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Multi-mode correlations and the entropy of turbulence in shell models

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We suggest a new focus for turbulence studies — multi-mode correlations — which reveal the hitherto hidden nature of turbulent state. We apply this approach to shell models describing basic properties of turbulence. The family of such models allows one to study turbulence close to thermal equilibrium, which happens when the interaction time weakly depends on the mode number. As the number of modes increases, the one-mode statistics approaches Gaussian (like in weak turbulence), the occupation numbers grow, while the three-mode cumulant describing the energy flux stays constant. Yet we find that higher multi-mode cumulants grow with the order. We derive analytically and confirm numerically the scaling law of such growth. The sum of all squared dimensionless cumulants is equal to the relative entropy between the full multi-mode distribution and the Gaussian approximation of independent modes; we argue that the relative entropy could grow as the logarithm of the number of modes, similar to the entanglement entropy in critical phenomena. Therefore, the multi-mode correlations give the new way to characterize turbulence states and possibly divide them into universality classes.

INTRODUCTION

We define turbulence as a state where many degrees of freedom are deviated from thermal equilibrium. This usually happens when wavenumbers or frequencies of the modes excited are vastly different from those of the modes that dissipate. A statistically steady state is then a long cascade of excitations.

Fluctuations in a wide interval of scales and times is the property shared by turbulence and critical phenomena. That analogy was discussed either in terms of power-law behavior of correlation functions [1, 2] or in terms of the probability distribution of macroscopic quantities [3]. However, the analogy was not advanced to reach deeper understanding of turbulence, in particular, reproduce the spectacular success of statistical physics in defining the universality classes of critical phenomena.

We suggest to focus on the entropic characteristics of turbulence, following the approach developed for critical phenomena by Wilczek, Kitaev, Cardy and others [4–6]. While the entropy itself is usually linear in the number of degrees of freedom, the quantum entanglement entropy or classical mutual information between different parts of the system or between a system and its environment could be logarithmic at criticality. The factor in front of the logarithm is the central charge of the conformal field theory, which determines the universality class of the transition [4]. That approach to critical phenomena brought most progress in two dimensions, here we shall apply it to one-dimensional model of turbulence.

We believe that an entropic approach is also natural for turbulence, which must have much lower entropy than thermal equilibrium (at the same energy). How does the entropy deficit depend on the number of modes deviated

from equilibrium or, in other terms, on the Reynolds number? Since the entropy deficit is information, where it is encoded?

Two natural possibilities exist here: correlations between points in space or between modes in Fourier space. The multi-point spatial correlations were first discovered in the simplest case of non-equilibrium imposed by a spatial gradient (of temperature, velocity, etc). So-called Dorfman-Cohen anomalies manifest themselves as infrared divergencies in the density expansion of kinetic coefficients (thermal conductivity, viscosity, diffusivity) [7–9]. However, turbulence deviates from equilibrium by forcing some modes and dissipating other, imposing different conditions in Fourier space rather than in real space. Therefore, the cascade nature of turbulence must manifest itself in inter-mode correlations.

Since the very existence of a cascade hinges on interaction between modes, the traditional approach is usually focused on the lowest two nonzero moments: the occupation numbers and the flux [10–15], which is typically a three- or four-mode correlation function. Yet it was argued recently from an entropic analysis that even for a weak wave turbulence two moments are not sufficient to describe the statistics of a multi-mode system, which could be far from Gaussian [16].

We suggest that the nature of a turbulent state is reflected in multi-mode correlations which can be quantified by the mutual information between upscale and downscale parts of the cascade or between all modes. We hope that the mutual information can play here the role that the entanglement entropy plays in critical phenomena. For the formidable task of analyzing the multi-mode correlations, it is natural to start from the class of systems where they are weak and can be treated perturba-

tively. We find one such class where occupation numbers are close to thermal equipartition, the single-mode statistics is close to Gaussian, and the (dimensionless) flux is small. We demonstrate that nonzero multi-mode dimensionless cumulants can be all of the same order in such systems. When this is the case, the mutual information must grow as a logarithm of the total number of modes in the cascade, thus establishing a direct quantifiable link with classical and quantum critical phenomena.

