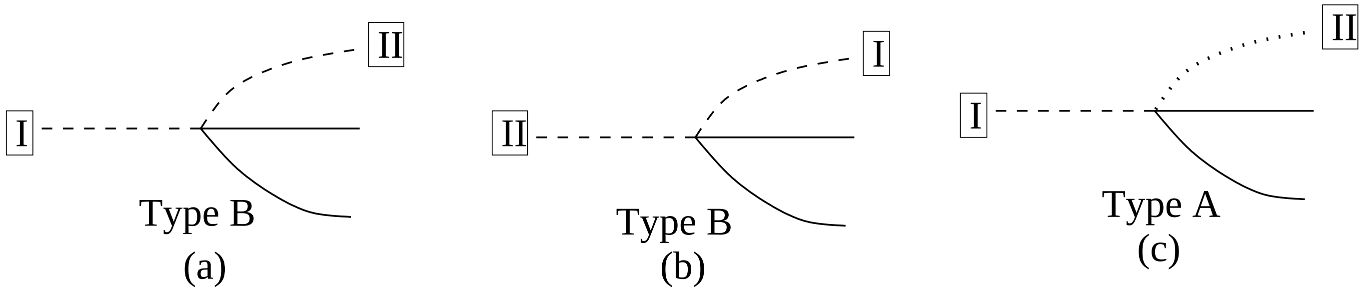

FIG. 4: Figures illustrating the rules for terminal vertices in the diagrams for the self-energy $\Sigma_{\hat{\psi}\hat{\psi}}$.

interchange $\Theta^{(n)}_{ib\ldots z} \to \Theta^{(n)}_{bi\ldots z}$. Since the Θ vertices are antisymmetric (equation 25), we find that the relations between the equations in figure 3(a) is $\chi_{il}\Sigma_{\psi_l\hat{\psi}_j}$ is equal to $-\Sigma_{\hat{\psi}_i\hat{\psi}_j}$. A similar relation holds between the figures 3(b) and 6(b). From this, we obtain equation 74 for the terminal Θ vertex. This shows that the self-energies in equation 72 satisfy the same reciprocal relations as the bare transport coefficients in equation 55.

Next, we come to the relation between the time derivative of the correlation function, equation 46, and the response function, 52. Comparing these equations, it is clear that the time derivative of the correlation function is equal to the response function if,

$$\begin{aligned} &\langle\psi_i(\mathbf{q}) \int_{\mathbf{q}'} \Gamma_{jk}(\{\psi\}, -\mathbf{q}, \mathbf{q}')\tilde{\psi}_k(-\mathbf{q}')\rangle - \langle\psi_i(\mathbf{q}) \int_{\mathbf{q}'} \Gamma'_{jk}(\{\psi\}, -\mathbf{q}, \mathbf{q}')\rangle \\ &= \langle\psi_i(\mathbf{q}) \int_{\mathbf{q}'} \Gamma_{jk}(\{\psi\}, -\mathbf{q}, \mathbf{q}')\hat{\psi}_k(-\mathbf{q}')\rangle. \end{aligned} \tag{84}$$

The most general diagram for the term on the right and the first term on the left of the above equation is shown in figure 7(a) and (b). Here, the extreme right vertex represents a typical term in the expansion of $\Gamma_{jk}(\{\psi\}, -\mathbf{q}, \mathbf{q}')\hat{\psi}_k(-\mathbf{q}')$ and $\Gamma_{jk}(\{\psi\}, -\mathbf{q}, \mathbf{q}')\tilde{\psi}_k(-\mathbf{q}')$ respectively, while the vertex on the left is due to the non-linear terms in the Langevin equation. The rules for the internal and terminal vertices discussed above for the self-energies apply to these diagrams as well.

1. The terms due to terminal [C] vertices, shown in figure 2 (a) and (c), are exactly cancelled by terms with terminal [A] vertices which have a bubble involving the [II] leg, shown in figure 2 (b) and (d). Therefore, there are no terminal [C] vertices, or terminal [A] vertices with bubbles involving the [II] leg, in the diagrams for the terms on the left and right sides of equation 84. It is easily seen that the same rule also applies to all internal vertices.

2. Diagrams with [A] vertices with a terminal [III] leg are exactly cancelled by equivalent diagrams with [B] vertices with a terminal [III] leg, as shown in figure 2(f) and 2(g).

3. The terms due to primordial internal vertices of [A] and [B] cancel, and so there are no contributions due to primordial internal vertices.

4. In a similar manner, all contributions due to primordial terminal [A] and [B] vertices on the left with terminal [I] legs, in figures 7(a) and 7(b), cancel.

