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Abstract

We apply a Langevin model by imposing additive and multiplicative noises to study thermally activated diffusion over a fluctuating barrier in underdamped case. The barrier fluctuation is characterized by Gaussian colored noise with exponential correlation. We present the exact solutions for the first and second moments. Furthermore, we calculate asymptotic probability for a Brownian particle passing over the fluctuating barrier by direct simulations. The results indicate that the correlation of fluctuating barrier is crucial for barrier crossing dynamics.

Keywords: Langevin equation; fluctuating barrier; multiplicative noise; Gaussian colored noise

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I. INTRODUCTION

The problem of thermally activated diffusion over various subsets of barrier potential is ubiquitous in physical contexts, chemical reactions, and biological transport [1, 2]. For past decades, dynamic processes across a fluctuating barrier potential have attracted much attention (for a review see [2]). In many complex environments, e.g., in dye lasers with pump noise [3], and chemical reactions in contact with fluctuating environment [4], the barrier height of the potential fluctuates over time stochastically instead of remaining static. Within Langevin equation description, the dynamics across a fluctuating barrier is usually modeled as an open system driven simultaneously by additive and multiplicative noises [1, 2, 5–9]. The internal thermal fluctuation is described by the additive noise and the fluctuation in the barrier height is characterized by multiplicative noise. Unlike the presence of additive noise only, the multiplicative noise with various correlation time may induce nontrivial results such as noise-induced nonequilibrium phase transition [10, 11], anomalous diffusion [12], and *stochastic activation* (SA) in stochastically fluctuating barrier systems [13].

It is noteworthy that much of abovementioned work focused mostly on the dynamic process in overdamped limit for sake of simplicity [2, 5–9, 11, 13]. However, barrier crossing dynamics have shown a significant difference between low and high friction limits [14–17]. In heavy-ion fusion reaction, compound nucleus can be formed if two colliding nuclei surmount the Coulomb barrier. The fusion probability increases with the increasing friction at low friction limit, while it decreases with the increasing friction in overdamped case owing to fast thermal equilibration and slowed diffusion [16, 17]. This leads to an evident peak value of fusion probability in the intermediate friction regime which divides two distinct dynamic behaviors. In the present work, we study analytically and numerically the problem of diffusion over a fluctuating barrier potential at underdamped limit. Barrier crossing processes in linear systems with time-dependent potential have been theoretically studied by several authors using Gaussian white noise [18–20], anomalous diffusion and interesting probability distribution have been shown in overdamped region.

Besides theoretical extension, our model has more practical consideration. Recently, a number of experimental data in synthesis of superheavy nuclei have shown remarkable enhancement of fusion cross sections at the energies below and near the barrier potential, which has been explained by the coupled-channel models by introducing a barrier distribution [21–

24]. It is considered that the barrier height encountered by each fusion event is stochastically distributed as a consequence of couplings of the relative motion to other degrees of freedom such as nuclear shape deformations [23, 24]. Due to the complex factors in heavy-ion collisions, such as neck fluctuation during neck formation [25], a time-dependent coupling which may lead to a fluctuating barrier is conceivable. Motivated by this assumption, in this work we consider a Langevin system similar to the one in [16], except a time-dependent barrier with random fluctuation described by Gaussian colored noise with exponential correlation. This more general description provides better understanding of barrier crossing dynamics in heavy-ion fusion reactions.

This paper is organized as follows: In Sec. II, we present a Langevin system subjected to additive and multiplicative noises to model the dynamics over a fluctuating barrier. In Sec. III, we study the stationary probability surmounting the fluctuating barrier by direct simulations for various situations. Finally the conclusion is given in Sec. IV.

II. THE STOCHASTIC DYNAMICS OVER A FLUCTUATING BARRIER

The dynamics of a Brownian particle in a fluctuating barrier potential is governed by the following Langevin equation,

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} - [1+\eta(t)]\Omega^2 x = \xi(t), \tag{1}$$

where γ is the damping parameter. The additive noise $\xi(t)$ comes from the environment in thermal equilibrium, satisfying fluctuation-dissipation theorem with the following statistical properties [16]

$$\langle\xi(t)\rangle = 0 \quad \text{and} \quad \langle\xi(t)\xi(t')\rangle = 2D\delta(t-t'), \tag{2}$$

where $D = \gamma k_B T/m$. The multiplicative noise $\eta(t)$ represents a time-dependent fluctuation in barrier height with

$$\langle\eta(t)\rangle = 0 \quad \text{and} \quad \langle\eta(t)\eta(t')\rangle = Q\exp\left(-|t-t'|/\tau\right), \tag{3}$$

where τ is the correlation time and Q is the variance of the colored noise, respectively. Eq. (3) defines an exponentially correlated Gaussian noise, which has been a typical consideration for barrier fluctuation [7, 9, 14, 26, 27]. Here, Gaussian colored noise $\eta(t)$ is assumed to be uncorrelated with Gaussian white noise $\xi(t)$.

The stochastic variable $\eta(t)$ can be described by the auxiliary stochastic equation [7, 9, 14, 26–28]

$$\frac{d\eta(t)}{dt} = -\frac{1}{\tau}\eta(t) + \sqrt{\frac{2Q}{\tau}}\zeta(t), \tag{4}$$

where $\zeta(t)$ is a standard Gaussian white noise with $\langle\zeta(t)\zeta(t')\rangle = \delta(t-t')$. The probability distribution has a form

$$P(\eta(t)) = \frac{1}{\sqrt{2\pi Q}}\exp\left[-\frac{\eta(t)^2}{2Q}\right], \tag{5}$$

which corresponds to a fluctuating barrier in Gaussian distribution. The variance Q and the correlation time τ dominate the power spectrum of the colored noise

$$S_\eta(\omega) = \frac{Q\tau}{\pi(1+\tau^2\omega^2)}. \tag{6}$$

For the case of small τ, a Gaussian white noise with weak noise intensity is revisited. The barrier height fluctuates fast within a finite interval, and hence the moving particle experiences an average barrier height. While the large value limit $\tau \to \infty$ ($S_\eta(\omega) \to \frac{Q}{\pi}\delta(\omega)$) characterizes slow fluctuation in barrier height of the potential.