Developed turbulence with the same scaling as in equilibrium occurs in many systems. One class is both direct and inverse cascades in the universal Nonlinear Schrodinger model in two dimensions [12], which describes cold atoms and all spectrally narrow wave spectra. Another example is a joint turbulence of interacting high and low-frequency waves abundant in geophysics and plasma physics [11, 17].

DEFINITION OF THE MODELS

We consider a very wide class of systems with strong interaction, described by the equations with quadratic nonlinearity,

$$i\dot{a}_j = \sum_{sq} (2V_{js}^q a_s^* a_q + V_{sq}^j a_s a_q), \quad (1)$$

conserving a quadratic integral of motion, $E = \sum \omega_j |a_j|^2$, as long as $V_{sq}^j \neq 0$ iff $\omega_s + \omega_q = \omega_j$. Such are the Euler equations for incompressible fluid flows and for solid body rotations, as well as the family of two-dimensional hydrodynamic models from geophysics, astrophysics and plasma physics, where a scalar field a (vorticity, temperature, potential) is transported by the velocity whose stream function ψ is linearly related to a : $\partial a / \partial t = -(\mathbf{v} \cdot \nabla) a$, $\mathbf{v} = (\partial \psi / \partial y, -\partial \psi / \partial x)$, $\psi(\mathbf{r}) = \int d\mathbf{r}' |\mathbf{r} - \mathbf{r}'|^{\beta-2} a(\mathbf{r}')$. For the 2D Euler equation, $\beta = 2$. Other cases include surface geostrophic ($\beta = 1$) [18], rotating shallow fluid or magnetized plasma ($\beta = -2$) [19], etc. After Fourier transform, $\dot{a}_{\mathbf{k}} = \sum_{\mathbf{q}} q^{-\beta} [\mathbf{k} \times \mathbf{q}] a_{\mathbf{q}} a_{\mathbf{k}-\mathbf{q}}$.

Recently, the family was expanded further by adding models describing waves or modes with a resonant triplet interaction determined by the Hamiltonian [20, 21]:

$$\mathcal{H}_w = \sum_j \omega_j |a_j|^2 + \sum_{j,s,q} V_{sq}^j (a_q^* a_s^* a_j + a_s a_q a_j^*).$$

Gauge transformation, $a_j \mapsto a_j \exp(i\omega_j t)$, turns $i\dot{a}_j = \partial \mathcal{H}_w / \partial a_j^*$ into (1).

All models are scale-invariant: $V(\lambda\omega_j, \lambda\omega_s, \lambda\omega_q) = \lambda^\alpha V(\omega_j, \omega_s, \omega_q)$; for hydrodynamic models $\alpha = 2 - \beta$. They all have exactly Gaussian statistics in thermal equilibrium [20, 21]: $\ln \mathcal{P}_{\mathcal{G}} \propto -E/T$ and $\langle |a_j|^{2m} \rangle = \Gamma(m+1) n_j^m$, where $n_j \equiv \langle |a_j|^2 \rangle = T/\omega_j$. Thermal equipartition takes place both in a closed system and under a contact with a thermostat, which pumps and dissipates all N modes equally, see below. Turbulence appears

when pumping and damping act on distant modes, p and d respectively. Nonlinear interaction provides for a cascade of excitations via the transparency window between p and d . Cascade is called direct for $p = 1, d = N$, and inverse for $p = N, d = 1$. In the transparency window,

