Inclusion statistics and particle condensation in two dimensions

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Inclusion statistics and particle condensation in 2 dimensions

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Abstract

We propose a new type of quantum statistics, which we call inclusion statistics, in which particles tend to coalesce more than ordinary bosons. Inclusion statistics is defined in analogy with exclusion statistics, in which statistical exclusion is stronger than in Fermi statistics, but now extrapolating beyond Bose statistics, resulting in statistical inclusion. A consequence of inclusion statistics is that the lowest space dimension in which particles can condense in the absence of potentials is $d = 2$, unlike $d = 3$ for the usual Bose-Einstein condensation. This reduction in the dimension happens for any inclusion stronger than bosons, and the critical temperature increases with stronger inclusion. Possible physical realizations of inclusion statistics involving attractive interactions between bosons may be experimentally achievable.

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1 Introduction

Bose-Einstein condensation is a highly nontrivial manifestation of quantum statistics: bosons condense in 3 dimensions when they are cooled below a critical temperature, whereas fermions never condense due to Pauli exclusion. A spectacular experimental confirmation [1, 2] of this stunning theoretical prediction has been achieved in 1995, and has since been reproduced by many groups. A salient feature of Bose-Einstein condensation is that, in the absence of external potentials, it is possible only in dimension $d \geq 3$.

On the other hand, and on another front, it is known [3] that in 2 dimensions intermediate statistics between Bose and Fermi statistics are possible due to the nontrivial topology of the configuration space of identical particles, i.e., to the topologically nontrivial braiding of spacetime paths of particles going around each other. It follows that the quantum N-body wavefunction acquires a multivalued phase, which, when gauged away, results in particles with ordinary Bose statistics but interacting through a two-body Aharonov-Bohm-type term, meaning that particles are endowed with a fictitious charge e and a fictitious flux ϕ. The statistical parameter is $g = \phi/\phi_o$ where the flux quantum is $\phi_o = h/e$. This is the famous anyon model that interpolates from Bose statistics ($g = 0$) to Fermi statistics ($g = 1$), and, by periodicity of the phase, returns to Bose statistics when $g = 2$. Interestingly, the spin and statistics of anyons satisfy a generalization of the spin-statistics theorem for fractional values of spin, namely $g = 2s$, which goes over to the standard spin-statistics relation for $s =$ integer (bosons) and $s =$ half-integer (fermions) (for a review of the spin-statistics theorem in quantum field theory see [4]).

A major hurdle in the anyon model is that the N-body problem is not solvable except for a class of exact eigenstates, the so-called linear states, which manifest themselves when a constant magnetic field and/or a harmonic potential are added. To make progress, a simplification is needed: one can add a magnetic field perpendicular to the plane, couple the anyons to it, and project the system on its lowest Landau level (LLL). The LLL-anyon model happens to be solvable, since there is a complete set of LLL-anyon states (the linear states) that interpolate between the LLL-Bose and LLL-Fermi basis states. Then, from the LLL-anyon spectrum, one can readily get the LLL-anyon thermodynamics [5], and as a consequence, the LLL-anyon occupation number, that is, the number of anyons per available one-body state, which in this degenerate case is nothing but the filling factor in the LLL.

Furthermore, aside from the microscopic anyonic formulation and space dimensionality considerations, an interesting feature of the LLL-anyon thermodynamics is that it allows for a combinatorial reformulation in terms of the occupation of single-particle quantum states, which leads to the so-called Haldane exclusion statistics [6]. Indeed, the LLL-anyons thermodynamics turn out to be identical to those derived from Haldane statistics [7], defined purely in terms of Hilbert space counting arguments. This combinatorial reformulation extends to positive values of the exclusion parameter g beyond the periodicity range $g \in [0, 2[$, i.e., $g \geq 2$. This results to an interparticle exclusion stronger than the standard Pauli exclusion, justifying the term exclusion statistics.

In this note we propose, instead of going beyond Fermi statistics, to go beyond Bose statistics by considering negative values of the exclusion parameter, i.e., $g < 0$. To do this, we take the combinatorial reformulation at face value and analytically continue it to negative values of g. This leads to particles for which the degeneracy of states *increases* when they occupy neighboring states, and therefore have a propensity to include, rather than exclude each other. We thus call this extension of quantum statistics *inclusion* statistics.