A. Exact solutions

To obtain the first moment of x, we rewrite Eq. (1) as two first-order differential equations

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -\gamma v + \Omega^2[1+\eta(t)]x + \xi(t). \end{aligned} \tag{7}$$

Taking the average of the above equations over the ensemble, one can obtain

$$\begin{aligned} \dot{\langle x\rangle} &= \langle v\rangle, \\ \dot{\langle v\rangle} &= -\gamma\langle v\rangle + \Omega^2\langle x\rangle + \Omega^2\langle \eta x\rangle, \end{aligned} \tag{8}$$

which contains a new correlator $\langle\eta x\rangle$. To find this correlator, we use Shapiro-Loginov theorem [29, 30] which reads

$$\frac{d\langle\eta g\rangle}{dt} = \left\langle \eta\frac{dg}{dt}\right\rangle - \lambda\langle\eta g\rangle \tag{9}$$

for exponentially correlated noise $\eta(t)$ where $\lambda = 1/\tau$. Here g is any differential function relevant to $\eta(t)$. Thus, by substituting $g = x$ or $g = v$ into Eq. (9), one can obtain the following linear matrix equation

$$\frac{d\mathbf{F}}{dt} = \mathbb{A}\cdot\mathbf{F}, \tag{10}$$

with

$$\mathbf{F} = \begin{pmatrix} \langle x \rangle \\ \langle v \rangle \\ \langle \eta x \rangle \\ \langle \eta v \rangle \end{pmatrix}, \qquad \mathbb{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \Omega^2 & -\gamma & \Omega^2 & 0 \\ 0 & 0 & -\lambda & 1 \\ \Omega^2 Q & 0 & \Omega^2 & -(\gamma+\lambda) \end{pmatrix}. \tag{11}$$

Thus a closed system is found for four variables of the first moment, and it is solvable for given initial condition. Similar scheme has been used in Refs. [31–33] to complete the system dynamics. To simplify the procedure, we have used the simplest version of the splitting of high order correlators in this paper, *i.e.* $\langle \eta^2 x \rangle = \langle \eta^2 \rangle \langle x \rangle = Q \langle x \rangle$.

The second moments can be obtained by similar procedure. Starting from Eq. (7) and making appropriate transformation, the differential equations for the second moments and cross correlation are written as

$$\begin{aligned} \frac{d\langle x^2 \rangle}{dt} &= 2\langle xv \rangle, \\ \frac{d\langle v^2 \rangle}{dt} &= -2\gamma \langle v^2 \rangle + 2\Omega^2 \langle xv \rangle + 2\Omega^2 \langle \eta xv \rangle + 2\langle \xi v \rangle, \\ \frac{d\langle xv \rangle}{dt} &= -\gamma \langle xv \rangle + \Omega^2 \langle x^2 \rangle + \langle v^2 \rangle + \Omega^2 \langle \eta x^2 \rangle + \langle \xi x \rangle. \end{aligned} \tag{12}$$

Notice there are some high order correlators $\langle \eta xv \rangle$ and $\langle \eta x^2 \rangle$ in the above equations, which can been treated by Shapiro-Loginov theorem, just like in the previous case. Similarly, two new correlators $\langle \xi x \rangle$ and $\langle \xi v \rangle$ need to be supplemented. We apply Shapiro-Loginov theorem again and find $\langle \xi x \rangle = 0$ and $\langle \xi v \rangle = D$ for Gaussian white noise ξ. The details of the derivation are provided in Appendix. Hence, the resulting dynamic system is six-component linear matrix equation in the form

$$\frac{d\mathbf{G}}{dt} = \mathbb{B}\mathbf{G} + \mathbf{K}, \tag{13}$$

where

$$\mathbf{G} = \begin{pmatrix} \langle x^2 \rangle \\ \langle v^2 \rangle \\ \langle xv \rangle \\ \langle \eta x^2 \rangle \\ \langle \eta v^2 \rangle \\ \langle \eta xv \rangle \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} 0 \\ 2D \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{14}$$

and $\mathbb{B}$ is 6×6 drift matrix

$$\mathbb{B} = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -2\gamma & 2\Omega^2 & 0 & 0 & 2\Omega^2 \\ \Omega^2 & 1 & -\gamma & \Omega^2 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 2 \\ 0 & 0 & 2\Omega^2 Q & 0 & -(2\gamma+\lambda) & 2\Omega^2 \\ \Omega^2 Q & 0 & 0 & \Omega^2 & 1 & -(\gamma+\lambda) \end{pmatrix}. \quad (15)$$

The formal solutions of linear matrix Eqs. (10) and (13) are given by

$$\mathbf{F}(t) = e^{\mathbb{A}t}\mathbf{F}(0), \quad (16)$$

$$\mathbf{G}(t) = e^{\mathbb{B}t}\mathbf{G}(0) - \left[\mathbf{I} - e^{\mathbb{B}t}\right]\mathbb{B}^{-1} \cdot \mathbf{K}, \quad (17)$$

where $\mathbf{I}$ is 6×6 unit matrix.