$$\omega_j d \langle |a_j|^2 \rangle / dt = \Pi_{j-1} - \Pi_j, \quad (2)$$

due to conservation of E . The right hand side is the discrete divergence of the flux $\Pi_j \simeq \sum_{s,q} V_{sq}^j \omega_j \langle C_3(j, s, q) \rangle$, where $C_3(j, s, q) = \text{Im} \{ a_j^* a_s a_q \}$ and the brackets $\langle \cdot \rangle$ denote time averaging. We also denote by C_m with $m = 3, 4, \dots, N$ generic multiplications of m modes, for which the averages are the correlation functions with no reducible parts (zero for Gaussian statistics). For scale-invariant systems, stationarity requires the flux independence on j , which sets the scaling $a_j^3 \propto \Pi \omega_j^{-1-\alpha}$. Strong turbulence of resonantly interacting waves is different from weak turbulence determined by quasi-resonances, where $a_j^4 \propto \Pi$ [11].

The rest of the article is devoted to the case of $\alpha = 1/2$, as it allows us to explore the difference between statistics of turbulence and of thermal equilibrium. One might expect a minimal difference, since the scaling of turbulence and equipartition coincide, $a_j \propto \omega_j^{-(1+\alpha)/3} = \omega_j^{-1/2}$. Yet, turbulence carries the flux from excess to scarcity, so the energy density $E_k = \omega_k n_k$ is not constant but must decrease from pumping to dissipation. This immediately brings the higher cumulants, beyond C_3 . Indeed, dC_3/dt is expressed via the fourth-order moments, whose cumulants are denoted collectively C_4 . In a steady state,

$$\begin{aligned} d \langle C_3(j, s, q) \rangle / dt &= \sum_{kl} (V_{kl}^q \langle a_j^* a_k a_l a_s \rangle + V_{kl}^s \langle a_j^* a_k a_l a_q \rangle \\ &+ 2V_{is}^k \langle a_j^* a_k a_i^* a_q \rangle + 2V_{lq}^k \langle a_j^* a_k a_i^* a_s \rangle - V_{kl}^j \langle a_i^* a_k^* a_s a_q \rangle \\ &- V_{jl}^k \langle a_k^* a_i a_s a_q \rangle) = \langle C_4 \rangle + 2V_{sq}^j (n_s n_q - n_j n_s - n_j n_q) = 0. \end{aligned} \quad (3)$$

In equilibrium, $n_k \propto \omega_k^{-1}$ and $\langle C_4 \rangle = -2V_{sq}^j (n_s n_q - n_j n_s - n_j n_q) \propto -V_{sq}^j (\omega_j - \omega_q - \omega_s) = 0$. The scaling of turbulence is the same, so the decrease of $\omega_k n_k$ towards the damping must be logarithmic [11]. For a direct cascade, we denote $\Lambda_k \equiv \log(\omega_d / \omega_k)$ and assume $\omega_k n_k = \Lambda_k^\xi$, with ξ to be determined. Considering $j - s \simeq j - q \ll j$ we expand n_s, n_q around n_j (see the Supplement for the details), which gives: $\langle C_4(j) \rangle \approx 2\xi V_{sq}^j n_j^2 / \Lambda_j \propto \Lambda_j^{2\xi-1}$ for $\Lambda_j \gg 1$. Stationarity of C_4, C_5 give respectively $\langle C_5 \rangle \simeq V n \langle C_3 \rangle$, $\langle C_6 \rangle \simeq V n \langle C_4 \rangle \simeq V n^3 / \Lambda$, and generally $\langle C_m(j) \rangle \simeq V n_j^{m/2} / \Lambda \propto \omega_j^{-m/2} \Lambda^{\xi m/2-1}$. The flux law requires that C_3 has no logarithm, which gives $\xi = 2/3$. We thus see that the dimensionless cumulants of all orders are suppressed by the same factor: $\langle C_m(j) \rangle n_j^{-m/2} \simeq \log^{-1}(\omega_d / \omega_j)$. Cumulants are of order unity at $j \simeq d$ [21]. As $d = N$ increases, the cumulants at $j \ll d$ are getting uniformly smaller but their number increases, since the cumulants of orders up to $m \simeq d - j$

involve the mode j .

