Solution of the Roth-Marques-Durian rotational abrasion model

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We solve the rotational abrasion model of Roth, Marques and Durian [Phys. Rev. E (2010)], a one-dimensional quasilinear partial differential equation resembling the inviscid Burgers equation with the unusual feature of a step function factor as a coefficient. The complexity of the solution is primarily in keeping track of the cases in the piecewise function that results from certain amputation and interpolation processes, so we also extract from it a model of an evolving planar tree graph that tracks the evolution of the coarse features of the contour.

What determines the shapes of pebbles is an intriguing physical question with interest not just to beachcombers out for walks but also geologists, who are interested in the history of erosion at a site [1], as well as mechanical engineers [2], who wish to understand wear processes. Recently several models have been proposed to explain these shapes. Two stochastic models are of note, a "cutting model" [3, 4] which accompanied an experimental measurement of pebbles rotating in a tray and an analytically tractable "chipping model" [5]. These models lead to distributions of non-circular shapes. More recently, deterministic erosion processes have been studied by Roth, Marques and Durian. They performed an experiment to measure the contours of linoleum tiles of fixed thickness and varying shape that they had rotated for differing amounts of time in a slurry of grit [6]. This paper describes the solution to their rotational abrasion model. Supposing $r(\theta, t)$ describes the radial distance of the contour from the rotational axis as a function of angle θ and time t, they proposed

$$\frac{\partial r}{\partial t} + Cr\frac{\partial r}{\partial \theta}H\left(\frac{\partial r}{\partial \theta}\right) = 0. \tag{1}$$

Here C is a positive proportionality factor with dimensions of Angle/(Time×Length) and $H(x)$ denotes a Heaviside step function defined so that $H(x) = 0$ for $x \leq 0$ and $H(x) = 1$ for $x > 0$. It is obvious that circles ($r(\theta, t) =$ constant) are stationary solutions to this equation, and the evolution of experimental contours computed in [6] by a finite differencing scheme also evolved towards circles unerringly. These solutions also matched quantitatively the evolution of several geometrical quantities extracted from their experimental data, such as area, perimeter, and the width of the curvature distribution.

Summarized here are the key ideas behind our exact solution of this equation. First, we exploit a connection to the Burgers equation at zero viscosity, a well-studied equation from gas dynamics [7]. Second, the solution $r(\theta, t)$ can be written in a piecewise fashion as a union of $r =$ constant (circular) arcs and certain "stretched" segments of the initial contour. More precisely, these segments are curves of the form $r_0(\theta(\theta_0, t))$ where $r_0(\theta_0)$ is the initial contour and $\theta(\theta_0, t)$ at fixed θ_0 is a linear function in t. The solution is constructed to be continuous, but will admit corners with discontinuous slope generically. Finally, the organization of the solution has a strong combinatorial flavor, and the evolution of the pattern of critical points in the contour is captured by a model of an evolving planar tree. This reduction suggests that discrete statistical models may capture the properties of ensembles of abraded pebbles. The final section discusses other possible extensions.

I. METHOD OF CHARACTERISTICS

In what follows, we will usually think of the variables r and θ in Eq. (1) as two dimensional rectangular coordinates (indeed, all the figures are plotted in this "unwrapped" fashion), though they do refer to polar coordinates, and we will usually refer to curves of constant r as circular arcs and to lengths in the θ direction as angular widths.

We first observe that Eq. (1) without the step function factor is precisely the inviscid Burgers equation, a quasilinear first-order partial differential equation. That equation may be solved by the method of characteristics, which we shall now adapt. See also the book of Melikyan on solutions via characteristics to nonsmooth first-order equations in the theory of optimal control and in differential games [8]. Let the initial contour be $r_0(\theta) \equiv r(\theta, 0) > 0$. We now search for "characteristics", or space-time curves $\theta(\sigma), t(\sigma)$ beginning at $\theta(0) = \theta_0$, $t(0) = 0$ (σ being some parameter) along which $r(\theta(\sigma), t(\sigma))$ remains constant, and hence equal to $r_0(\theta_0)$. In other words, each point on the initial contour $r_0(\theta_0)$ evolves forward in time along its characteristic.