Due to the above rules, non-zero contributions are only due to terminal [A] or [D] vertices which are not primordial, with terminal [I] or [II] legs, so that time increases monotonically from right to left or vice-versa in these diagrams. The typical diagram for the term on the right side of equation 84, shown in figure 7(a), has a terminal [A] or [D] vertex with a terminal [I] leg, with time increasing from right to left. The equivalent diagram for the term on the left side of equation 84, shown in figure 7(b), has a terminal [A] or [D] vertex with a terminal [II] leg. The latter is obtained by interchanging all the [I] and [II] legs in the former. In this transformation, if the terminal vertex is type [A] , the values of all the vertices are unchanged (equation 17), due to the Onsager reciprocal relations 24. In addition, the internal correlation functions for the hatted and unhatted fields also remain unchanged, due to the relations 69 to 71. Therefore, the non-zero contributions in the expressions for the time derivative of the correlation

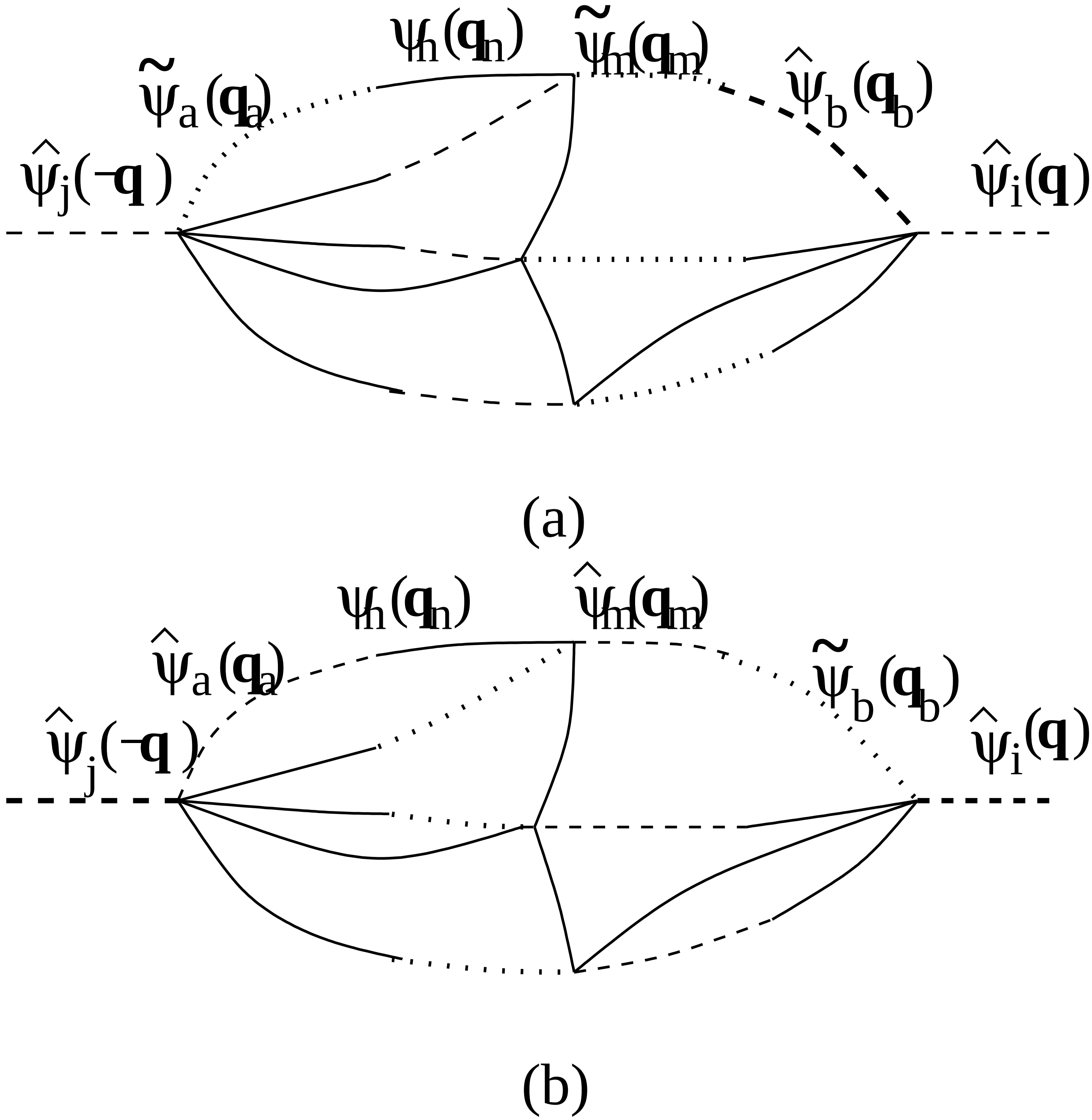

FIG. 5: Diagrams for the self energy $\Sigma_{\hat{\psi}_i \hat{\psi}_j}(\mathbf{q})$ with a terminal $\boxed{\text{B}}$ vertex.

function (46) and response functions (52), which are the left and right sides of equation 84, are equal at each order in the perturbation expansion. In case the a terminal or internal vertex is $\boxed{\text{D}}$, the value of the vertex changes sign when we go from figure 7(a) to figure 7(b), due to the antisymmetry condition 18. In addition, the value of the non-linear term represented by the vertex on the left side of equation 7(a) also changes sign due to time reversal. Therefore, we obtain equality of the terms shown in diagrams in figures 7(a) and 7(b). This proves that the fluctuation-dissipation theorem is valid at each order in the perturbation expansion.