COMPUTATION OF CUMULANTS

We now present a direct computation for two models, defined respectively by the Hamiltonians [20, 21]:

$$\mathcal{H}_2 = \sum_j 2^{j\alpha} (a_j^2 a_{j+1}^* + a_j^{*2} a_{j+1}), \quad (4)$$

$$\mathcal{H}_3 = \sum_j F_j^\alpha (a_j^* a_{j+1}^* a_{j+2} + a_j a_{j+1} a_{j+2}^*). \quad (5)$$

The first (doubling) model describes a chain of modes resonantly interacting with their second harmonics. The second (Fibonacci) model describes a chain of resonantly interacting triplets; here F_j are the Fibonacci numbers, defined by $F_j + F_{j+1} = F_{j+2}$. The Hamiltonians are invariant with respect to the gauge transformation with $\omega_j = 2^j$ and $\omega_j = F_j$ respectively. Since $F_j = [\phi^j - (-\phi)^{-j}]/\sqrt{5}$, then at $j \gg 1$, the wave frequency also depends exponentially on the mode number, $F_j \propto \phi^j$, where $\phi = (1 + \sqrt{5})/2$ is the golden mean. Both (4,5) are so-called shell models defined in logarithmically discretized Fourier space [20–23].

To study turbulence, we pass to $b_j = a_j \omega_j^{(1+\alpha)/3}$, which makes the triple moment j -independent in the transparency window. We also renormalize time to bring the dynamical equations to the universal form (see numerical factors in the Supplement):

$$i\dot{b}_j = 2^j \frac{2\alpha-1}{3} (b_{j-1}^2 + b_j^* b_{j+1}). \quad (6)$$

$$i\dot{b}_j = \phi^j \frac{2\alpha-1}{3} [\phi b_{j-1} b_{j-2} + b_{j-1}^* b_{j+1} + \phi^{-1} b_{j+1}^* b_{j+2}]. \quad (7)$$

Remarkably, both equations have an exact stationary solution $b_j = \iota, \forall j$. It corresponds to the turbulent scaling $a_j \propto \omega_j^{-(1+\alpha)/3}$. The solution is linearly unstable and cannot be matched with pumping and damping, yet we find that turbulence fluctuates around it. The prefactors $\omega_j^{(2\alpha-1)/3}$ ($= 2^{j(2\alpha-1)/3}$, $\phi^{j(2\alpha-1)/3}$) determine the dependence of the interaction time on the mode number j . For $\alpha = 1/2$, the equations (6,7) are translation invariant along j , the interaction time is the same for all modes, and scaling laws of turbulence and equilibrium coincide. Both direct and inverse cascades exist in this case [20, 21].

Analytic derivations of cumulants for (4) are presented here and for (5) in the Supplement. Consider the products $C_{kl\bar{m}\dots}(j) = b_{j-k} b_{j-l} b_{j-m}^* \dots b_j^*$, having non-zero mean values when gauge invariant: $2^{-k} + 2^{-l} - 2^{-m} + \dots - 1 = 0$. Every such product of order m can be obtained from that of $m-1$ replacing b_j by either b_{j-1}^2 or $b_j^* b_{j+1}$. Denote $A_m(j) = \langle |b_j|^m \rangle$. Stationarity of A_2

similar to (2) gives $\langle C_3 \rangle = \langle C_{1\bar{1}\bar{0}}(j) \rangle = 1$. Next,

$$\begin{aligned} d\langle b_j^2 b_{j+1}^* \rangle / dt &= 2\langle |b_j b_{j+1}|^2 \rangle - \langle |b_j|^4 \rangle + 2\langle C_{22\bar{1}\bar{0}}(j+1) \rangle \\ &- \langle C_{22\bar{1}\bar{0}}(j+2) \rangle = 2n_j(n_{j+1} - n_j) + \langle \langle C_4(j) \rangle \rangle = 0, \end{aligned} \quad (8)$$