As an example, we discuss a special case of static barrier potential with no barrier fluctuation ($Q = 0$). In this situation, the system is driven by the additive noise only. Eq. (10) is reduced to

$$\frac{d}{dt}\begin{pmatrix} \langle x \rangle \\ \langle v \rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} \langle x \rangle \\ \langle v \rangle \end{pmatrix}, \quad (18)$$

which is linear and exactly solvable. The first component of the vector in the above equation is given by

$$\langle x \rangle = a_1 e^{p_1 t} + a_2 e^{p_2 t}, \quad (19)$$

where $p_1 > 0$ and $p_2 < 0$ are eigenvalues of the drift matrix in Eq. (18)

$$p_1 = \frac{1}{2}(\gamma' - \gamma),$$

$$p_2 = -\frac{1}{2}(\gamma' + \gamma), \quad (20)$$

and

$$a_1 = \frac{-p_2 x_0 + v_0}{\gamma'},$$

$$a_2 = \frac{p_1 x_0 - v_0}{\gamma'}. \quad (21)$$

Here $\gamma' = \sqrt{\gamma^2 + 4\Omega^2}$, x_0 and v_0 are initial position and velocity of the particle. At large time $t \to \infty$, the mean position $\langle x \rangle$ is dominated mostly by the first term of Eq. (19) due to positive p_1.

Similarly, Eq. (13) in static potential field reads

$$\frac{d}{dt}\begin{pmatrix} \langle x^2\rangle \\ \langle v^2\rangle \\ \langle xv\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -2\gamma & 2\Omega^2 \\ \Omega^2 & 1 & -\gamma \end{pmatrix}\begin{pmatrix} \langle x^2\rangle \\ \langle v^2\rangle \\ \langle xv\rangle \end{pmatrix} + \begin{pmatrix} 0 \\ 2D \\ 0 \end{pmatrix}. \tag{22}$$

To determine the variance of the motion of the particles, we use the covariance matrix [34, 35]

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xv} \\ \sigma_{vx} & \sigma_{vv} \end{pmatrix} = \begin{pmatrix} \langle x^2\rangle - \langle x\rangle^2 & \langle xv\rangle - \langle x\rangle\langle v\rangle \\ \langle xv\rangle - \langle x\rangle\langle v\rangle & \langle v^2\rangle - \langle v\rangle^2 \end{pmatrix}. \tag{23}$$

The linear equation for the elements of covariance matrix from Eqs. (18) and (22) takes the form

$$\frac{d}{dt}\begin{pmatrix} \sigma_{xx} \\ \sigma_{vv} \\ \sigma_{xv} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -2\gamma & 2\Omega^2 \\ \Omega^2 & 1 & -\gamma \end{pmatrix}\begin{pmatrix} \sigma_{xx} \\ \sigma_{vv} \\ \sigma_{xv} \end{pmatrix} + \begin{pmatrix} 0 \\ 2D \\ 0 \end{pmatrix}. \tag{24}$$

The component of interest in the above equation is

$$\sigma_{xx} = \frac{k_B T}{m\Omega^2}(c_1 e^{q_1 t} + c_2 e^{q_2 t} + c_3 e^{q_3 t} - 1), \tag{25}$$

where $q_1 > 0$, $q_2 < 0$ and $q_3 < 0$ are eigenvalues of the drift matrix in Eq. (24) with

$$\begin{aligned} q_1 &= -\gamma + \gamma', \\ q_2 &= -\gamma - \gamma', \\ q_3 &= -\gamma, \end{aligned} \tag{26}$$

and

$$\begin{aligned} c_1 &= \frac{\gamma^2 + \gamma\gamma'}{2\gamma'^2}, \\ c_2 &= \frac{\gamma^2 - \gamma\gamma'}{2\gamma'^2}, \\ c_3 &= \frac{4\Omega^2}{\gamma'^2}. \end{aligned} \tag{27}$$

In Eq. (25), the first term in right hand is the leading one since $q_1 > 0$, which is divergent in the large time limit. The equations (19) and (25) are in good agreement with the results in [16]. Other components such as $\langle v\rangle$, σ_{vv} and σ_{xv} are easily obtained if needed.

In what follows we discuss the PDF of particle position. For the case where the system is subjected to additive noise only, namely, $Q = 0$, the time-evolutionary PDF has Gaussian form [10, 16, 20, 36]

$$p(x, t; x_0, v_0) = \frac{1}{\sqrt{2\pi\sigma_{xx}(t)}} \exp\left\{ -\frac{[x(t) - \langle x(t)\rangle]^2}{2\sigma_{xx}(t)} \right\}, \tag{28}$$

with the initial condition $p(x, t = 0) = \delta(x - x_0)\delta(v - v_0)$.

When the barrier fluctuation can not be neglected, we restrict our consideration to the overdamped case for simplicity. Thus, the stochastic system in the large γ limit reads

$$\gamma \frac{dx}{dt} = [1 + \eta(t)]\Omega^2 x + \xi(t). \tag{29}$$

According to the result given by [9, 37], we get effective Fokker-Planck equation (FPE) corresponding to Eq. (29) in the form

$$\frac{\partial}{\partial t} p(x, t) = -\frac{\Omega^2}{\gamma} \frac{\partial}{\partial x} x p(x, t) + D_0 \frac{\partial^2}{\partial x^2} p(x, t) + D_1 \frac{\partial}{\partial x} x \frac{\partial}{\partial x} x p(x, t), \tag{30}$$

where

$$\begin{aligned} D_0 &= \frac{D}{\gamma^2} = \frac{k_B T}{\gamma m}, \\ D_1 &= \frac{Q\tau\Omega^4}{\gamma^2}. \end{aligned} \tag{31}$$

Here D_0 and D_1 attribute to additive and multiplicative noises, respectively. So far, it is difficult to find out the exact expression of the PDF in Eq.(30), especially the non-stationary solution [10].

