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Correction-to-scaling exponent for two-dimensional percolation

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We show that the correction-to-scaling exponents in two-dimensional percolation are bounded by $\Omega \leq 72/91$, $\omega = D\Omega \leq 3/2$, and $\Delta_1 = \nu\omega \leq 2$, based upon Cardy's result for the crossing probability on an annulus. The upper bounds are consistent with many previous measurements of site percolation on square and triangular lattices and new measurements for bond percolation, suggesting they are exact. They also agree with exponents for hulls proposed recently by Aharony and Asikainen. A corrections scaling form evidently applicable to site percolation is also found.

In percolation, a quantity of central interest is the size distribution $n_s(p)$, which gives the number of clusters (per site) containing s sites, as a function of the site or bond probability p. In the scaling limit, in which s is large and $p - p_c$ is small such that $(p - p_c)s^\sigma$ is constant, $n_s(p)$ behaves as [1]

$$n_s(p) \sim As^{-\tau} f(B(p - p_c)s^\sigma) \ , \tag{1}$$

where τ, σ, and $f(z)$ are universal, while the metric factors A and B and the threshold p_c are system-dependent. For two-dimensional systems, $\tau = 187/91$, $\sigma = 36/91$.

For finite systems, there are corrections to (1). Here we are concerned with the corrections precisely at p_c, where asymptotically $n_s(p_c) \sim As^{-\tau}$. Many studies are carried out at the critical point, so knowing the behavior there is useful. Also knowing the nature of the corrections helps one to determine the thresholds precisely for systems where they are not known exactly [2, 3].

It is generally hypothesized that the corrections to this behavior are of the form

$$n_s(p_c) \sim As^{-\tau}(1 + Cs^{-\Omega} + \ldots) \tag{2}$$

where Ω is the correction-to-scaling exponent. In terms of a length scale $L \sim s^{1/D}$, the correction term is of the form $(1 + cL^{-\omega} + \ldots)$, where $\omega = D\Omega$ and $D = 91/48$ is the fractal dimension.

Studies of Ω go back to the mid-1970's and are summarized in Table I; this is an updated version of a table given in [11], which surveyed the work up to 1983. The values were found by Monte-Carlo simulation, analyses of series expansions, and theoretical arguments. Theoretical analyses of corrections to scaling were also given in [24]. Derrida and Stauffer [25] studied ω on strips, and their values for two different orientations extrapolate to a value of ω in the range of 1 to 2. Studying backbones, Bhatti et al. find Δ_1 in the range $\approx 1 - 3$ [26]. Also, $\omega = 1.2(2)$ was found by Normand, Herrmann and Hajjar [27] for percolation conductivity on a cylindrical geometry, which may share the same finite-size corrections. In more recent work, Ziff and Babalievski [18] found $\Omega = 0.77(2)$, and independently Kammerer et al. [23] found the identical value, $\Omega = 0.77(4)$. The value 0.73(2) has also been given recently by Tiggemann [22]. Corrections have been studied recently for percolation on the XY-model [28].

In this Letter, we derive a value for Ω by relating Cardy's result [29] for crossing in an annulus to the cluster-size distribution. For the probability $\Pi(\tau)$ that a crossing occurs in an annulus or a cylinder, Cardy found

$$\Pi(\tau) = \frac{\eta(-1/3\tau)\eta(-4/3\tau)}{\eta(-1/\tau)\eta(-2/3\tau)} = (3/2)^{1/2}\frac{\eta(3\tau)\eta(3\tau/4)}{\eta(\tau)\eta(3\tau/2)} \tag{3}$$

where $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24}$ is the Dedekind η function, with $q = e^{2\pi i\tau}$. For an annulus of outer and inner radii R and R_1 respectively, $\tau = (i/\pi)\ln(R/R_1)$. For a cylinder of circumference ℓ and length L, $\tau = 2iL/\ell$, since by conformal transformation R/R_1 corresponds to $e^{2\pi L/\ell}$. Note that $\eta(\tau)$ can be directly evaluated in Mathematica using `DedekindEta[τ]`.