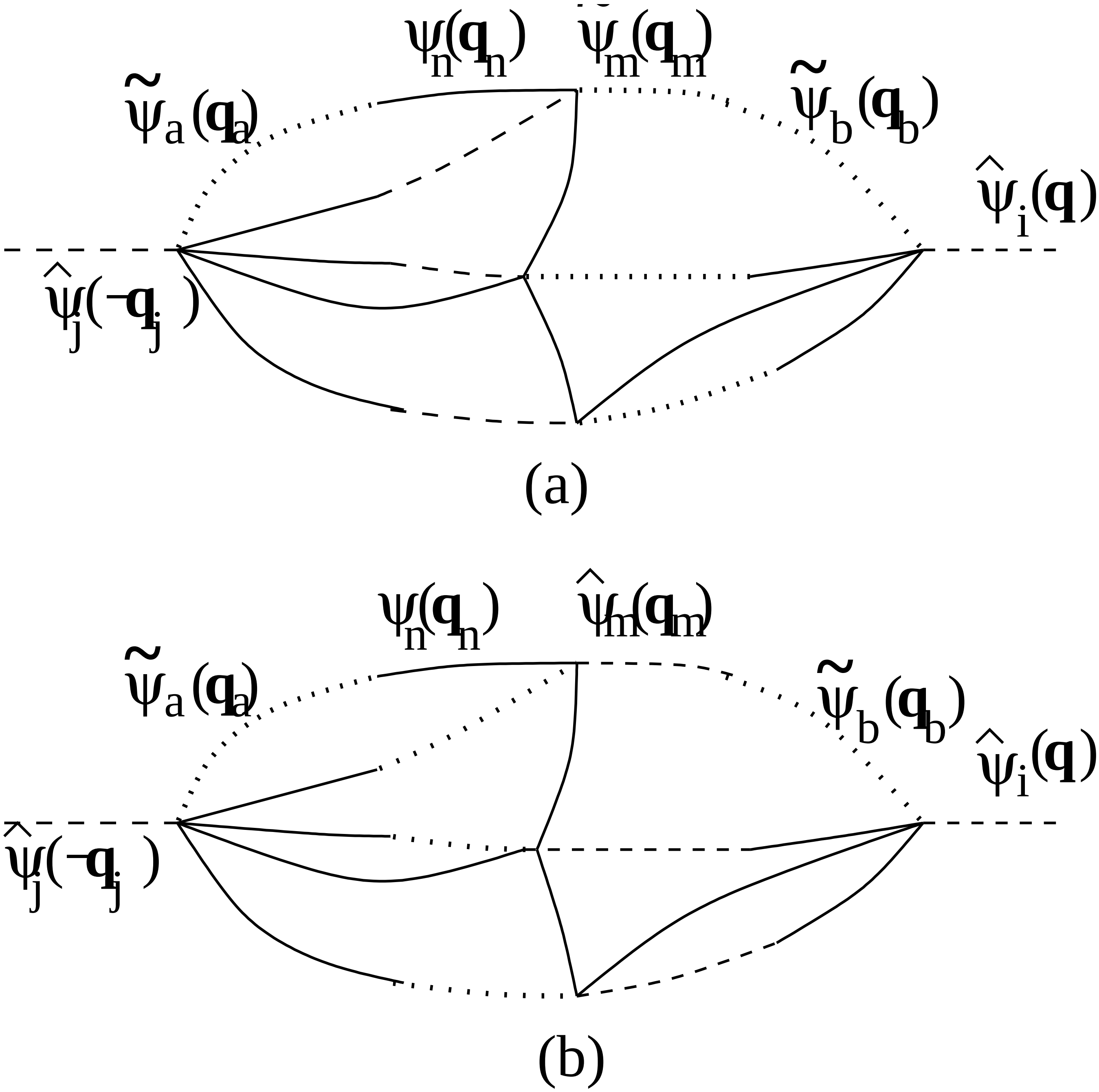

FIG. 6: Diagrams for the self energy $\Sigma_{\hat{\psi}_i\hat{\psi}_j}(\mathbf{q})$ with a terminal $\boxed{\text{D}}$ vertex.

VI. FIELD-DEPENDENT SUSCEPTIBILITY:

VII. CONCLUSION:

The non-linear Langevin equations have been analysed using the functional integral formalism, with the Ito interpretation of the noise correlations. It is shown that these equations satisfy the fluctuation-dissipation relations, at each order in the perturbation expansion, when the non-linearities in the Langevin equation are due to field-dependent kinetic coefficients, and the free energy functional is quadratic in the fields (field-independent susceptibility). This is regardless of the form of the kinetic coefficient and degree of non-linearity, provided each term in the expansion of the kinetic coefficient satisfies the Onsager reciprocal relations for the irreversible terms in the Langevin equation, and