B. Numerical simulations

In order to find out the nonstationary PDF of the system subjected to additive and multiplicative noises, We perform numerical integration of Eq. (1) by Heun's method [38, 39]. The exponentially correlated noise $\eta(t)$ is generated by integrating Eq. (4), which is driven by a Gaussian white noise ζ [40]. The time step Δt is set to 0.001 for simulations. The Brownian particles start at the left of the fluctuating barrier potential $x_0 = -2$ and the simulations are performed by averaging over 200 000 trajectories to calculate the dynamical properties. For simplicity, we have set Boltzmann constant k_B and the particle mass m

as unity. The parameter $\Omega = 1$ is set. The friction coefficient $\gamma = 0.1$ is taken in the simulations, unless otherwise stated.

The numerical results of non-stationary PDF are shown in Fig. 1. One can see clearly a non-Gaussian distribution, even in this linear Langevin system but driven by additive and multiplicative noises simultaneously [9, 41]. The non-Gaussian PDF indicates that it can not be determined by the first and second moments only.

We calculate the time-evolutionary ratio of average position and the variance, $r = -\langle x \rangle / \sqrt{\sigma_{xx}}$. For Gaussian distribution, the ratio approaches constant at large times, which indicates a stable spreading with respect to the center of the Gaussian distribution. As expected, this non-Gaussian distribution results in a nonstable ratio which decreases monotonously, as shown in Fig. 2. For the case of large correlation time and low friction coefficient, the ratio allows a fast decaying, implying a sharp spreading of the distribution.

III. PROBABILITY PASSING OVER THE BARRIER

A. the static barrier

Barrier crossing problem has been considered as a simplified model to describe fusion mechanism of the synthesis of the heavy elements in heavy-ion reactions [16, 36, 42]. For the static barrier potential with $Q = 0$, Eq. (1) results in a Gaussian PDF (28) for particle position. The asymptotic probability for a particle passing over the static barrier has been given in Refs. [16, 17, 20, 43–45]

$$P(t \to \infty) = \int_0^\infty p(x, t; x_0, v_0) dx = \frac{1}{2} \mathrm{erfc} \left[-\frac{1}{\sqrt{T}\sqrt{1 - y^2}} (\sqrt{B} - y\sqrt{K}) \right], \qquad (32)$$

where $y = p_1/\Omega$ is a dimensionless parameter, with p_1 is the positive eigenvalue in Eq. (20). Here $B = \frac{1}{2} m\Omega^2 x_0^2$ is the barrier height measured from the initial position, and $K = \frac{1}{2} m v_0^2$ the initial kinetic energy.

The critical kinetic energy K_c, which is defined as the energy necessary for half of the particles passing over the barrier, has the form [16, 42]

$$\frac{K_c}{B} = \frac{1}{y^2} = \left[\frac{\tilde{\gamma}}{2} + \sqrt{(\frac{\tilde{\gamma}}{2})^2 + 1} \right]^2 . \qquad (33)$$