Evidently, Cardy's result (3) has not been discussed much in the literature since it appeared five years ago. Here we show that it is consistent with several previous measurements of crossing. For a cylinder of aspect ratio $L/\ell = 1/2$, which corresponds to an annulus with $R/R_1 = e^\pi$, $\tau = i$ and (3) gives $\Pi(i) \approx 0.876\,631\,451\ldots$, in agreement with the precise value 0.876 657(45) found by de Oliveira, Nóbrega, and Stauffer [30]. For $L/\ell = 1$ or $R/R_1 = e^{2\pi}$, $\tau = 2i$ and (3) gives $\Pi(2i) \approx 0.636\,454\,001$, which agrees closely with the measured values 0.636 65(8) of Hovi and Aharony [31], 0.63 of Gropengiesser and Stauffer [32], 0.638 of Acharyya and Stauffer [33], 0.64(1) of Ford, Hunter, and Jan [34], 0.6365(1) of Shchur [35], and 0.6363(3) (average) by Pruessner and Moloney [36].

Eq. (3) also implies $\Pi = 1/2$ for $L/\ell \approx 1.368\,800$, and the maximum in $-\Pi'(L/\ell)$ occurs at $L/\ell \approx 0.540\,652$, where $-\Pi' \approx 0.522\,282$. In comparison, for a system with open boundaries, the maximum in $-\Pi'(L/\ell)$ occurs at $L/\ell \approx 0.523\,522$ with value 0.737 322 [37], as follows from Cardy's original crossing formula [38].

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TABLE I. History of determinations of Ω, $\omega = D\Omega = (91/48)\Omega$, and $\Delta_1 = \Omega/\sigma = (91/36)\Omega$. Numbers in parentheses represent errors in last digit(s), and are shown on original values only.

year	author	method	Ω	ω	Δ_1
1976	Gaunt & Sykes [4]	series	0.75(5)	1.42	1.90
1978	Houghton, Reeve & Wallace [5]	field theory	0.54–0.68	0.989–1.28	1.32–1.71
1979	Hoshen et al. [6]	MC	0.67(10)	1.27	1.69
1980	Pearson [7]	conjecture	$64/91 \approx 0.703$	1.333	1.778
1980	Nakanishi & Stanley [8]	MC	0.6–1		
1982	Nienhuis [9]	field theory	$96/91 \approx 1.055$	2	2.667
1982,3	Adler, Moshe & Privman [10, 11]	series $p < p_c$	0.5	0.95	1.26
	”	series	0.66(7)	1.25	1.67
1983	Aharony & Fisher [12, 13]	RG theory	$55/91 \approx 0.604$	$55/48 \approx 1.15$	$55/36 \approx 1.53$
1983,4	Margolina et al. [13, 14]	MC	0.64(8)	1.21	1.62
	”	series	0.8(1)	1.52	2.02
1985	Adler [15]	series	0.63(5)	1.19	1.59
1986	Rapaport [16]	MC	0.71–0.74		
1998	MacLeod & Jan [17]	MC	0.65(5)	1.23	1.64
1999	Ziff & Babalievski [18]	MC	0.77(2)	1.46	1.95
2001	Tiggemann [19]	MC	0.70(2)	1.33	1.77
2003	Aharony & Asikainen [20, 21]	theory (hulls)	72/91	1.5	2
2007	Tiggemann [22]	MC	0.73(2)	1.38	1.85
2008	Kammerer, Höfling & Franosch [23]	MC	0.77(4)	1.46	1.95
2010	this work	theory	$72/91 \approx 0.791$	1.5	2