where we denoted the combination of the fourth-order cumulants: $\langle \langle C_4 \rangle \rangle = \langle C_{22\bar{1}\bar{0}} \rangle + \langle \langle 2|b_j b_{j+1}|^2 - |b_j|^4 \rangle \rangle$. Double brackets denote subtraction of the reducible parts. Assuming self-consistently that the statistics is close to Gaussian and $n_j \propto \log^\xi(2^{j-d}) \propto |j-d|^\xi$, we obtain $\langle \langle C_4(j) \rangle \rangle \approx 2n_j(n_j - n_{j+1}) \approx 2\xi n_j^2 / |j-d|$. Numerics described below confirm that. Next, stationarity of $\langle |b_j|^4 \rangle$ gives $\langle |b_j|^2 C_3(j) \rangle = \langle |b_j|^2 C_3(j+1) \rangle$. Since the reducible contributions are respectively 3 and 2 times $n_j \langle C_3 \rangle$, then the difference of the cumulants is $\langle \langle |b_j|^2 [C_3(j) - C_3(j+1)] \rangle \rangle = A_2(j) \langle C_3 \rangle = n_j$, that is the fifth-order cumulants are comparable to n_j . Numerics give $\langle |b_j|^2 C_3(j) \rangle = \langle |b_j|^2 C_3(j+1) \rangle \approx 2.25n_j$. Stationarity of $\langle |b_j b_{j+k}^*|^2 \rangle$ by induction over k imposes the identities: $\langle |b_j|^2 [C_3(j+k+1) - C_3(j-k)] \rangle = 0$, the cumulants are negligible for $k > 1$ when there is no overlap (see Supplement). Next orders give $\langle \langle C_6 \rangle \rangle \simeq A_2 \langle \langle C_4 \rangle \rangle \simeq A_2 A_4 / |j-d| \simeq A_6 / |j-d|$ and $\langle \langle C_m \rangle \rangle \simeq A_m / |j-d|$, supporting the above estimates in the general case. Since $A_m \propto n_j^{m/2}$, then $\langle \langle C_m \rangle \rangle = n_j^{m/2} / |j-d| \propto |j-d|^{m\xi-1}$. Applying it to $\langle C_3 \rangle = 1$ we obtain $\xi = 2/3$, that is $n_j \propto |j-d|^{2/3}$, $A_m(j) \simeq n_j^{m/2} \propto |j-d|^{m/3}$ and $\langle \langle C_m(j) \rangle \rangle \simeq A_m(j) / |j-d| \propto |j-d|^{(m-3)/3}$. We see that cumulants with $m > 3$ actually grow with increasing the length of the cascade. The dimensionless cumulants $D_m = \langle \langle C_m \rangle \rangle / \langle |C_m^2| \rangle^{1/2}$ all decay by the same law:

$$D_{kl\bar{s}\dots}(j) \equiv \frac{\langle \langle b_{j-k} b_{j-l} b_{j-s}^* \dots b_j^* \rangle \rangle}{\langle |b_{j-k} b_{j-l} b_{j-s}^* \dots b_j^*|^2 \rangle^{1/2}} \simeq \frac{1}{|j-d|}. \quad (9)$$

NUMERICAL RESULTS

To check the assumptions in our derivations, we solved numerically (6,7) for $\alpha = 1/2$ and both direct and inverse cascades. Exponentially large ω_i do not appear for the variables b_i , which allows us to enlarge N without shortening the time step. Compared with [20–22], this novel approach gives much larger transparency window, up to a record $N = 200$. Figure 1 shows that the single-mode probability distribution is close to Gaussian for all modes in the transparency window, see also [20, 21].