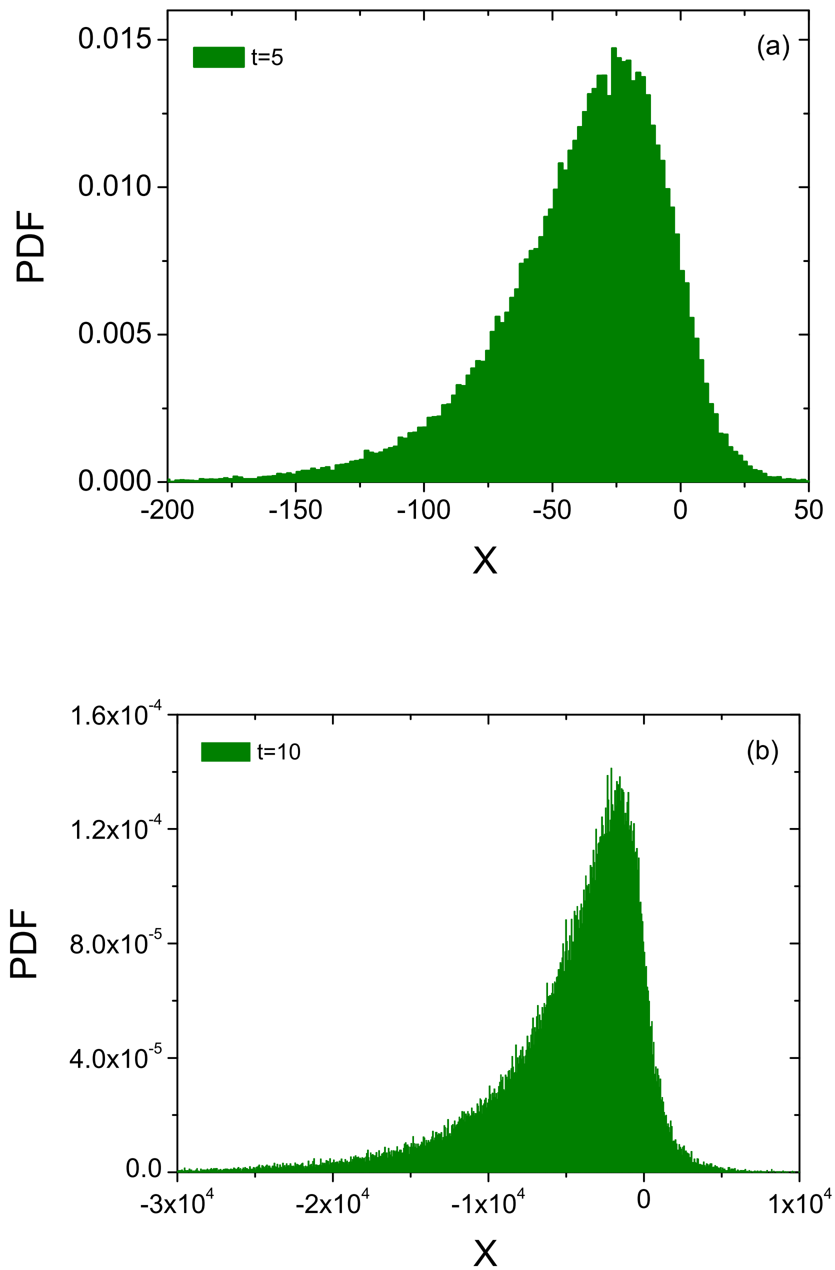

FIG. 1: Normalized PDF in the presence of additive and multiplicative noises. The initial position $x_0 = -2$ and initial velocity $v_0 = 1.5$. $B = \frac{1}{2}m\Omega^2 x_0^2$ is the barrier height measured from the initial position. Here $T/B = 0.5$, $\gamma = 0.1$, $Q = 0.1$ and $\tau = 1$. (a) $t = 5$. (b) $t = 10$.

Notice that the ratio of critical kinetic energy to the barrier height is just determined by the dimensionless friction coefficient $\tilde{\gamma} = \gamma/\Omega$, regardless of the environment temperature T.

We are interested in temperature effect on the over-passing probability. The asymptotic

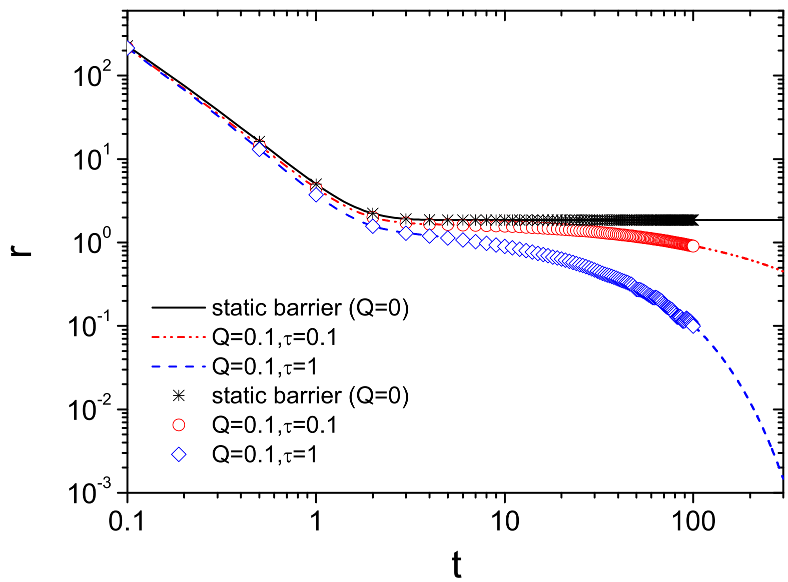

FIG. 2: (Color online) Log-log plot of time-dependent ratio $r = -\langle x \rangle / \sqrt{\sigma_{xx}}$ in the system with static barrier $Q = 0$ and fluctuating barrier of $Q = 0.1$, $\tau = 0.1$ and $\tau = 1$, respectively. The lines are the analytical results of Eqs. (16) and (17), and the scatters denote numerical integration results. Other parameters are same as those in Fig. 1.

probability Eq. (32) as a function of the environment temperature T is shown in Fig. 3. In weak friction coefficient regime, there are clearly three distinct behaviors. For low initial kinetic energy $K < K_c$, a monotonously increasing probability with increasing temperature is observed, and close to the asymptotic value $P = 0.5$ in the high temperature limit. For the case of $K > K_c$, the over-probability decreases as the temperature increases, which is not expected intuitively. In fact, all particles with initial kinetic energy larger than barrier height will pass over the barrier in nondissipative system of $\tilde{\gamma} = 0$. For the system with small $\tilde{\gamma}$, most of the particles can surmount the barrier due to low energy dissipation, and thus a large over-passing probability $P > 0.5$ is exhibited. When the environment temperature T rises, on an average some of the particles which have passed over the barrier potential will return back to the starting side, resulting in a reduced over-passing probability, namely, negative dP/dT. When $K = K_c$, the asymptotic probability takes the uniform value of $P = 0.5$ regardless of the environment temperature. Thus, the energy regimes can be divided into three parts, as shown in Fig. 4. In this figure one can find $dP/dT < 0$ for

$K > K_c$, $dP/dT > 0$ for $K < K_c$, and $dP/dT = 0$ at $K = K_c$.