By applying the chain rule, we find

$$\frac{d}{d\sigma}r(\theta(\sigma), t(\sigma)) = 0$$
$$\frac{d\theta}{d\sigma}\frac{\partial r}{\partial \theta} + \frac{dt}{d\sigma}\frac{\partial r}{\partial t} = 0.$$

Comparing this to Eq. (1),

$$\frac{dt}{d\sigma} = 1$$
$$\frac{d\theta}{d\sigma} = CrH\left(\frac{\partial r}{\partial \theta}\right).$$

Integrating these equations with the initial condition $t = 0$, $\theta = \theta_0$ and using the fact that $r(\theta(\sigma), t(\sigma))$ is
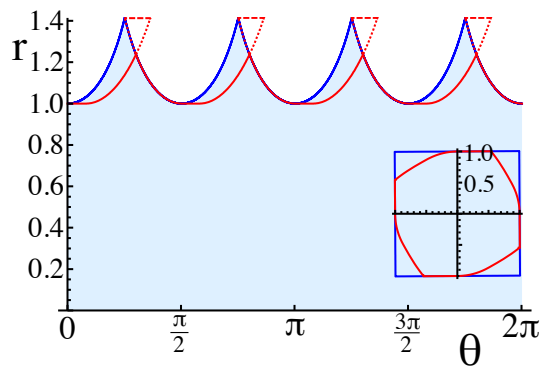
FIG. 1. (Color online) Evolution of a square, r vs θ (C=1) via the method of characteristics. $t = 0$ (blue) and $t = 0.04$ (red). Dotted portions of the curve indicate pieces of the evolved contour which are amputated, and the horizontal dashes indicate discontinuities of the curve before amputation. The inset shows the two contours plotted in polar coordinates.

constant and equal to $r_0(\theta_0)$, we obtain

$$r(\theta(\theta_0, t), t) = r_0(\theta_0) \qquad (2)$$

$$\theta(\theta_0, t) = \theta_0 + Cr_0(\theta_0)tH\left(\frac{\partial r}{\partial \theta}\right). \qquad (3)$$

This equation yields a (possibly multi-valued) formal solution for $r(\theta, t)$ via $r(\theta(\theta_0, t), t)$. To plot the evolution of a curve with the solution in this form, begin with points distributed on the initial curve and move each of those points along its characteristic arc an angular distance $d\theta = Cr_0(\theta_0)H(\partial_\theta r)dt$ in each time step dt. See Fig. 1 showing this evolution in the case of an initially square contour.

II. MULTIVALUEDNESS AND AMPUTATION

The solution given in the previous section is not yet well-defined; we must deal with the multivaluedness of the evolution along characteristics. To see how this arises, consider Fig. 1. Points on the contour slightly behind the local maxima will quickly overtake the points with the same r-values but slightly ahead of the maximum, as those have nonpositive $\partial_\theta r$ and hence are frozen by the step function factor. This causes the two pieces of the contour to overlap, and is analogous to the formation of shocks in the inviscid Burgers equation. The "horizontal" discontinuity this creates is depicted in the figure as dotted lines. Based on the physical interpretation of the equation, the way to deal with this multivaluedness is to amputate the portion of the contour where this has occurred, as in the figure. This generates a corner in $r(\theta, t)$ (though in the pictured example, the contour began with corners at the maxima).

There are also problematic points around a local minimum of r. Let (r_{min}, θ_{min}) be the coordinates of the local minimum on the initial contour. Since $r_{min} > 0$, all

points $\theta > \theta_{min}$ ahead of this minimum will have traveled a nonzero angular distance along their characteristics at any $t > 0$, which results in a growing gap of undefined points (i.e. points such that Eq. (2) has no solution in θ_0 at a given t) between θ_{min} and $\theta_{min} + Cr_{min}t$. The obvious thing to do is to interpolate by setting $r = r_{min}$ for all θ in this interval, as this is the only natural way to ensure that the shape remains continuous. Thus intervals of constant r (circular arcs) are continually growing at local minima.