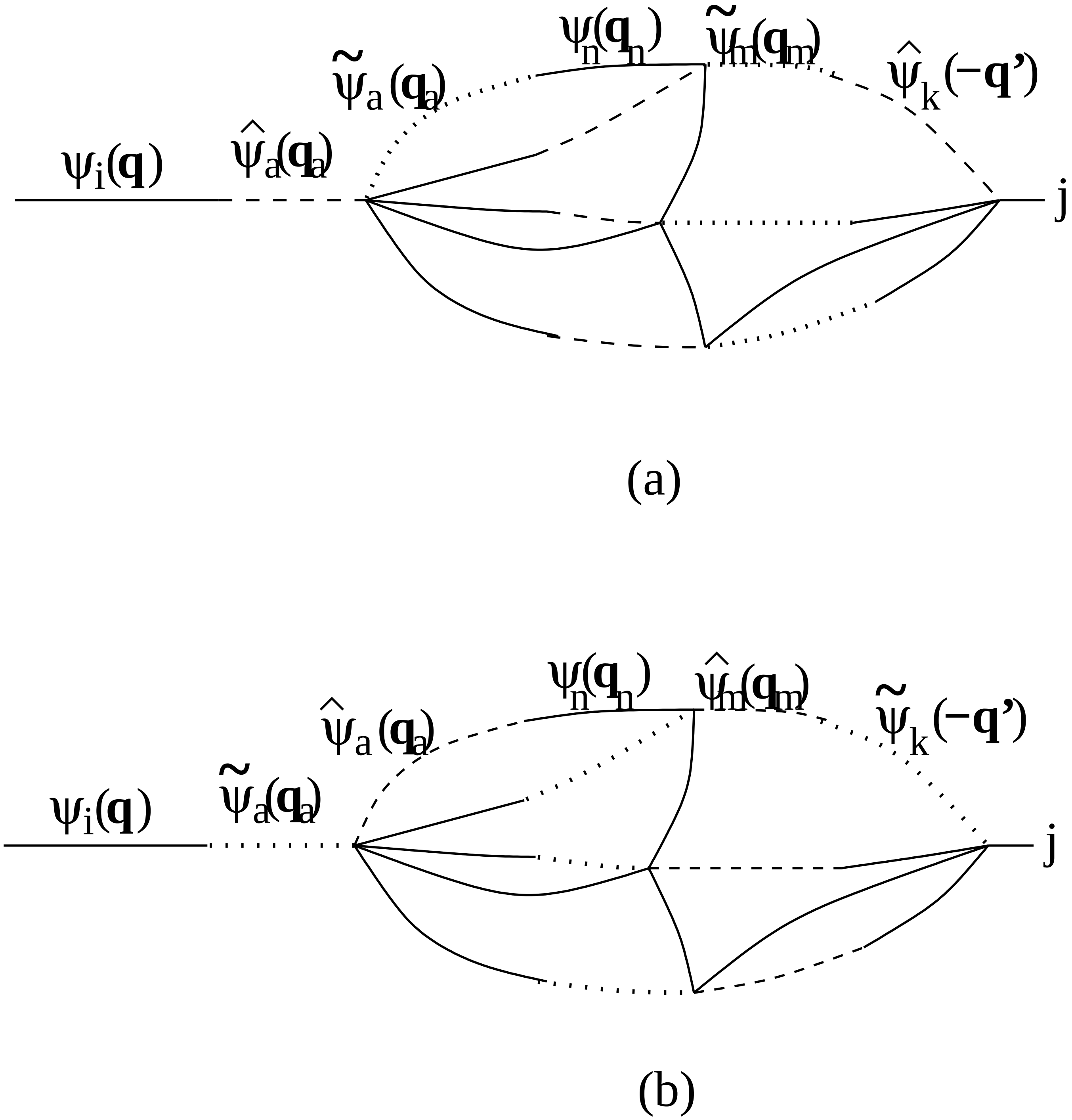

FIG. 7: Equivalent diagrams for the correlation and response functions.

the antisymmetry relation for the reversible terms. This settles the issue of validity of fluctuation-dissipation relations for systems with field-dependent kinetic coefficients and a quadratic free energy functional.

When the kinetic coefficients are field-independent, and the susceptibility is field-dependent, the fluctuation-dissipation relation is still valid, provided the renormalised susceptibility is used in the Langevin equation. This is a direct result from the ergodic hypothesis, because if the the equilibrium and dynamical averages are equal, the fluctuation dissipation theorem is satisfied at all orders.

In the more complicated case where both the kinetic coefficients and susceptibility are field-dependent, it is much more difficult to prove that the fluctuation-dissipation relations hold. This is because there are two-distinct types of vertices, and it is virtually certain that the fluctuation-dissipation relations do not hold at each order in the perturbation expansion. In this case, a trivial extension of our analysis is that when the renormalised susceptibility is used in the Langevin equation (pre-averaging approximation), the fluctuation-dissipation relations are valid at each order in the expansion. However, there are several coupling terms that are neglected in the pre-averaging approximation for the susceptibility, and it is a formidable challenge to prove that the sum of all these terms is equal in both the correlation and response functions. Despite this, the pre-averaging approximation may be a useful practical approximation in solving non-linear Langevin equations, since it ensures that the fluctuation-dissipation relations are satisfied.

This research was supported in part by the National Science Foundation under Grant No. NSF PHY05-51164, and in part by the J. C. Bose fellowship, Department of Science and Technology, Government of India. The author would like to thank Prof. S. Ramaswamy for useful discussions.