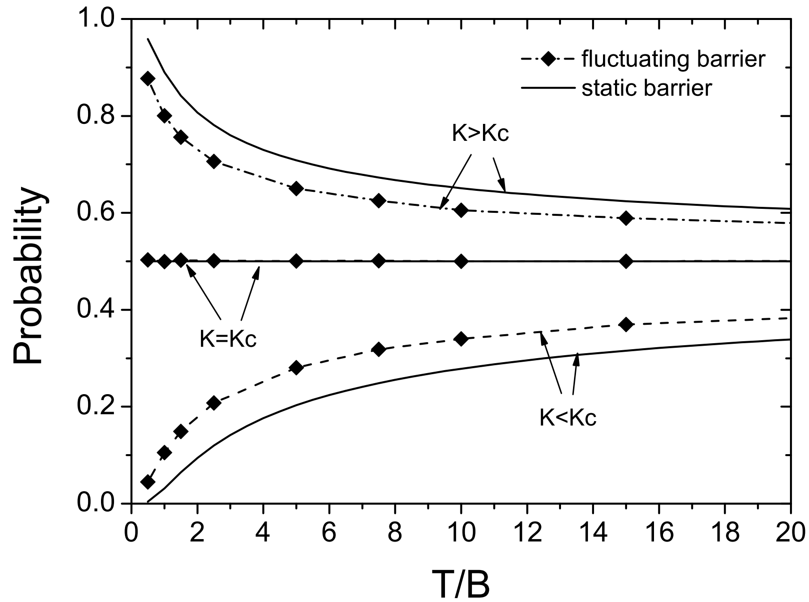

FIG. 3: Temperature dependence of the probability (32) for weak friction case of $\gamma = 0.1$. Here the critical kinetic energy $K_c/B = 1.1051$. The initial kinetic energy from upper to bottom correspond to $K/B = 1.5625, 1.1051, 0.5625$ respectively. The barrier height $B = 2$. For fluctuating barrier, $Q = 0.1$ and $\tau = 0.1$.

B. the fluctuating barrier

Now we consider the probability of the particles passing over a fluctuating barrier, their motion is described by Eqs. (1−3). The fluctuating barrier has Gaussian distribution dominated by two parameters, the variance Q and the correlation time τ. For $Q = 0$, the barrier remains static, corresponding to the situation of no barrier fluctuation. While large Q, such as $Q \sim 1$, might lead to temporarily harmonic oscillator potential. On the other hand, $\tau \to 0$ corresponds to Gaussian white noise, and therefore, a moving particle feels fast fluctuation in barrier height. While slow fluctuation of barrier height is shown for $\tau \to \infty$. In the latter case, each particle starting at initial position feels almost a fixed barrier when moving along its trajectory, which is stochastically distributed in the form

$$P(\eta) = \frac{1}{\sqrt{2\pi Q}} \exp\left(-\frac{\eta^2}{2Q}\right). \tag{34}$$

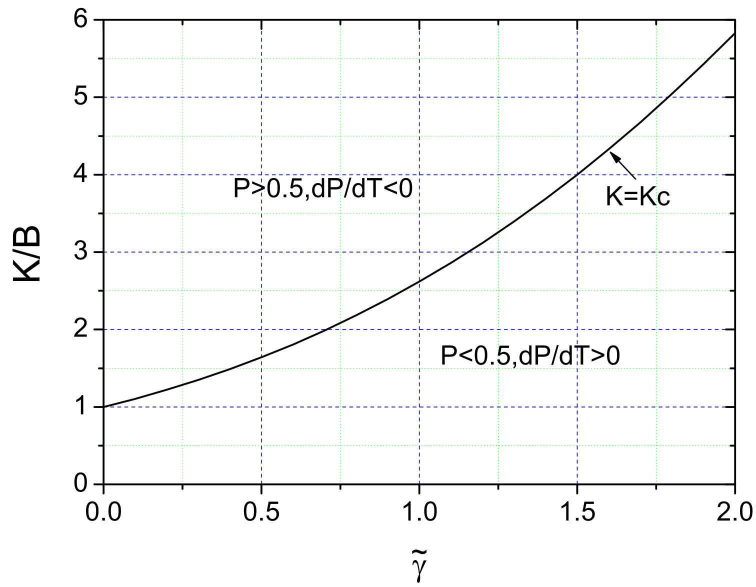

FIG. 4: The energy regimes with completely different characteristics of temperature dependence. The solid line is the result of Eq. (33), acting as the boundary of the two regimes. The barrier height $B = 2$ is taken.

Here η is time-independent stochastic value. At the first sight one might take it for granted that Eq. (34) is intuitively same as Eq. (5). Actually, Eq. (5) is true for a wide range of τ, leading to a time-dependent barrier, while Eq. (34) is only appropriate for $\tau \to \infty$, implying a time-independent barrier encountered by each particle.

Next we turn to numerical simulations to study the over-passing probability. We have investigated the time evolutionary probability extensively and have demonstrated a stationary probability after a period of time for either small Q or small τ, as shown in Fig. 5 (a). In addition for large Q and large τ, the probability evolves to the asymptotic value of $P = 0.5$ after a very long time, as plotted in Fig. 5 (b).

The asymptotic probability as a function of Q for small τ in low friction regime is shown in Fig. 6. If the initial kinetic energy is below or near the barrier potential, $K < B$, the over-probability is less than the value of 0.5 and increases slowly with growing variance Q. While for $K > B$ the probability is larger than 0.5, implying overwhelming majority of particles have surmounted the fluctuating barrier, as in expected. Furthermore, larger Q, and hence large barrier fluctuation, will not help more particles to pass over the fluctuating barrier. On the contrary, part of the particles returned to starting point if Q increases, and