The two cases just described are the simplest cases where the evolution along characteristics must be repaired to become continuous. There are several more similar cases involving intervals of constant r which lead to multivaluedness (or no-valuedness), but they are all treated by either amputation or interpolation, as described above. In the terminology of Melikyan [8], the points of amputation are "equivocal" and the points of interpolation are "dispersal".

III. PIECEWISE SOLUTION FOR $r(\theta, t)$

From the considerations above, giving an explicit formula for the solution would involve several layers of if-then constructs. We describe the full piecewise solution $r(\theta, t)$ to Eq. (1) instead by decomposing the contour into strictly monotonic intervals, and within each of these the solution depends continuously on the initial contour. We also give the positions of the endpoints separating these intervals as a function of time.

We define *rising* and *falling faces* of the contour to be connected components of points on the contour with $\partial_\theta r > 0$ and $\partial_\theta r < 0$, respectively. See Fig. 2 for illustrations. The step function in Eq. (2) forces falling faces to be pieces of the original contour, i.e. $r(\theta, t) = r_0(\theta)$. The rising faces will be intervals from the original contour "sheared" by the evolution along characteristics. More precisely, $r(\theta, t)$ is defined implicitly by solving for the value of θ_0 in Eq. (2) such that $\theta = \theta(\theta_0, t)$, and then setting $r(\theta, t) = r_0(\theta_0)$. The shearing is caused by points moving with speed proportional to their radius.

We will call intervals of constant r (circular) *arcs*, and we will classify these into four types. *Rising* and *falling arcs* are those that are adjacent to rising, respectively falling faces on both sides. *Min* and *max arcs* are those which contain local minima, respectively maxima of the contour. Thus, $r(\theta, t)$ at any fixed time decomposes into a set of faces and arcs. The evolution of each face or arc can be treated independently of the others for almost all times except for a discrete set of events when a face or arc changes into another type or disappears.

The endpoints of faces or arcs fall into three categories named according to their behavior under time evolution: *stationary endpoints*, *interpolating endpoints*, and *amputating endpoints*. Stationary endpoints are those that do not move under time evolution. There are two types, those at the right of a falling face and at the left of a
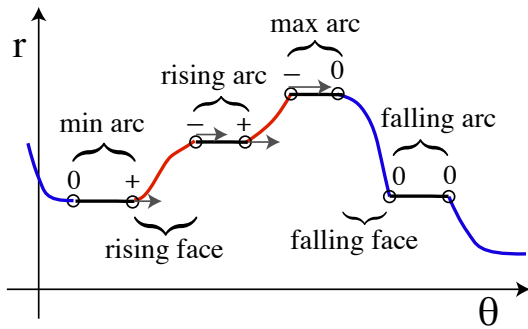
FIG. 2. (Color online) Decomposition of the contour into faces and arcs; see text. The different types of endpoints are labeled with symbols: "0" are stationary, "+" are interpolating, "−" are amputating.

min or falling arc, and those at the right of a max or falling arc and at the left of a falling face. Interpolating endpoints are those that move to the right under time evolution and are the sites of new interpolation. These are always to the right of rising or min arcs and to the left of rising faces. These endpoints move at constant speed Cr where r is the radius of the arc. Finally, amputating endpoints move to the right and are the sites of new amputation. They are always to the right of rising faces and are on the left of max arcs, rising arcs, or falling faces.

To calculate the motion of the amputating endpoints we will need the survival time of each point of the pebble. The intersection of the area under the contour with a circle of radius r will be several disjoint arcs. From Eq. (2) these shrink in time as the left endpoint of each moves with speed Cr towards the right, until the moment this value of r is amputated and the entire interval has vanished. The remaining lifetime of a point (r, θ) inside the contour at time t is thus $\frac{\Delta(r,\theta,t)}{Cr}$ where $\Delta(r,\theta,t)$ is the angular distance along the circle of radius r from (r, θ) to the left endpoint of its interval on the contour at time t. At fixed r, $\Delta(r,\theta,t) = \theta - \theta_L - Crt$, where θ_L is the position of this left endpoint on the initial contour, i.e. θ_L is an appropriate solution of $r_0(\theta_L) = r$.