---

[1] S. P. Das and G. F. Mazenko, *Phys. Rev. A*, **34**, 2265, 1986.
[2] W. Gotze and L. Sjogren, *Rep. Prog. Phys.*, **55**, 241, 1992.
[3] V. Kumaran, *J. Chem. Phys.*, **104**, 3120-3133, 1996.
[4] V. Kumaran and G. H. Fredrickson, *J. Chem. Phys.* , **105** , 8304 - 8313, (1996).
[5] P. C. Hohenberg and B. I. Halperin, Theory of dynamical critical phenomena, *Rev. Mod. Phys.* **49**, 435 - 479 (1977).
[6] P. C. Martin, E. D. Siggia and H. A. Rose, *Phys. Rev. A*, **8**, 423, 1973.
[7] U. Deker and F. Haake, *Phys. Rev. A*, **11**, 2043, 1975.
[8] B. Kim and K. Kawasaki *Journal of Statistical Mechanics: Theory and Experiment*, P02004, **2008**, 2008.
[9] A. Basu and S. Ramaswamy, *Journal of Statistical Mechanics: Theory and Experiment*, P11003, **2007**, 2007.
[10] T. H. Nishino and H. Hayakawa, *Physical Review E*, **78**, 061502, 2008.
[11] G. Szamel, J. Chem. Phys. **127**, 084515, 2007.
[12] D. Dean, *J. Phys. A*, **29**, L613, (1996).
[13] K. Kawasaki, Physica A **208**, 35, 1994.
[14] K. Miyazaki and D. R. Reichman, *J. Phys. A: Math. Gen.* **38**, L343, (2005).
[15] A. Andreanov, G. Biroli and A. Lefevre, *J. Stat. Mech.* **2006**, P07008, (2006).
[16] R. V. Jensen, *J. Stat. Phys.*, **25**, 183 (1981).
[17] H. K. Janssen, Z. Phys. B, **23**, 377 (1976).
[18] R. Phythian, J. Phys. A, **10**, 777 (1977).
[19] N. van Kampen, *Stochastic Processes in Physics and Chemistry*, North- Holland, New York, (1981).
[20] R. L. Stratonovich, *Topics in the Theory of Random Noise*, Gordon-Breach, New York, (1963).

Appendix A: Average of time derivative of correlation function over noise realisations:

Here, we provide the details of the calculation of the average of the right side of equation 45 over noise realisations to obtain equation 46. It is more convenient to carry out the averaging in Fourier space.

$$\mathcal{G}(\mathbf{q}) = G_i(\{\psi\}, \mathbf{q})\theta(\omega) \tag{A1}$$

The average over the noise realisations is defined as,

$$\langle \bullet \rangle_{\text{noise}} = c_{\mathcal{G}} \int_{\mathcal{G}} \bullet \exp\left(-\frac{1}{2} \int_{\mathbf{q}^\dagger, \mathbf{q}^\ddagger} \mathcal{G}^{\ddagger T} \cdot (\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger, \mathbf{q}^\dagger))^{-1} \mathcal{G}^\dagger\right) \tag{A2}$$

where $c_{\mathcal{G}}$ is the normalisation constant, $\mathcal{G}^\dagger$ is the column vector whose elements are $\mathcal{G}_i(-\mathbf{q}^\dagger)$, $\int_{\mathcal{G}} \equiv \prod_i \int d\mathcal{G}_i$, and $\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\dagger, \mathbf{q}^\ddagger)$ is the matrix whose elements are the averages of the noise correlations,

$$\begin{aligned} \mathcal{T}_{ij}^{\mathcal{G}}(\mathbf{q}^\dagger, \mathbf{q}^\ddagger) &= \langle \mathcal{G}(\mathbf{q}^\dagger)\mathcal{G}(\mathbf{q}^\ddagger)\rangle_{\text{noise}} \\ &= 2\Gamma_{ij}(\{\psi\}, \mathbf{q}^\dagger, \mathbf{q}^\ddagger) \end{aligned} \tag{A3}$$

The average over noise correlations in equation 45 is of the form,

$$c_{\mathcal{G}} \int_{\mathcal{G}} \mathcal{G}_i(\mathbf{q}) \exp\left(-\int_{\mathbf{q}^\dagger, \mathbf{q}^\ddagger} \hat{\Psi}^{\ddagger T} \cdot \mathcal{G}^\dagger \delta(\mathbf{q}^\dagger + \mathbf{q}^\ddagger)\right) \exp\left(-\frac{1}{2} \int_{\mathbf{q}^\dagger, \mathbf{q}^\ddagger} \mathcal{G}^{\ddagger T} \cdot (\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger, \mathbf{q}^\dagger))^{-1} \cdot \mathcal{G}^\dagger\right) \tag{A4}$$

where $T_{\mathcal{G}}$ is the noise correlation, $\hat{\psi}_i^\dagger = \hat{\psi}_i(-\mathbf{q}^\dagger)$ and $\hat{\psi}_i^\ddagger = \hat{\psi}_i(-\mathbf{q}^\ddagger)$. The above average can be symmetrised and expressed in matrix form as follows,