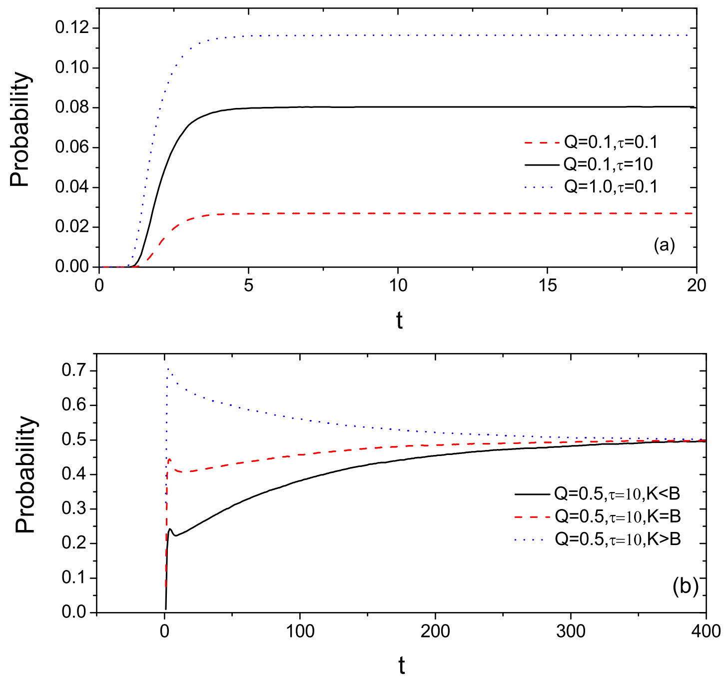

FIG. 5: (Color online) The probability passing over a fluctuating barrier as a function of time. The temperature $T/B = 0.5$ and friction $\gamma = 0.1$ are taken in the simulations. (a) Stationary probability obtained after a time for either small Q or small τ. Three lines from bottom to upper are $Q = 0.1$ and $\tau = 0.1$ (small Q and small τ), $Q = 0.1$ and $\tau = 10$ (small Q), $Q = 1.0$ and $\tau = 0.1$ (small τ), respectively. The initial kinetic energy $K/B = 0.5$ is adopted. (b) Non-stationary probability for large Q and large τ. Notice the time scale in this figure is dozens of times as compared to (a).

hence a decreasing probability with increasing Q is observed in large K and low γ.

In what follows we restrict our consideration to the case of $Q \ll 1$ to ensure a barrier potential, avoiding a harmonic oscillator, even temporarily. Extensive numerical simulations show completely distinct behaviors in three energy regimes with respect to the case of static barrier: the enhanced probability in energy area $K < K_c$, a suppressed one for $K > K_c$, and invariable one at $K = K_c$, as shown in Fig. 3. It is interesting that the energy regimes in Fig. 4 are still in effect even in the presence of fluctuating barrier. Hence we come to a conclusion that neither the environment temperature nor the fluctuating barrier will affect critical kinetic energy.

Finally, it is instructive to study the effect of correlation time τ on the asymptotic probability. A wide range of correlation times have been used in our simulations to catch the characteristics. The over-probability as a function of initial kinetic energy is shown in Fig. 7.

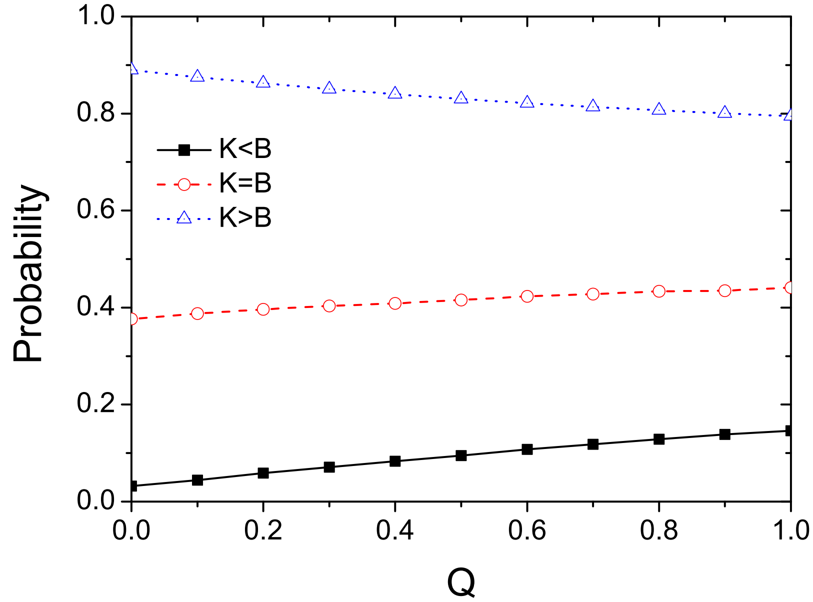

FIG. 6: (Color online) The asymptotic probability passing over a fluctuating barrier as a function of the variance Q for three cases. Small correlation time $\tau = 0.1$ is taken to ensure a stationary probability within simulation time. Other parameters are same as those in Fig. 5.

Large τ leads to dozens of times enhancement of probability at the energies well below the barrier. At the energies near the barrier, a prominently increasing probability can still be seen when τ increases. Whereas a slight suppression happens at high energies, even for the case of large τ, see the inset in Fig. 7. The results are qualitatively consistent with those of previous work on the enhanced fusion cross sections for heavy-ion synthesis by introducing barrier distribution, especially at deep sub-barrier energies [21–24, 46], and reduced one at high energies with respect to barrier potential [46, 47]. Indeed, the fusion cross section is dominated by fusion probability [42, 48] and realistic fusion reaction includes more complex potential and specific parameters. Our results have demonstrated that both enhancement and suppression of the fusion probability are relevant to not only barrier distribution, but the correlation time of fluctuating barrier as well.