For an amputating endpoint that lies between a rising and falling face, the angular position as a function of time follows the contour of the falling face. Its position is a solution $(r(t), \theta(t))$ of $\Delta(r, \theta, t) = Crt$. One limit to keep in mind is when the falling face is vertical. Then the angular position of the amputating endpoint will be stationary for a period of time. The opposite limit is when the amputating endpoint lies between a rising face and an arc with radius r, then the endpoint moves to the right with constant speed Cr. The nature of the above decomposition of the contour into faces and arcs changes precisely when endpoints collide with each other and faces and arcs merge. There are three basic processes: vanishing of a max arc, vanishing of a falling face, and vanishing of a rising face. See Fig. 3 for illustrations of these. If a rising face comes directly before a falling face, the radius r_r of the arc before the rising face and the radius r_f of the arc
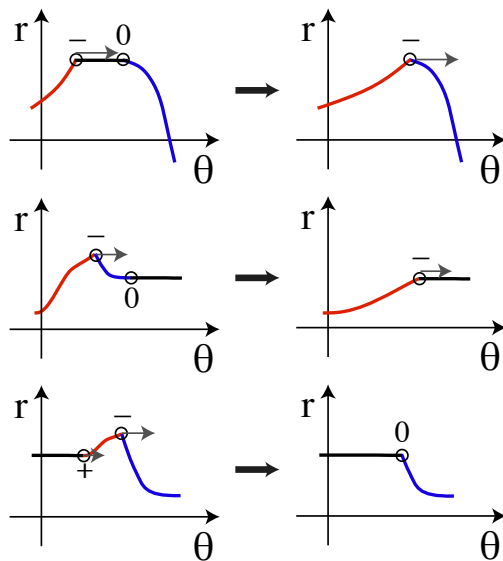

FIG. 3. (Color online) The face and arc merging processes, from top to bottom: vanishing of a max arc, vanishing of a falling face, and vanishing of a rising face. Endpoints are labeled as in Fig. 2

after the falling face determines which of the two faces vanish. When $r_r < r_f$, the falling face vanishes, when $r_r > r_f$ the rising face vanishes, and when $r_r = r_f$, they vanish simultaneously and the two arcs are joined.

To make the above solution a bit more concrete, we sketch what happens in the case of a rectangular contour with side lengths $2a$ and $2b$ with $a \leq b$. Let $\gamma = \arctan(b/a)$. The initial contour (for $0 \leq \theta < 2\pi$) takes the form

$$r_0(\theta) = \begin{cases} a \sec\theta & 0 \leq \theta < \gamma \text{ or } 2\pi - \gamma \leq \theta < 2\pi \\ b \csc\theta & \gamma \leq \theta < \pi - \gamma \\ -a \sec\theta & \pi - \gamma \leq \theta < \pi + \gamma \\ -b \csc\theta & \pi + \gamma \leq \theta < 2\pi - \gamma. \end{cases} \quad (4)$$