Expanding the η functions in (3) for large L/ℓ, we find,

$$\Pi(\tilde{q}) = \sqrt{\frac{3}{2}}\,\tilde{q}^{5/48}\left(1 - \tilde{q}^{3/2} + \tilde{q}^2 - \tilde{q}^{7/2} + 2\tilde{q}^4 - \tilde{q}^{9/2} - \ldots\right) \tag{4}$$

where $\tilde{q} = R_1/R = e^{-2\pi L/\ell}$ as in [29]. Thus, for the annulus, we have

$$\Pi\left(R/R_1\right) = \sqrt{\frac{3}{2}}\left(\frac{R}{R_1}\right)^{-5/48}\left\{1 - \left(\frac{R}{R_1}\right)^{-3/2} + \left(\frac{R}{R_1}\right)^{-2} - \left(\frac{R}{R_1}\right)^{-7/2} + 2\left(\frac{R}{R_1}\right)^{-4} - \ldots\right\} \tag{5}$$

and for the cylinder, we have

$$\Pi\left(L/\ell\right) = \sqrt{\frac{3}{2}}\,e^{-(5/24)\pi L/\ell}\left\{1 - e^{-3\pi L/\ell} + e^{-4\pi L/\ell} - e^{-7\pi L/\ell} + 2e^{-8\pi L/\ell} - \ldots\right\} \tag{6}$$

Shchur [35] has verified the leading term above numerically, finding 0.654 48(5) for the exponent $5\pi/24 \approx 0.654\,498$, and the intercept of the asymptotic line in his Fig. 6 is consistent with the predicted coefficient $\sqrt{3/2} \approx 1.224\,745$. Preussner and Moloney [36] also measure this coefficient and find 1.2217(4) for bond percolation and 1.2222(4) for site percolation on a square lattice, where the errors bars represent statistical errors of a single set of system sizes.

For the annular result, we can imagine that the system is actually infinite with an inner boundary of radius R_1; then, $\Pi(R/R_1)$ is the probability that a cluster connected to the inner boundary has a maximum radius greater than or equal to R. That is, if, in the infinite system, the cluster connected to the center extends beyond a circle of radius R, then in the annulus there will be a crossing cluster between the two circles of radii R_1 and R, and these two events will occur with the same probability. Thus Π gives a measure of the size distribution of the clusters connected to the inner circle, where the size is characterized by the maximum cluster radius. We relate this to the cluster size distribution by associating the inner radius to the discreteness of the lattice.

Given a size distribution n_s, the probability that an occupied site is connected to a cluster of size greater than or equal to s is given by:

$$P_{\geq s}(p) = \sum_{s'=s}^{\infty} s' n_{s'} \approx \int_s^{\infty} s' n_{s'} ds' \tag{7}$$

Using (2) for $n_s(p_c)$, we thus find that, at p_c,

$$P_{\geq s}(p_c) \sim A' s^{2-\tau}(1 + B' s^{-\Omega} + \ldots) \tag{8}$$

where $2 - \tau = -\beta\sigma = -5/91$ in 2-d. Because critical percolation clusters are fractal, s are R are related by

$$s \sim s_0(R/\epsilon)^D \tag{9}$$

where D is the fractal dimension, s_0 is a constant, and ϵ is of the order of the lattice spacing and represents the lower size cutoff of the system, similar to a boundary extrapolation length [39]. Putting (9) into (8), assuming $\epsilon = R_1$, we find that the probability a cluster has a radius greater than R is given by

$$P_{\geq R}(p_c) \sim a R^{D(2-\tau)}(1 + b R^{-\Omega D} + \ldots) \tag{10}$$

where a and b are constants. By hyperscaling $D(2-\tau) = D - d = -\beta/\nu = -5/48$. Comparing this with (5), we see that $\Omega D = \omega = 3/2$ or

$$\Omega = 3/(2D) = 72/91 \tag{11}$$

implying also $\Delta_1 = \nu\omega = 2$.