It also shows the dependence of the single-mode moments $A_m(j)$ on j counted from dissipation. The occupation numbers are of order unity at $j \simeq d$; away from dissipation, the moments are indeed Gaussian, that is the renormalized moments $A_m(j) A_2^{-m/2}(j) / \Gamma(1 + m/2)$ are all equal to unity, independent of j . That means that the single-mode distribution is scale invariant, the scaling is indeed $A_m(j) \propto |j-d|^{m/3}$. The same is true for the

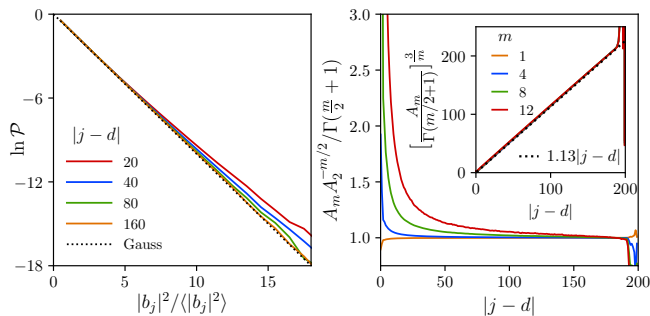


FIG. 1. Direct Fibonacci cascade. Left: away from damping, single-mode distribution approaches Gaussian. Right: single-mode moments $A_m = \langle |b_j|^m \rangle$.

doubling model.

Both large transparency window and massive statistics are crucial for the first-ever computation of high-order cumulants in turbulence presented in Figure 2. Due to symmetry $b_j \rightarrow -b_j^*$, the cumulants with even m are real and those with the odd m are imaginary (they are non-zero due the breakdown of time reversibility in turbulence). The cumulants of $b'_j = ib_j$ are all real and proportional to the flux, that is have opposite signs for direct and inverse cascades (see also Supplement). Figure 2 shows dimensionless cumulants multiplied by $|j-d|$. The data confirm the scaling (9) for all pure cumulants, like $\langle C_{2210} \rangle$, and those comparable to the reducible part, like $\langle |b_j|^2 C_3(j) \rangle$. The few cumulants due to amplitude-only correlations, like $\langle |b_j b_{j-1}|^2 \rangle$, are small differences of large values; the data are irregular, don't contribute the mutual information, and left out of the consideration.

Numerics give the squared flux $\langle C_3^2 \rangle = \langle (b_{j-1}^* b_j)^2 \rangle \approx 2.44 n_j^3 / |j-d|$ for (4). Two things are noteworthy. First, $\langle C_3^2 \rangle \propto |j-d| \gg \langle C_3 \rangle^2 = 1$, that is the flux fluctuates strongly, its variance grows unbounded with N , while its mean stays constant. The same is true for (5), and reproduces the experimental and numerical data on direct and inverse cascades in fluid turbulence [24]. Second, for both models, we find no systematic decay of cumulants with m . We cannot presently prove this for $m \rightarrow \infty$, nor we are able to classify all gauge-invariant cumulants of an arbitrary order. We managed it up to the seventh order: (4) gives 26 distinct types of cumulants and (5) gives 70 for $m = 7$ (see the tables in the Supplement). For example, cumulants of the type $\langle \langle C_3 C_4 \rangle \rangle$ are among the largest and are comparable to $\langle C_4 \rangle \langle C_3 \rangle = \langle C_4 \rangle$ — another sign of no decay with m (see also the Table).

To summarize the findings: the dimensionless cumulants are uniformly small far from dissipation: $D_m(j) \simeq |j-d|^{-1}$. In the thermodynamic limit $d = N \rightarrow \infty$ (direct cascade), all dimensionless cumulants tend to zero and we have an asymptotic equipartition with the temperature $T = N^{2/3}$ for all finite j . Yet the number of cumulants grow with N , so it is not clear if the full multi-

mode statistics approaches Gaussian in the limit.