$$\begin{aligned}
& c_{\mathcal{G}} \int_{\mathcal{G}} \mathcal{G}_i(\mathbf{q}) \exp\,(-\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} \hat{\Psi}^{T\ddagger}\cdot\mathcal{G}^\dagger \delta(\mathbf{q}^\dagger+\mathbf{q}^\ddagger)) \exp\left(-\frac{1}{2}\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} \mathcal{G}^\ddagger\cdot(\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger,\mathbf{q}^\dagger))^{-1}\cdot\mathcal{G}^\dagger\right) \\
&= c_{\mathcal{G}} \int_{\mathcal{G}} \mathcal{G}_i(\mathbf{q}) \exp\,(-\frac{1}{2}\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} (\hat{\Psi}^{\ddagger T}\cdot\mathcal{G}^\dagger + \mathcal{G}^{\ddagger T}\cdot\hat{\Psi}^\dagger)\delta(\mathbf{q}^\dagger+\mathbf{q}^\ddagger)) \\
&\quad \times \exp\left(-\frac{1}{2}\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} (\mathcal{G}^{\ddagger T}\cdot\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger,\mathbf{q}^\dagger)\cdot\mathcal{G}^\dagger)\right) \\
&= c_{\mathcal{G}} \int_{\mathcal{G}} \mathcal{G}_i(\mathbf{q}) \\
&\quad \times \exp\left(-\frac{1}{2}\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} (\mathcal{G}^\ddagger - (\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger,\mathbf{q}^\dagger)\cdot\hat{\Psi}^\ddagger))^T\cdot(\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger,\mathbf{q}^\dagger))^{-1}\cdot(\mathcal{G}^\dagger - (\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger,\mathbf{q}^\dagger)\cdot\hat{\Psi}^\dagger))\right) \\
&\quad \times \exp\,(-\frac{1}{2}\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} \hat{\Psi}^{\ddagger T}\cdot\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger,\mathbf{q}^\dagger)\cdot\hat{\Psi}^\dagger) \\
&= c_{\mathcal{G}} \int_{\mathcal{G}}\int_{\mathbf{q}'} ((\mathcal{G}(-\mathbf{q}')\delta(\mathbf{q}+\mathbf{q}') - \mathcal{T}^{\mathcal{G}}(\mathbf{q},\mathbf{q}')\cdot\hat{\Psi}(-\mathbf{q}'))_i + (\mathcal{T}^{\mathcal{G}}(\mathbf{q},\mathbf{q}')\cdot\hat{\psi}(-\mathbf{q}'))_i) \\
&\quad \times \exp\left(-\frac{1}{2}\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} (\mathcal{G}^\ddagger - (\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger,\mathbf{q}^\dagger)\cdot\hat{\Psi}^\dagger))^T\cdot(\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger,\mathbf{q}^\dagger)^{-1}\cdot(\mathcal{G}^\dagger - (\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger,\mathbf{q}^\dagger)\cdot\hat{\Psi}^\dagger))\right) \\
&\quad \times \exp\,(-\frac{1}{2}\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} \hat{\Psi}^{\ddagger T}\cdot\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger,\mathbf{q}^\dagger)\cdot\hat{\Psi}^\dagger)
\end{aligned} \tag{A5}$$

The first term in the pre-exponential in the above equation averages to zero, while the second term, upon averaging over the noise realisations, gives,

$$\begin{aligned}
& c_{\mathcal{G}} \int_{\mathcal{G}} \mathcal{G}_i(\mathbf{q}) \exp\left(-\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} \hat{\psi}_i^\ddagger \mathcal{G}_i^\dagger\right) \exp\left(-\frac{1}{2}\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} \mathcal{G}_i^\ddagger (T_{\mathcal{G}}^\dagger)_{ij}^{-1} \mathcal{G}_j^\dagger \delta(\mathbf{q}^\dagger+\mathbf{q}^\ddagger)\right) \\
&= c_{\mathcal{G}} \int_{\mathbf{q}'} (\mathcal{T}^{\mathcal{G}}(\mathbf{q},\mathbf{q}')\cdot\hat{\psi}(-\mathbf{q}'))_i \exp\,(-\frac{1}{2}\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} \hat{\Psi}^{\ddagger T}\cdot\mathcal{T}^{\mathcal{G}}(\mathbf{q}^\ddagger,\mathbf{q}^\dagger)\cdot\hat{\Psi}^\dagger)
\end{aligned} \tag{A6}$$

In the above equation, $((\mathcal{T}^{\mathcal{G}}(\mathbf{q},\mathbf{q}')\cdot\hat{\Psi}(-\mathbf{q}'))_i = 2\sum_j \int_{\mathbf{q}'} \Gamma_{ij}(\{\psi\},\mathbf{q},\mathbf{q}')\hat{\psi}_j(-\mathbf{q}')$.

Appendix B: Bare correlation and response functions:

The bare correlation and response functions can be determined by defining the generating functional for the auxiliary fields, ξ_i and $\hat{\xi}_i$,

$$F[\Xi,\hat{\Xi}] = c\int D[\Psi]D[\hat{\Psi}] \exp\,(-L_0) \exp\left(\int_{\mathbf{q}} (\Xi^{*T}\cdot\Psi + \hat{\Xi}^{*T}\cdot\hat{\Psi}(\mathbf{q}))\right), \tag{B1}$$

where Ξ and $\hat{\Xi}$ are column vectors whose elements are $\xi_i(\mathbf{q})$, and $\hat{\xi}_i(\mathbf{q})$ respectively, and Ξ^* and $\hat{\Xi}^*$, the complex conjugates, are column vectors whose elements are $\xi_i(-\mathbf{q})$ and $\hat{\xi}_i(-\mathbf{q})$ respectively. The bare averages can be evaluated from the generating functional B1 as,

$$\langle\hat{\psi}_i(-\mathbf{q})\psi_j(\mathbf{q})\rangle_0 = \left.\frac{\delta^2 F}{\delta\hat{\xi}_i(\mathbf{q})\delta\xi_j(-\mathbf{q})}\right|_{\Xi=0,\hat{\Xi}=0}. \tag{B2}$$

