Power-law flow statistics in anisometric (wedge) hoppers

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Power-law flow statistics in anisometric (wedge) hoppers

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We find the probability for N particles to exit an anisometric (having unequal dimensions) hopper before jamming to have a broad power-law decay with exponent $\alpha = -2$, in marked contrast to the exponential decay seen in hoppers with symmetric apertures. The transition from exponential to power-law is explained by a new model that assumes particle motion is correlated over a distinct length scale. Hoppers with lengths larger than this length are modeled as a series of adjacent, statistically independent, "cells". Experiments with apertures 27-37 particle diameters D long are well-fit by a 3-cell model, implying that the correlation length is $\approx 9 - 12D$.

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I. INTRODUCTION

The clogging of particles at a hopper outlet has been termed "the canonical example of jamming" [1]. It is a matter of practical import, affecting a wide range of industries that transport granular media through pipes, silos, and hoppers [2, 3]. While early research focused on the steady-state flow rate [4], more recent attention has turned to the probability that flow stops, i.e. the transition to the jammed state. The wealth of research on jamming and the glass transition in various systems (e.g. [5–8]) has been summarized elsewhere [9–11].

Flow through a hopper stops when an arch, capable of supporting the weight of the grains above, forms at the exit aperture. To et al. [12–14] investigated the jamming of two-dimensional systems and measured the relative probability for an arch consisting of n particles to form. In a particularly beautiful piece of theory relying only on the requirement that arches be concave down, To et al. modeled this probability as a random walk, implying that the particles independently sampled all possible locations relative to their neighbors.

Zuriguel et al. [1, 15] measured the probability for three-dimensional flows to jam as a function of aperture diameter, independent of arch geometry. The probability $P(N)$ for N grains to exit the hopper before flow stopped decayed exponentially with N, which Zuriguel explained by assuming that the probability p for a particle to exit was statistically independent from that of other particles. The probability for N particles to exit is then

$$P(N) = p^N(1-p) \propto e^{N \ln p}. \quad (1)$$

$p < 1$, and so Eq.1 represents an exponential decay.

More recently, Janda et al.[16] connected the probability of forming an arch of specific size with the exit-mass probability distribution function $P(N)$ and also investigated how the mean flow $\langle N \rangle$ scaled with aperture diameter. A key finding was the absence of a critical aperture size above which the mean flow diverges. Rather, $\langle N \rangle$ grows with aperture diameter D as $\langle N \rangle \propto \exp[D^2]$.

Baxter et al. [17] and Choi et al. [18] tracked individual particles flowing through two- and three-dimensional hoppers and observed correlated motion over a well-defined length scale, with particles remaining in contact with their neighbors for long periods of time. Choi et al. compared the correlated motion with that of colloidal and molecular particles approaching the glass transition. Confocal microscopy of colloidal glasses [19] and molecular dynamics simulations of molecular glasses [20] have shown that the cooperative motions involve one dimensional strings of particles. The notion of a critical length scale of correlated motion becomes important when considering hopper apertures with different length scales.

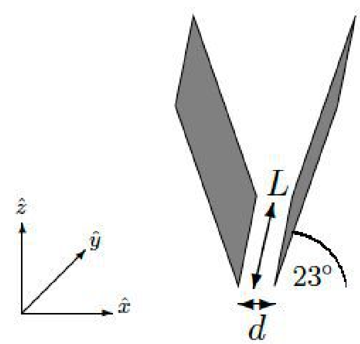

FIG. 1: Wedge-hopper geometry. The hopper angle (of the x-sidewalls) can be varied independently, as can the width of the exit aperture and depth (in $\hat{y}$) of the pile. All reported experiments are for fixed sidewall angle of 23° and $d = 1.00 \pm 0.05$ cm. The length L varies from 10-31 cm.

II. EXPERIMENTAL RESULTS

Our experiment involves a wedge-shaped hopper, with geometry shown in Fig. 1, consisting of two 31 cm x 38 cm Plexiglas sheets that can be angled independently to end in a rectangular aperture. The thickness d of the exit aperture can be changed without varying the angle by sliding one of the sidewalls in the $\hat{x}$ direction; all reported results are for $d = 1.00 \pm 0.05$ cm with a sidewall angle of $23° \pm 1°$. A Plexiglas insert allows us to vary the length L