In the sequel we will examine the properties of particles obeying inclusion statistics and derive their thermodynamics. We will see that an interesting physical consequence of these statistics is a lowering of the lowest dimension at which particles can condense compared to the Bose case from $d = 3$ to $d = 2$. We leave to a separate publication [8] the technical details pertaining to inclusion statistics, and in particular the case of a discrete one-body spectrum.

2 g-exclusion thermodynamics

Following Haldane's Hilbert space counting argument[1], exclusion statistics of order $g > 0$ can be formulated by postulating that the degeneracy G of N particles occupying K degenerate states is

$$G_g(K, N) = \frac{[K - (g-1)(N-1)]!}{N![K - g(N-1) - 1]!}$$

Clearly $g = 0$ reproduces the bosonic result $\binom{K+N-1}{N}$ and $g = 1$ the fermionic one $\binom{K}{N}$. For integer $g \geq 0$ this can be interpreted as particles placed on a linear one-body spectrum with the constraint that no more than 1 particle can occupy any set of g adjacent states. In this sense, each particle "excludes" other particles from occupying its nearby states. An alternative definion of the degeneracy can be adopted [9]

$$G_g(K, N) = \frac{K[K - (g-1)N - 1]!}{N!(K - gN)!} \tag{1}$$

which corresponds to placing particles in a periodic spectrum, i.e., on a circle. The two definitions are equivalent in the $K \gg 1$ limit and lead to the same thermodynamics, but the second one is more convenient for deriving this limit. Note that for integer $g > 0$ there is a finite number of particles $\lfloor K/g \rfloor$ that can be placed in K states, beyond which $G_g(K, N)$ vanishes, which implies the upper bound for the filling fraction $N/K \leq 1/g$.

The grand partition function for exclusion-g particles in K states of energy ϵ, at inverse temperature β and chemical potential μ

$$\mathcal{Z}(K, z) = \sum_{N=0}^{\infty} G_g(K, N) z^N \ , \quad z = e^{\beta(\mu - \epsilon)}$$

becomes extensive in the thermodynamic limit $K \gg 1$ and acquires the form

$$\ln \mathcal{Z} = K \ln y + \mathcal{O}(K^{-1})$$

(in fact, for the circular counting (1) the perturbative corrections $\mathcal{O}(K^{-1})$ vanish and there are only nonperturbative corrections of order $\mathcal{O}(e^{-K})$). The function y can, thus, be interpreted as an effective grand partition function for g-exclusion particles at chemical potential μ on a single state with energy ϵ. It satisfies the equation

$$y^g - y^{g-1} = z \tag{2}$$

[1]As stated in the introduction the g-exclusion thermodynamics can be as well directly obtained from the microscopic LLL-anyon model.

The physically relevant solution of this equation is for $y > 1$. The single-state cluster coefficients c_n, defined by

$$\ln y = \sum_{n=1}^{\infty} c_n z^n$$

are derived to be

$$c_n = \frac{1}{n!} \prod_{k=1}^{n-1} (k - ng) = \frac{(-1)^{n-1}}{ng} \binom{ng}{n}$$

the second expression valid for $g \geq 1$. The c_n never vanish for integer $g \geq 0$ and alternate in sign for $g \geq 1$.

All this generalizes to the grand partition function Z for particles with g-exclusion and 1-body density of states $\rho(\epsilon)$

$$\ln Z = \int_0^{\infty} \rho(\epsilon) \ln y \, d\epsilon$$

where we assumed without loss of generality that $\rho(\epsilon)$ starts at $\epsilon = 0$. The occupation number at energy ϵ follows from

$$n = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln y = z \frac{\partial}{\partial z} \ln y$$

which, in combination with (2), implies

$$n = \frac{y - 1}{gy + 1 - g} \; , \quad z = \frac{n}{(1 + (1-g)n)^{1-g} (1 - gn)^g} \tag{3}$$

This gives the grand potential as

$$\ln Z = \beta PV = \int_0^{\infty} \rho(\epsilon) \ln \left(1 + \frac{n}{1 - gn} \right) d\epsilon$$

which determines the equation of state and reaffirms that there is a maximum critical occupation number

$$n \leq 1/g$$

For bosons ($g = 0$) there is no maximum, while for fermions ($g = 1$) we have the standard Pauli exclusion principle with at most one fermion per quantum state. For $g > 1$ we have that the critical occupation number should be smaller than 1, corresponding to exclusion.