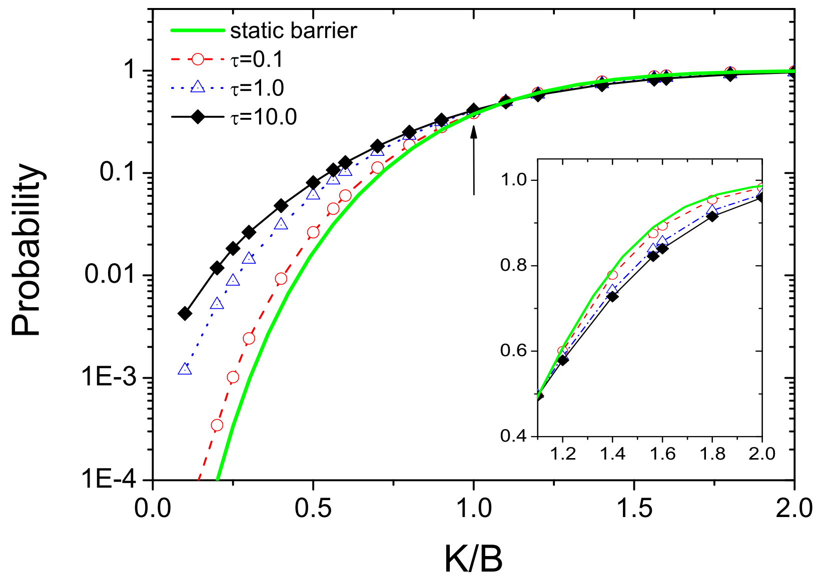

FIG. 7: (Color online) Semi-logarithmic plot of over-passing probability as a function of initial kinetic energy. The temperature $T/B = 0.5$. The solid line is the result of Eq. (32), corresponding to the static barrier potential ($Q = 0$), and the scatters from bottom to upper correspond to fluctuating barrier with the variance $Q = 0.1$ and the correlation time $\tau = 0.1, 1.0, 10.0$, respectively. The arrow indicates average barrier height. The inset: the probability at higher energies in a linear scale.

IV. CONCLUSION

In conclusion, we have proposed a Langevin model subjected to additive and multiplicative noises to describe thermally activated diffusion over a fluctuating barrier. The analytical predictions of the first and second moments and relevant dynamical characteristics have been obtained by two linear matrix equations. Numerical analysis of the Langevin model provides a nonstationary PDF in non-Gaussian form. Our results can also be applied to stochastically modulated harmonic oscillators (Kubo oscillator) only if the inverted parabolic barrier is replaced by harmonic oscillator potential.

We also showed interesting dynamics in underdamped region where the exchange of energy with the environment is limited due to the low friction. For the case of static barrier with $Q = 0$, there are three regimes depending on initial kinetic energy with respect to the critical energy Eq. (33). For $K < K_c$, the asymptotic probability P passing over a static barrier is less than 0.5 and increases with the increasing temperature. In contrast, the asymptotic probability stands above the value of 0.5 and reduces with the increasing temperature for $K > K_c$. This indicates that higher temperatures are not always helpful for the particles to surmount the barrier, contrary to one's expectation. At the energy $K = K_c$, the probability keeps unchanged at $P = 0.5$ with varying environment temperatures. Furthermore, the simulations also showed this holds true even for fluctuating barrier. Consequently, it implies that the critical kinetic energy K_c has nothing to do with the environment temperature as well as the stochastic modulation of the barrier.

We have explored the influence of barrier fluctuation in relevant scales of correlation time on the probability passing over the fluctuating barriers. For large correlation time τ, the over-passing probability is enhanced remarkably at sub-barrier energies, while a slight suppression takes place at high energies, as compared to the case of the static barrier. Similar enhancement of fusion cross sections has been found in previous work on experimental studies in heavy-ion fusion reactions and explained by theoretical analysis of coupled-channel effect using barrier distribution. It should be noted that our model presents a qualitative prediction of dynamics in fluctuating barrier with various scales of correlation. Our results indicates that the correlation time of fluctuating barrier plays an important role in barrier crossing dynamics. We hope this model will provide a better understanding of the barrier crossing problem in heavy-ion fusion reactions, particularly at energies below and near the Coulomb barrier.

Appendix

For the stochastic variables x and v described by two-component Langevin equation (7), one can not find directly the correlators $\langle \xi x \rangle$ and $\langle \xi v \rangle$ using Shapiro-Loginov theorem for Gaussian white noise ξ. In Stratonovich representation, which supposed δ-correlated process as an approximation to a colored one with zero mean and finite correlation time $\langle \xi(t)\xi(t') \rangle = (D/\tau_G) \exp(-|t-t'|/\tau_G)$. In the very small time scale τ_G, Gaussian white noise ξ with noise intensity D is revisited [10, 27, 28, 49]. This is considered as appropriate interpretation for most physical situations. With such assumption, applying Shapiro-Loginov theorem yields two differential equations,

$$\frac{d\langle\xi x\rangle}{dt} = -(1/\tau_G)\langle\xi x\rangle + \langle\xi v\rangle,$$
$$\frac{d\langle\xi v\rangle}{dt} = \Omega^2\langle\xi x\rangle - [\gamma + (1/\tau_G)]\langle\xi v\rangle + D/\tau_G, \tag{35}$$

where the approximation of $\langle\xi\eta x\rangle = \langle\xi\eta\rangle\langle x\rangle = 0$ has been used. Eq. (35) is exactly solvable but in a very complicated form. Here we give a concise result of $\langle\xi x\rangle \to 0$ and $\langle\xi v\rangle \to D$ in the limit of $\tau_G \to 0$.

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