ENTROPIC CONSIDERATION

Let us argue that the growth of cumulants makes the full multi-mode probability distribution $\mathcal{P}\{b_j\}$ very different from the Gaussian distribution of independent modes with the same occupation numbers, $\mathcal{P}_G\{b_j\} = \prod_j n_j^{-1} \exp(-|b_j|^2/n_j)$. Since the dimensionless cumu-

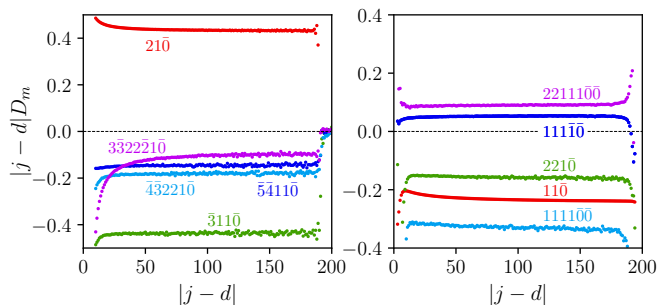


FIG. 2. Largest compensated cumulants for every order. Left: Direct cascade in the Fibonacci model. Right: Inverse cascade in the doubling model.

lants are small, the probability distribution can be approximated as follows:

$$\mathcal{P}\{b_j\} = Z^{-1} \exp \left[-\sum_{j=1}^N \alpha_j |b_j|^2 + B \right], \quad (10)$$

$$Z \approx (1 + \langle B^2 \rangle_0 / 2) \prod_j \alpha_j^{-1}. \quad (11)$$

The average $\langle \dots \rangle_0$ is with \mathcal{P}_G , and B is the sum of all products giving nonzero cumulants, symbolically

$$B = \sum_m \frac{C_m \langle C_m^* \rangle + C_m^* \langle C_m \rangle}{\langle |C_m|^2 \rangle}. \quad (12)$$

Define $S_N = \langle \ln 1/\mathcal{P} \rangle$ as the total entropy. Relative entropy, $\mathcal{D}(\mathcal{P}|\mathcal{P}_G) = \langle \ln(\mathcal{P}/\mathcal{P}_G) \rangle$ measures the difference between \mathcal{P} and \mathcal{P}_G . Since the latter is a product, $\mathcal{P}_G \propto \prod_j \exp(-\alpha_j |b_j|^2)$, then $\mathcal{D} = \sum_{j=1}^N S_j - S_N$ is also the difference of the entropies, that is the multi-mode mutual information I_N , quantifying correlations in the system.

In the leading order in $1/|j-d|$, \mathcal{D} is the sum of all squared dimensionless cumulants. If we had only triple cumulant, then \mathcal{D} would be given by a converging series and thus independent of the number of modes: $\mathcal{D} = \sum_{j=3}^N D_3^2(j)/4 \simeq \sum_{j=3}^N (N-j)^{-2} \simeq 1$. The contributions of multi-mode cumulants makes \mathcal{D} a double sum. Our computations suggest that the sum of squared dimensionless cumulants do not decrease with the order: Table I shows quite irregular dependence on m , but no overall decay: contributions of $m = 4$ is comparable to $m = 6$, while $m = 5$ to that of $m = 7$. This is all the more remarkable since only nearest neighbors inter-

act. Assuming that indeed the sum of squared cumulants do not decay with the order and asymptotically saturates to some number (which we cannot yet compute) $c = \lim_{m \rightarrow \infty} |D_m(j)|^2 |j - d|^2 / 4$, we obtain the relative entropy logarithmic in the number of modes:

$$\mathcal{D} = \sum_{m=3}^N \sum_{j=m}^N \frac{|D_m(j)|^2}{4} = c \sum_{m=3}^N \sum_{j=m}^N \frac{1}{j^2} = c \log N. \quad (13)$$