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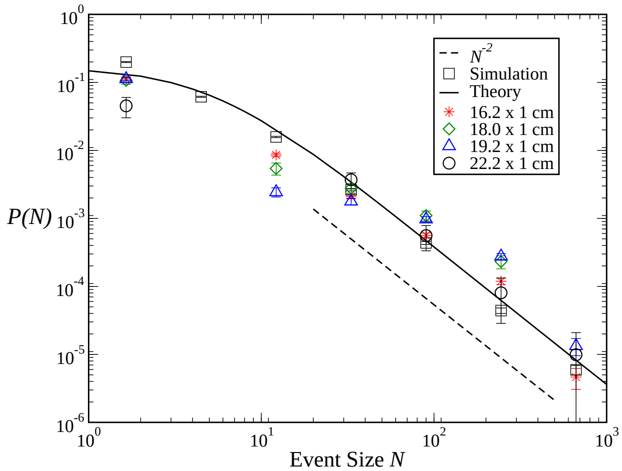

FIG. 2: (Color online) The probability for N particles to exit a wedge-hopper has a broad power-law tail with $P(N) \sim N^{-2}$ (dashed line), seen in experiments, Monte Carlo simulation, and analytic theory. This is in distinct contrast to the distribution function found in conical hoppers, which decays exponentially. Shown are distribution functions for experimental hoppers of lengths $L = 16.2-22.2$ cm. Simulation and theory assume $n_c = 3$ adjacent, statistically independent cells.

in $\hat{y}$ from of order a few particle diameters to a maximum of 31 cm, several times the particle diameter.

Acrylic spheres (diameter $D = 6$ mm, mass $m = 0.5$ g) are poured into the hopper while the aperture is blocked by a flat plate. The plate is then removed and particles fall onto an Ohaus Digital Balance with a resolution of 0.1 g, sufficient to distinguish the number of individual particles that fall through the hopper before the pile jams. A computer detects when the scale reading stabilizes and, when the weight has been constant for 0.5 seconds, triggers a solenoid valve to quickly open and close, diverting air to a Clippard pneumatic cylinder. The cylinder acts as an impact hammer, jolting the hopper and re-initiating particle flow. This process continues until the hopper has emptied, at which point the last scale reading is discarded (since flow never actually stopped), the hopper refilled, and the process repeated.

Figure 2 shows the exit-mass probability distribution function for 4 different aperture lengths: 16.2, 18.0, 19.2, and 22.2 cm. After an initial plateau, the distribution falls off as a power law with $P(N) \sim N^{-2}$. Each distribution shown is the result of at least 1500 separate events; error bars indicate the standard deviation of all events that fall within the respective (logarithmically spaced) bin. We have confirmed that the exit-mass distribution functions obtained are unaffected by the refilling process. Distributions made from events immediately after refilling, for example, are indistinguishable from those made from events when the hopper is nearer to being empty.

Zuriguel et al. [1, 15] used air to break up the arch, and always had at least 1 particle exit. We suspect that the plateau ($P(0) \neq 0$) occurs because our impact does not always break up the arch. Chen et al. [21] reported that lower magnitude impacts shift the distribution at low N, but do not change the scaling at large N.

III. NUMERICAL MODEL

In order to modify Eq. 1 to include anisometric apertures, we recall experiments [17, 18] that found correlated motion in clusters of particles moving through hoppers. Assuming that these correlations are "string-like" we claim that the probability for a correlated string of particles to exit p is a function of the string orientation relative to the hopper aperture. Strings aligned along the length of the aperture ($\hat{y}$ in Fig.1) have a high probability of passing through, while those aligned across the width ($\hat{x}$ in Fig. 1) have a smaller probability. We relate p to the ratio of the projected length of the string along the $\hat{x}$ and $\hat{y}$ axes with the aperture size in that direction, with a maximum value of $p = 1$:

$$p(\theta) \;=\; \min\Big(\max\Big[\frac{l^x_{\text{proj}}}{d}, \frac{l^y_{\text{proj}}}{L}\Big], 1\Big) \qquad (2)$$

The probability is a maximum $p = 1$ when $l^x_{\text{proj}} > d$ (forming an arch across the short length) or $l^y_{\text{proj}} > L$ (spanning the hopper length). The experimentally observed distribution function is an average over the range of individual string exit probabilities.

An implicit assumption is that the granular strings are independent of the specific shape of the opening aperture. That is, the distribution of string size and orientation is largely determined by the characteristics of the granular material and not the experimental geometry. New experiments studying the transition from exponential to power-law statistics are underway to validate this assumption. For now, we claim only that our results are consistent with such an assumption, not definitive proof.