Alternately, if we put $(R/R_1) = (s/s_0)^{1/D}$ into (5), we find

$$P_{\geq s}(p_c) = \sqrt{\frac{3}{2}}\left(\frac{s}{s_0}\right)^{-\frac{5}{91}}\left\{1 - \left(\frac{s}{s_0}\right)^{-\frac{72}{91}} + \left(\frac{s}{s_0}\right)^{-\frac{96}{91}} - \left(\frac{s}{s_0}\right)^{-\frac{168}{91}} + 2\left(\frac{s}{s_0}\right)^{-\frac{192}{91}} - \ldots\right\} \tag{12}$$

which gives the higher-order corrections also. However, in deriving (12) we have ignored finite-size corrections to (9). Say we have to next order

$$s \sim s_0(R/\epsilon)^D(1 + cR^{-x} + \ldots) \tag{13}$$

then this will lead to a term of order $(s/s_0)^{-x/D}$ in the expansion of $P_{\geq s}(p_c)$, and where x/D lies within the exponents 72/91, 96/91, ... will determine its importance. It is of course possible that $x/D < 72/91$, in which case that would be the dominant correction. There have been studies in the past on the finite-size corrections to the radius of gyration [40], but not to our knowledge to the maximum radius with respect to an arbitrary point within a cluster, which is needed here.

Our result for Ω is consistent with the very first determination 0.75(5) [4] as well as many of the recent measurements. Some previous results gave lower values, such as the series results of [10, 11] that were based upon studying the scaling of quantities like the mean cluster size away from p_c. Perhaps these lower values were due to other corrections which are relevant away from p_c.

In (12), we see that the next-order correction term is $s^{-\Omega_2}$ with $\Omega_2 = 96/91$. This interestingly is exactly the value of Ω proposed by Nienhuis [9]. The closeness of this exponent to 72/91 would make its determination numerically difficult, and in any case higher-order corrections in the relation (13) between s and R might mask it.

It turns out that also a few years ago, Aharony and Asikainen [20, 21] proposed the same two leading correction exponents $\omega = 3/2$ and 2 for the correction-to-scaling for fractal properties of the complete hulls of percolation clusters, relating them to earlier results of den Nijs [41]. Evidently, these same corrections to apply to the cluster statistics. Note that the results of [20] have not been verified numerically.

Assuming the other correction terms are small, (12) implies that $P_{\geq s}$ should be a universal function of s/s_0, where s_0 varies from system to system. This may explain an observation made in [18] that the quantity Ω(est) defined by

$$\Omega(\text{est}) = -\log_2\left(\frac{C_s - C_{s/2}}{C_{s/2} - C_{s/4}}\right) \tag{14}$$

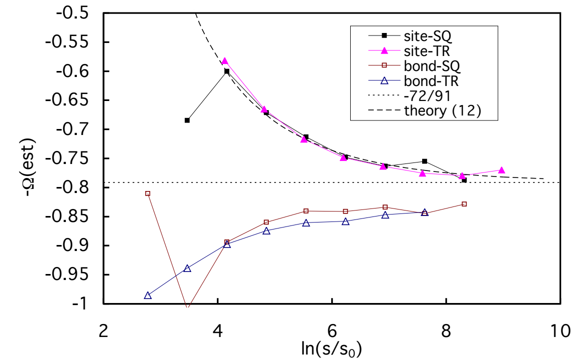

FIG. 1. Plot of $-\Omega$(est) as a function of $\ln(s/s_0)$ for $s = 2$, 4, ... 1024. The two upper curves are a replotting of the data given in Fig. 6 of Ref. [18], using $s_0 = 0.25$ (square lattice) and $s_0 = 0.13$ (triangular lattice) to line the data up with the theoretical prediction from (12). No vertical rescaling was done. (Note: there was a minus sign missing in the label of the vertical axis in Ref. [18].) The two lower curves are new plots for bond percolation on the square and triangular lattices, using $s_0 = 0.25$ and 0.5, respectively.