Higher order correlation functions can also be calculated in a similar manner, for example,

$$\langle\hat{\psi}_i(\mathbf{q}_i)\hat{\psi}_m(\mathbf{q}_m)\psi_j(\mathbf{q}_j)\psi_n(\mathbf{q}_n)\rangle_0 = \left.\frac{\delta^4 F}{\delta\hat{\xi}_i(-\mathbf{q}_i)\delta\hat{\xi}(-\mathbf{q}_m)\delta\xi_j(-\mathbf{q}_j)\delta\xi_n(\mathbf{q}_n)}\right|_{\Xi=0,\hat{\Xi}=0}. \tag{B3}$$

In order to simplify the calculation, we rewrite L_0 in equation 54 as,

$$L_0 = \frac{1}{2}\int_{\mathbf{q}^\dagger}\int_{\mathbf{q}^\ddagger} \left(\Psi^T(\mathbf{q}^\ddagger) \;\; \hat{\Psi}^T(\mathbf{q}^\ddagger) \right) \left(\quad \mathbf{M}_0^{-1}(\mathbf{q}^\dagger)\delta(\mathbf{q}^\dagger + \mathbf{q}^\ddagger) \quad \right) \begin{pmatrix} \Psi(\mathbf{q}^\dagger) \\ \hat{\Psi}(\mathbf{q}^\dagger) \end{pmatrix}. \tag{B4}$$

The equation B1 can be reformulated by first symmetrising the last two terms in the equation,

$$\begin{aligned} &\int_{\mathbf{q}} \Xi^{*T}\cdot\Psi + \hat{\Xi}^{*T}\cdot\hat{\Psi} \\ &= \frac{1}{2}\int_{\mathbf{q}} \Xi^{*T}\cdot\Psi + \Psi^{*T}\cdot\Xi + \hat{\Xi}^{*T}\cdot\hat{\Psi} + \hat{\Psi}^{*T}\cdot\hat{\Xi} \\ &= \frac{1}{2}\int_{\mathbf{q}} (\Xi^{*T}\cdot\bar{\mathbf{M}}\cdot\bar{\mathbf{M}}^{-1}\cdot\Psi + \Psi^{*T}\cdot\bar{\mathbf{M}}^{-1}\cdot\bar{\mathbf{M}}\cdot\Xi \\ &\quad +\hat{\Xi}^{*T}\cdot\bar{\mathbf{M}}\cdot\bar{\mathbf{M}}^{-1}\cdot\hat{\Psi} + \hat{\Psi}^{*T}\cdot\bar{\mathbf{M}}^{-1}\cdot\bar{\mathbf{M}}\cdot\hat{\Xi}). \end{aligned} \tag{B5}$$

Since the matrix $\bar{\mathbf{M}}(\mathbf{q})$ is Hermetian, $\bar{\mathbf{M}}(-\mathbf{q})^T = \bar{\mathbf{M}}^{*T} = \bar{\mathbf{M}}(\mathbf{q})$, the above equation can be written as,

$$\begin{aligned} &\int_{\mathbf{q}} \Xi^{*T}\cdot\Psi + \hat{\Xi}^{*T}\cdot\hat{\Psi}(\mathbf{q}) \\ &= \frac{1}{2}\int_{\mathbf{q}^\dagger} ((\bar{\mathbf{M}}\cdot\Xi)^{*T}\cdot\bar{\mathbf{M}}^{-1}\cdot\Psi + \Psi^{*T}\cdot\bar{\mathbf{M}}^{-1}\cdot(\bar{\mathbf{M}}\cdot\Xi) \\ &\quad +(\bar{\mathbf{M}}\cdot\hat{\Xi})^{*T}\cdot\bar{\mathbf{M}}^{-1}\cdot\hat{\Psi} + \hat{\Psi}^{*T}\cdot\bar{\mathbf{M}}^{-1}\cdot(\bar{\mathbf{M}}\cdot\hat{\Xi})). \end{aligned} \tag{B6}$$

For calculating the averages, it is convenient to rewrite the above equation in a manner similar to equation B3,

$$\begin{aligned} &\int_{\mathbf{q}} \Xi^{*T}\cdot\Psi + \hat{\Xi}^{*T}\cdot\hat{\Psi}(\mathbf{q}) \\ &= \frac{1}{2}\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} ((\bar{\mathbf{M}}^\ddagger\cdot\Xi^\ddagger)^T\cdot(\bar{\mathbf{M}}^\dagger)^{-1}\cdot\Psi^\dagger + {\Psi^\ddagger}^T\cdot(\bar{\mathbf{M}}^\dagger)^{-1}\cdot(\bar{\mathbf{M}}^\dagger\cdot\Xi^\dagger) \\ &\quad +(\bar{\mathbf{M}}^\ddagger\cdot\hat{\Xi}^\ddagger)^T\cdot(\bar{\mathbf{M}}^\dagger)^{-1}\cdot\hat{\Psi^\dagger} + \hat{\Psi^\ddagger}^T\cdot(\bar{\mathbf{M}}^\dagger)^{-1}\cdot(\bar{\mathbf{M}}^\dagger\cdot\hat{\Xi}^\dagger)) \\ &\quad \times\delta(\mathbf{q}^\dagger + \mathbf{q}^\ddagger), \end{aligned} \tag{B7}$$

where $\Xi^\dagger = \Xi(\mathbf{q}^\dagger)$, $\Xi^\ddagger = \Xi(\mathbf{q}^\ddagger)$, $\Psi^\dagger = \Psi(\mathbf{q}^\dagger)$, $\Psi^\ddagger = \Psi(\mathbf{q}^\ddagger)$, $\bar{\mathbf{M}}^\dagger = \bar{\mathbf{M}}(\mathbf{q}^\dagger)$ and $\bar{\mathbf{M}}^\ddagger = \bar{\mathbf{M}}(\mathbf{q}^\ddagger)$.