We label the probability limits p_y and p_x and, for simplicity, assume a uniform distribution of exit probability probability between $p_x < p < p_y$:

$$O(p) = \frac{\theta(p - p_x)\theta(p_y - p)}{\Delta p}. \qquad (3)$$

$\theta(p-p_x)$ and $\theta(p_x-p)$ are Heaviside step functions, and $\Delta p \equiv p_y - p_x$ is a normalizing factor. Averaging over all the allowable exit probabilities involves the integral

$$\langle P(N)\rangle = \int_{p_x}^{p_y} p^N (1-p) O(p)\ \mathrm{d}p. \qquad (4)$$

Isometric (round) apertures correspond to the case where $p_y = p_x$. The orientational probability $O(p)$ then approximates a Dirac delta function at p_x, and so

$$\langle P(N)\rangle \approx \int p^N (1-p)\delta(p-p_x)\ \mathrm{d}p = p_x^N (1-p_x) \qquad (5)$$

recapturing the exponential form of Eq. 1.

We can readily calculate the integral of Eq. 4

$$\int_{p_x}^{p_y} p^N(1-p)O(p)\ \mathrm{d}p = \frac{1}{\Delta p}\left[\frac{p^{N+1}}{N+1} - \frac{p^{N+2}}{N+2}\right]\Bigg|_{p_x}^{p_y} \quad (6)$$

Eq. 6 represents a $1/N$ decay with an exponential cutoff (since $p^N = \exp[N \ln p]$). As $p_y \to 1$, the exponential cutoff occurs at larger N, eventually disappearing when $p_y = 1$. Physically, $p_y \to 1$ means that strings always exit the hopper; intuitively, we associate this with the aperture length approaching the string length. In this case, a string aligned with the long edge cannot bridge the gap between the sidewalls to form an arch. In the limit of $p_y = 1$, Eq. 6 has the asymptotic value of $P(N \to \infty) \propto 1/N^{-2}$, as seen in experiment (Fig. 2).

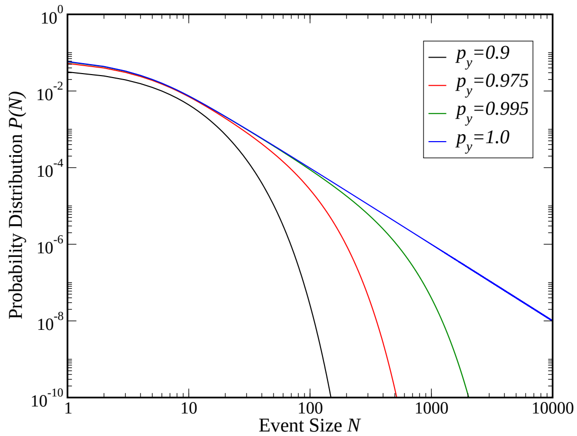

FIG. 3: (Color online) The exit-mass probability distribution function $P(N)$ shows a transition from exponential to power-law decay as the upper limit (p_y in Eq. 4) approaches 1 (reading curves left to right). This corresponds to a hopper aperture with one length-scale the size of a granular string. For smaller aperture lengths, the distribution function has the form of a power-law with an exponential cutoff.

Figure 3 shows the probability distribution for several values of p_y approaching 1. The exponential cutoff occurs at larger values of N for p_y nearer 1, finally disappearing at $p_y = 1$ and leaving behind a power-law decay. The transition from exponential to power-law distribution occurs as the aperture's long length approaches the length scale over which correlated motion occurs [18]. We are currently conducting experiments with less anisometric apertures to capture this transition.

To extend the model to longer apertures, we note that the flow through anisometric hoppers is spatially and temporally inhomogeneous. It is not uncommon for a subsection of the hopper to jam, with flow continuing unabated through the rest of the hopper. While the global flow rate decreases, we observed no effect on the local flow rate, nor did we see any preferential jamming at the sidewalls; jamming was equally likely to occur in the middle of the hopper as at the ends. Our data, however, are not sufficient to draw quantitative conclusions.