Since we disregard amplitude-only correlations, the conclusion is about $\mathcal{D}(\mathcal{P}\{b_j\}|\mathcal{P}\{|b_j|\})$ which bounds $\mathcal{D}(\mathcal{P}|\mathcal{P}_G)$ from below. The mutual information between the two parts of the cascade, $I(A, B) = S(A) + S(B) - S(A, B)$, must have the same logarithmic dependence, since it is given by the double sum over j, m of the squared cumulants that involve modes from both $A = \{p \dots N/2\}$ and $B = \{1 + N/2 \dots d\}$. The longer is the transparency window, the smaller are the cumulants, yet larger is their number, so that the mutual information grows logarithmically with the number of modes excited, that is with the effective Reynolds number. The reason for the growth here is the nonlocality of correlations. Another source of a logarithmic dependence of the mutual information on the degree of non-equilibrium was found for a minimal (two-mode) model of a cascade [25]. It is a task for the future to derive the logarithmic law with the factor c for different models, which requires solving a formidable problem of classification of cumulants of an arbitrary order. After that, we may be able distinguish universality classes of turbulent cascades.

DISCUSSION

The next step will be to study multi-mode correlations for $\alpha \neq 1/2$, when non-Gaussianity of a single mode grows exponentially along the cascade: $A_{2m}/A_2^m \propto \omega_{j-p}^{\Delta_{2m}}$ [20–22]. Preliminary (short-interval) data show a linear growth with $|j - p|$ for $I(b_{j-1}, b_j)$ and $I(b_{j-1}, b_j, b_{j+1})$ [20, 21], which bound $I(b_1, \dots, b_N)$ from below due to monotonicity. It is tempting to treat $\alpha = 1/2$ logarithmic case as that of critical phenomena in dimension 4 [26] and develop a Wilson-type ϵ -expansion in $\epsilon = \alpha - 1/2$ [27]. That could be non-trivial, since the perturbation expansions, regular in thermal equilibrium, tend to be singular in non-equilibrium states, especially in turbulence [7, 8, 16].

Note the dramatic difference between our turbulence and nonuniform dilute gases described in [7–9]. There, even though naive expansion encounters divergencies, renormalized expansion gives higher cumulants proportional to higher powers of the small parameter (density), which leaves the multi-particle mutual information small. Our (9) gives all cumulants proportional to the same (first) power of the small parameter, which may lead to logarithmically large multi-mode mutual information.

m	\mathcal{H}_3 direct	\mathcal{H}_3 inverse	\mathcal{H}_2 direct	\mathcal{H}_2 inverse
3	0.17986	0.18992	0.06522	0.05660
4	0.22636	0.22748	0.02818	0.02558
5	0.07100	0.07102	0.00600	0.00356
6	0.11829	0.12089	0.19594	0.17387
7	0.06500	0.06670	0.02553	0.02333

TABLE I. Sum of squared dimensionless cumulants, $D_m^2(j)|j - d|^2$, in the transparency window for both models.

Our models deal only with resonant modes and respective cumulants. In a general wave turbulence, both resonant and non-resonant interactions are present, so the analysis of multi-mode correlations will be more complicated. It was argued in [16] that cumulants might be substantial for resonant modes in the weak turbulence, even when the statistics of mode amplitudes is close to Gaussian. It is likely that in a continuous limit, determined by quasi-resonances rather than resonances, the higher cumulants are proportional to higher powers of the small parameter, and the mutual information is small. As far as real-world fluid turbulence is concerned, we believe that our work shows importance of measuring multi-mode correlations and the entropy of multi-mode distributions. While only treatment of a moderate number of modes is feasible, it may provide an important insight.

CONCLUSION

We have shown that turbulent cascade necessary involves multi-mode correlations even in systems with local interaction and close to equilibrium. The entropy-lowering information about turbulence is encoded in the overlapping sets of multi-mode correlations. As far as we were able to compute, the degree of correlation does not decay with the number of modes. If true, multi-mode cumulants would make the multi-mode statistics very different from Gaussian; the logarithmic growth of the relative entropy with the number of modes establishes a promising analogy with critical phenomena. It remains to be seen how universal are multi-mode correlations across classes of turbulent systems and whether they can be related to coherent structures.

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