In the first instant of time, the local minima at $\theta = 0, \pi/2, \pi, 3\pi/2$ expand by interpolation into min arcs, so the initial contour consists of a min arc of zero width at each of these local minima, each sandwiched between a falling face and a rising face. Fig. 1 depicts the first interval of time in the case $b/a = 1$, where the amputating endpoints at the local maxima (initially at positions $\theta = \gamma, \pi \pm \gamma, 2\pi - \gamma$) move to the right and the interpolating endpoints at the local minima do as well. The solution in this first time interval consists 12 piecewise smooth curves, four each of min arcs, rising faces, and falling faces (though there is a π periodicity of the contour in θ which is preserved by the evolution, simplifying matters somewhat). This proceeds until time $\tau_1 = \frac{\pi/2 - \operatorname{arcsec}(b/a)}{Cb}$ when the points on the contour above $r = b$ have all been amputated, and we remove the corresponding rising and falling faces. If $a \neq b$, the contour now consists of 8 piecewise smooth curves, two each of
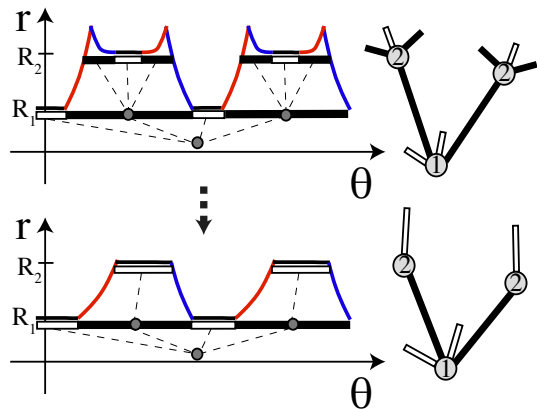
FIG. 4. (Color online) (top) A schematic contour for a rectangular contour after some abrasion and its associated tree. The two levels correspond to the distance R_1 from the origin to the long sides and the distance R_2 to the short sides. The lengths of the edges, corresponding to the widths of radial slices of the pebble, change in time according to their color and position – black edges at level i shrink with rate CR_i, white edges sandwiched between black edges grow with rate CR_i. (bottom) The contour and tree after the four original corners of the rectangle have been amputated.

max arcs, min arcs, rising faces, and falling faces, as in the bottom left of Fig. 4. The max arcs vanish at time $\tau_2 = \frac{\mathrm{arcsec}(-b/a) - \mathrm{arcsec}(b/a)}{Cb}$. During the final phase of the evolution, the contour consists of 6 piecewise smooth curves, two each of min arcs, rising faces and falling faces. The rising faces and falling faces all vanish at time $\tau_3 = \frac{\pi}{Ca}$ and for all later times the contour is a circle with radius $r = a$.

IV. COARSE EVOLUTION AND TREE MODEL

The constructions in the previous section are a bit unwieldy to write out by hand, though it is straightforward to program a computer to map (θ, t) to $r_0(\theta_0)$ and thus solve the evolution to the precision of the initial data $r_0(\theta_0)$. The complexity is all in how the pattern of arcs and faces changes over time. In this section, we extract from the solution above a more combinatorial model of the evolution that focuses on this pattern and may make it more intuitive.

First, identify all values of r such that the initial contour has a point where $\partial_\theta r = 0$ (do not count corners at maxima). Order these values from minimum to maximum to define $R_1 < R_2 < \cdots < R_N$. Geometrically, these critical values are the radii of circles centered at the origin which are tangent to the contour. These will be the N "levels" of a planar tree graph which we are constructing, which represents something like a skeleton of the contour. For example, regular polygons have only one level, the distance from the origin to any edge, and the rectangle discussed earlier has two levels, a and b.

Consider the set of intervals arising from the intersec-

tion of a circle of radius R_i with the area below the contour. Cut each interval in this set into black and white edges as follows: every subinterval which contains arcs (i.e. a segment on the contour coincident with the circle of radius R_i) becomes a white edge, and all other subintervals become black edges. Note that the black edges correspond to subintervals which support some hump of the contour. Each edge is assigned a length equal to the angular width of its subinterval. The edges just constructed constitute all the edges at level i. An example of white and black edge assignments can be seen on the left of Fig. 4.

We will now glue these edges into a planar tree. First, create a root vertex at level 1. Attach one end of every edge at level 1 to this root vertex, preserving the cyclic ordering of edges. Next, create a vertex at every black edge whose corresponding hump has intervals at level 2 above it, and then attach one end of every edge at level 2 to the appropriate vertex, preserving the linear ordering. Repeat this process of creating vertices and gluing for each remaining level. See Fig. 4 for the example of a rectangle. Roughly speaking, the tree captures the pattern of the protrusions of the contour as we move from the origin outwards. Note that the edges at level i are not necessarily all attached to edges at level $i-1$.