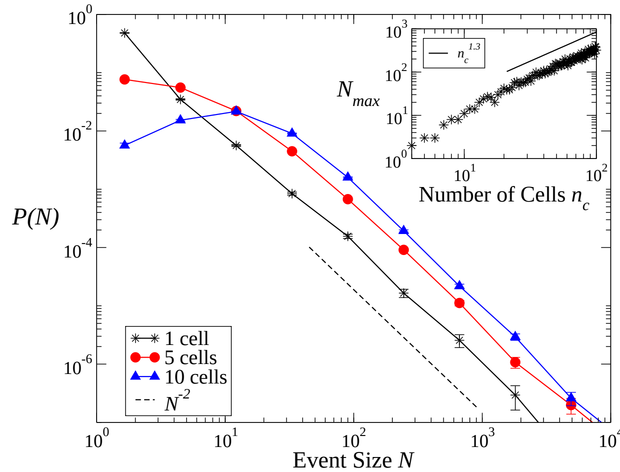

FIG. 4: (Color online) Adapting the model to account for lengths longer than that of a granular string changes the shape of the probability distribution function. A plateau forms and, for increasing number of cells n_c, the peak of the distribution shifts to larger values of N as $N_{max} \sim n_c^{1.3}$ (inset). All distributions show the characteristic N^{-2} decay.

The spatial inhomogeneity suggests it appropriate to model long apertures as a series of n_c adjacent "cells", each as long as a granular string. The probability for n_i particles to exit the ith cell is given by Eq. 6 with $p_y = 1$, and we assume that the probabilities for different cells are statistically independent. The average exit mass probability distribution is calculated by summing over all the ways that N particles can exit n_c cells:

$$\langle P(N) \rangle = \sum_{n_1, n_2, \ldots = 0}^{N} \prod_{i=1}^{n_c} P(n_i)\delta\Big(N - \sum_{i=1}^{n_c} n_i\Big), \qquad (7)$$

where the delta function forces the sum of the particles exiting the individual cells to total N: $n_1 + n_2 + \ldots = N$.

We set $p_x = 0$ for computational and analytic simplicity, although there is some justification for this choice in two-dimensional experiments [12] that found flow to stop through round apertures when the aperture diameter was approximately 1.5 times the particle diameter. As a result, $P(n_i) = [(1+n_i)(2+n_i)]^{-1}$, and we plot in Fig. 4 the distribution that results from $n_c = 1, 5$, and 10. The plateau around $P(0)$ gives way to a distinct peak in the distribution. The location of this peak, the most likely number of particles to exit, increases as $n_c^{1.3}$ (Fig. 4(inset)), although the origin of this behavior is unknown. All distributions show a N^-2 decay (dashed line), consistent with experiment and analytic solution.

Equation 7 can be calculated numerically for low values of n_c; for larger numbers of cells we use a Monte Carlo simulation. The simulation is actually quite simple, where each string is modeled by two random numbers, the first describing the orientation (and hence jamming probability), the second to determine whether the string actually jams or not. The simulation runs until all cells are jammed and then totals the number of attempts required to reach the jammed state. We have confirmed that the two agree for $n_c < 5$ (c.f. Fig. 2).

We vary n_c as the lone fitting parameter in our simplified model to best fit the plateau at low N. Figure 2 shows experimental data from wedge hoppers of four different geometries (1 cm x $16.2, 18.0, 19.2$, and 22.2 cm), the Monte Carlo simulation, and the theoretical calculation (Eq. 7) for $n_c = 3$. Experiment, simulation, and theory all show a plateau for low exit masses and N^{-2} decay for large N. The excellent quantitative agreement of all experimental data with simulation and theory for $n_c = 3$ implies that the the experimental length of 27-37 particle diameters corresponds to three "cells", where each cell has a length equal to the average length of the granular strings. We therefore tentatively suggest that the strings themselves are 9-12 particle diameters long.

IV. CONCLUSIONS

We have experimentally measured a power-law decay with exponent $\alpha = -2$ of exit-mass probability in flow through anisometric wedge hoppers, in distinct contrast to the exponential decay found in isometric round hoppers. We argue that this arises from the orientational alignment with the aperture of clusters or strings of granular particles whose motion is correlated. The hopper anisometry requires an average over the orientation-dependent probability for a single string to exit. When the upper limit of this probability is 1, as it must be for very long aperture lengths, the exit-mass distribution assumes a power-law tail.

Motivated by the observed spatial inhomogeneity of the hopper flow, we extend the model to consider adjacent, statistically independent cells of a well-defined length. The exit-mass probability can be calculated analytically (for low numbers of cells) or with Monte Carlo simulation (for larger numbers of cells), and find $n_c = 3$ to give an excellent fit with the experimental data. That experiments over a range of 27-37 particle diameters D in lengths are fit by $n_c = 3$ implies an estimate for the granular string length of $9D - 12D$.

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