Using the above transformations, the generating functional $F[\Xi,\hat{\Xi}]$ can be written as,

$$\begin{aligned} F[\Xi,\hat{\Xi}] &= c\int D[\Xi]D[\hat{\Xi}]D[\Psi]D[\hat{\Psi}] \\ &\exp\left(-\frac{1}{2}\left(\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} (\Psi^\ddagger - \bar{\mathbf{M}}^\ddagger\cdot\Xi^\ddagger)^T \;\; (\hat{\Psi^\ddagger} - \bar{\mathbf{M}}^\ddagger\cdot\hat{\Xi}^\ddagger)^T \right)\left(\bar{\mathbf{M}}^{\dagger-1}\delta(\mathbf{q}^\dagger + \mathbf{q}^\ddagger)\right)\begin{pmatrix} (\Psi^\dagger - \bar{\mathbf{M}}^\dagger\cdot\Xi^\dagger) \\ (\hat{\Psi^\dagger} - \bar{\mathbf{M}}^\dagger\cdot\hat{\Xi}^\dagger) \end{pmatrix}\right) \\ &\times\exp\left(-\frac{1}{2}\int_{\mathbf{q}^\dagger,\mathbf{q}^\ddagger} \left(\Xi^{\ddagger T} \;\; \hat{\Xi}^{\ddagger T} \right)\left(\bar{\mathbf{M}}^\dagger\delta(\mathbf{q}^\dagger + \mathbf{q}^\ddagger)\right)\begin{pmatrix} \Xi^\dagger \\ \hat{\Xi}^\dagger \end{pmatrix}\right). \end{aligned} \tag{B8}$$

The integrals over the ψ_i and $\hat{\psi}_i$ fields are explicitly performed, to obtain,

$$F[\Xi,\hat{\Xi}] = c\int D[\Xi]D[\hat{\Xi}]\exp\left(\frac{1}{2}\left(\Xi^{\ddagger T} \;\; \hat{\Xi}^{\ddagger T} \right)\left(\bar{\mathbf{M}}^\dagger\delta(\mathbf{q}^\dagger + \mathbf{q}^\ddagger)\right)\begin{pmatrix} \Xi^\dagger \\ \hat{\Xi}^\dagger \end{pmatrix}\right). \tag{B9}$$

The correlation functions can now be calculated using equation B2,

$$\begin{aligned} \langle\psi_i(\mathbf{q})\psi_j(\mathbf{q}')\rangle_0 &= \frac{\delta^2 F[\Xi,\hat{\Xi}]}{\delta\xi_i(-\mathbf{q})\delta\xi_j(-\mathbf{q}')} \\ &= M_{ij}(\mathbf{q}^\dagger)\delta(\mathbf{q}^\dagger + \mathbf{q}^\ddagger)\delta(\mathbf{q}' + \mathbf{q}^\ddagger)\delta(\mathbf{q} + \mathbf{q}^\dagger) \\ &= M_{ji}(-\mathbf{q})\delta(\mathbf{q} + \mathbf{q}') \\ &= M_{ij}(\mathbf{q})\delta(\mathbf{q} + \mathbf{q}') \\ &= ((-\imath\omega + \bar{\mathbf{\Gamma}}(\mathbf{k})\cdot(\chi(\mathbf{k}))^{-1})\cdot(2\bar{\mathbf{\Gamma}}(\mathbf{k}))\cdot(\imath\omega + (\chi(\mathbf{k}))^{-1}\cdot\bar{\mathbf{\Gamma}}(\mathbf{k})))_{ij}\delta(\mathbf{q} + \mathbf{q}'), \end{aligned} \tag{B10}$$

and

$$\begin{aligned}
\langle \psi_i(\mathbf{q})\hat{\psi}_j(\mathbf{q}')\rangle_0 &= \frac{\delta^2 F[\Xi,\hat{\Xi}]}{\delta\xi_i(-\mathbf{q})\hat{\xi}_j(-\mathbf{q}')} \\
&= M_{ji}(\mathbf{q}^\dagger)\delta(\mathbf{q}^\dagger+\mathbf{q}^\ddagger)\delta(\mathbf{q}'+\mathbf{q}^\ddagger)\delta(\mathbf{q}+\mathbf{q}^\dagger) \\
&= M_{\hat{j}i}(-\mathbf{q})\delta(\mathbf{q}+\mathbf{q}') \\
&= M_{i\hat{j}}(\mathbf{q})\delta(\mathbf{q}+\mathbf{q}') \\
&= (-\imath\omega+\bar{\mathbf{\Gamma}}(\mathbf{k})\cdot(\chi(\mathbf{k}))^{-1})_{ij}\delta(\mathbf{q}+\mathbf{q}'),
\end{aligned} \tag{B11}$$

where $\hat{\jmath} = j + N$, where N is the total number of elements in the Ψ and $\hat{\Psi}$ column matrices.