The dynamics of endpoints, faces and arcs yields the following rules for the evolution of the tree. As time progresses, every black edge at level i shrinks at the rate CR_i. White edges at level i which happen to be sandwiched between black edges (in the cyclic ordering around the root if $i = 1$, or the linear ordering above a vertex if $i > 1$) grow at the same rate CR_i. White edges at level $i > 1$ which sit alone on a vertex shrink at the rate CR_i. The lengths of all other white edges are held constant. If the length of an edge shrinks to zero, we remove it; if two white edges become adjacent on a vertex, we merge them into one white edge with length equal to the sum of their lengths.

Under this evolution, the tree contracts from the leaves inwards; the edges supporting a branch will never vanish before the edges at higher levels connected to it. The total time of evolution is thus determined by the length of the longest black edge at level 1. Translating back to the original contour, this means we just need to measure the angular width of the base of the largest hump. Therefore the contours with fixed minimum radius which take longest to evolve to a circle are those with a single minimum radius. The tree picture also makes it clear that all contours evolve to a circle with radius equal to the minimum radius of the initial shape.

At all times the pattern of white edges and black edges at different levels on the tree may be used to reconstruct a coarse version of the contour at that point in time. This is not a one-to-one correspondence between trees and contours, as there many possible contours that lead to the same tree. More explicitly, one can place at the left and right of the intervals corresponding to each black edge an arbitrary increasing function (rising face) between the

radii R_i to R_{i+1}, respectively decreasing function (falling face), provided the angular widths of these two functions is consistent with the lengths of the edges above and the length of this black edge. Indeed, not even the maximum height of each hump enters this description. However, all contours that lead to the same tree have the same pattern of face and arc disappearances. This property gives some stability to the evolution – if noise is added to the initial contour, this will only affect the long term behavior insofar as it might change the widths of the bases of the large scale features. Small humps coming from the short-wavelength part of the noise will correspond to short edges on the tree which quickly vanish or merge with the large edges.

V. DISCUSSION

The solution in this paper generalizes easily to the case where the equation takes the form $\partial_t r + f(r)\partial_\theta r H(\partial_\theta r) = 0$ with nondecreasing f in place of Cr. If f is not nondecreasing the solution above will be modified significantly as then some points of the contour would propagate in the opposite direction. However, a choice like this would also seem physically unmotivated. The function f allows for more general radius-speed relations, and the tree picture makes it clear that all that changes is the relative rate of growth or shrinkage of each edge, and not the overall qualitative picture; in particular, this may explain the observation in [6] that the model was robust to changing r to r^α with $\alpha = 1/2, 1, 2, 3$.

We speculate next on some possible choices for f. The rectangle is the most interesting case studied by Roth, Marques and Durian, as it is the only contour with two widely-separated levels. In their data (see rightmost panel of Fig. 2 in [6]), two of the corners are abraded before the other two, whereas the model predicts that all four corners vanish at the same time (τ_2 in the nota-

tion of the end of Sec. III). Thus the "constant" C may differ between the rising faces, perhaps being larger if the radii of the preceding minimum is smaller, and more generally, the speed f might in general be a function not just of r but also of r_{min} of the face as well. In principle, (different branches of) $f(r)$ can be extracted from experimentally measured curve contours by computing $\partial_t r$, $\partial_\theta r$ and comparing them at fixed r, but preliminary analysis of data provided by Roth, Marques and Durian was not conclusive due to the difficulty of estimating $\partial_t r$ accurately.

We did not carry out an extensive comparison here with the numerical solution of [6], but the plotted curves appeared indistinguishable in a few checks. Indeed, it would be interesting to place this work on firmer mathematical ground along the lines of [8] by analyzing how the corners generated by the amputation and interpolation processes are smoothed by the addition of higher derivative terms, and how this happens in the finite difference scheme of Roth, Marques and Durian. We also did not yet attempt to calculate the typical evolution of area, perimeter and other geometric quantities from our exact solution. Finally, we leave open the question whether erosion or shape evolution models in general may be simplified by posing them as laws for evolving tree graphs. In particular, it may be easier to construct models for the evolution of an ensemble of pebbles in terms of a mean-field model on trees, rather than attempt a more direct description of ensembles of interacting contours.

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