where $C_s = s^{\tau-2}P_{\geq s}$, when plotted vs. $\ln s$, appears to be a universal curve for site percolation on the square and triangular lattices, except for a horizontal shift. Ω(est) equals Ω if the correction to scaling has only one term as in (8), and otherwise can be used to estimate Ω by taking s relatively large. The shift that was needed in [18] to line up the data just reflects the difference in s_0 between the two lattices, because of the logarithmic scale in the plot. In Fig. 1 we have replotted the data of Ref. [18] along with the results of using the Ω(est) that follow from (12) (using the first three terms), and adjusted s_0 to match the theoretical prediction. The behavior can be seen to match the theory very well, with only the single adjustable parameter s_0. In Ref. [18], we generated $6\cdot10^9$ clusters up to size $s = 1024$ for each system, to obtain these data.

Here we have generated additional data for bond percolation on square and triangular lattices, in which we characterize s by the number of sites wetted by the clusters. We generated $5 \cdot 10^{10}$ clusters for the square, and $2.5 \cdot 10^{11}$ clusters for the triangular lattice, both with $s \leq 1024$ using the R(9689) random-number generator of [42]. The corresponding Ω(est) are also plotted in Fig. 1. For both lattices Ω(est) approaches ≈ 0.8 from above for larger s, implying that (11) is again the leading correction exponent, but for smaller s the data do not follow the behavior implied by (12), meaning here other finite-size corrections must be significant. While (12) implies asymptotic deviations from the leading behavior are negative, for bond percolation in fact the deviations for small

s are positive. Similar positive finite-size effects are seen in the closely related SIR epidemic model [43].

We can find the behavior of n_s from (12) as follows:

$$n_s\ (p_c) = \frac{p_c}{s}\left(P_{\geq s} - P_{\geq(s+1)}\right) = \frac{5p_c}{91 s_0^2}\sqrt{\frac{3}{2}}\left(\frac{s}{s_0}\right)^{-\frac{187}{91}} \times \left\{1 - \frac{77}{5}\left(\frac{s}{s_0}\right)^{-\frac{72}{91}} - \frac{48}{91}\left(\frac{s}{s_0}\right)^{-1} + \frac{101}{5}\left(\frac{s}{s_0}\right)^{-\frac{96}{91}} ...\right\} \tag{15}$$

where the factor of p_c is included for site percolation only [1]. Thus, for (1), we have $A = (5p_c/91)\sqrt{3/2}\, s_0^{5/91}$ and $B = (77/5)s_0^{72/91}$, and we have picked up the analytic term s^{-1}. Likewise,

$$n_{\geq s}(p_c) = \frac{5p_c}{96 s_0}\sqrt{\frac{3}{2}}\left(\frac{s}{s_0}\right)^{-\frac{96}{91}}\left\{1 - \frac{44}{5}\left(\frac{s}{s_0}\right)^{-\frac{72}{91}} + \dots\right\} \tag{16}$$

For example for site percolation on the triangular lattice ($p_c = 0.5$), we found $s_0 = 0.13$ (from Fig. 1), which implies a coefficient of 0.0285 to $n_{\geq s}(p_c)$. This value agrees closely with ≈ 0.028 given in Fig. 2 of [13].

Eq. (16) also gives that the number of clusters whose covering area πR^2 is greater than or equal to A is given by the Zipf's-law form [44, 45] $n_{\geq A}(p_c) \sim (5p_c/96)\sqrt{3/2}(\pi R_1^2/s_0)/A$, and the correction to scaling is predicted to be proportional to $A^{-3/4}$.

In conclusion, we found that the behavior of crossing on an annulus implies $\Omega = 72/91$ or $\omega = 3/2$, which appears to represent the dominant exponent for the systems we studied by simulations. For future work, it would be interesting to study these corrections on additional lattices as well as higher dimensional systems.

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