Other thermodynamic quantities, such as the particle number N, energy E, and entropy S are given by

$$N = \int_0^{\infty} \rho(\epsilon) \, n \, d\epsilon \; , \qquad E = \int_0^{\infty} \rho(\epsilon) \, n \, \epsilon \, d\epsilon$$

$$S = \ln Z + \beta E - \beta \mu N = \int_0^{\infty} \rho(\epsilon) \ln \frac{(1 + n(1-g))^{1+n(1-g)}}{(1 - ng)^{1-ng} \; n^n} d\epsilon$$

3 Inclusion statistics and particle condensation

As we stated in the introduction, we define inclusion statistics by taking $g < 0$ in the expressions of section (2). The combinatorial formulae for the number of states, expression for cluster coefficients, and equation for the single-level grand partition function remain the same, understanding that g is negative.

Noting that the equation for y can be rewritten as

$$\left(\frac{1}{y}\right)^{1-g} - \left(\frac{1}{y}\right)^{(1-g)-1} = -z$$

we deduce the relation between g-statistics and $(1-g)$-statistics

$$y(z, 1-g) = \frac{1}{y(-z, g)} \tag{4}$$

Therefore, inclusion statistics of order $g < 0$ are related to exclusion statistics of order $1-g > 1$ upon changing the sign of the grand potential and of the fugacity. In particular, the cluster coefficients c_n remain the same but now become all positive:

$$c_n(g) = (-1)^{n-1} c_n(1-g) > 0 \quad \text{for} \quad g < 0$$

The crucial new feature of the $g < 0$ case is that the branch of the equation corresponding to $z > 0$ admits real solutions only for z below some maximal value, $z < z_{max}$. Indeed, the left-hand-side $y^g - y^{g-1} = z$ of (2), viewed as a function of y, vanishes for $y = 1$ (as it should) and has a critical point at $y = y_c$ with

$$y_c = \frac{g-1}{g}$$

For $g < 0$ this occurs at a physical value of $y > 1$ and corresponds to a maximum, at which z takes its maximum value z_{max}

$$z_{max} = \frac{(-g)^{-g}}{(1-g)^{1-g}} > 0$$

Therefore, for $g < 0$ the solution $y = 1 + z + \cdots > 1$ exists as long as $0 < z < z_{max}$. For each such z, (2) has two solutions, but only the smaller one, $y < y_c$, reached from $y = 1$ and $z = 0$ by increasing z, is physical.

This is already enough to demonstrate that a system of inclusion particles will undergo condensation at a nonzero temperature. Near z_{max}

$$z - z_{max} \simeq -C(y - y_c)^2 \quad \Rightarrow \quad y \simeq y_c - \sqrt{\frac{z_{max} - z}{C}}$$

with

$$C = \frac{(-g)^{3-g}}{2(1-g)^{2-g}} > 0$$

and the average particle number $n = z\frac{\partial}{\partial z}\ln y$

$$n \simeq \frac{z_{max}}{2y_c\sqrt{C(z_{max}-z)}}$$

The maximal number of particles that the system can accommodate under a normal thermodynamic distribution (without condensation) is achieved for $\mu = \mu_{max}$ at $\epsilon = 0$, that is

$$z_{max} = e^{\beta\mu_{max}} \quad \Rightarrow \quad \mu_{max} = \frac{1}{\beta}\ln z_{max}$$

So, if

$$N_{max} := \int_0^\infty n(\epsilon)|_{\mu=\mu_{max}}\,\rho(\epsilon)d\epsilon < \infty \tag{5}$$

the system will undergo particle condensation when the actual number of particles $N > N_{max}$. It follows that condensation will occur if the integral (5) does not diverge. The integral cannot diverge as $\epsilon \to \infty$ since $n(\epsilon)|_{\mu=\mu_{max}}$ goes to zero exponentially in that limit, so it can only diverge as $\epsilon \to 0$, that is, $z \to z_{max}$. For $\mu = \mu_{max}$, $z = z_{max}e^{-\beta\epsilon}$ and

$$n \simeq \frac{\sqrt{z_{max}}}{2y_c\sqrt{C(1-e^{-\beta\epsilon})}}$$

and thus, for $\epsilon \to 0$

$$n \sim \epsilon^{-1/2}$$

This has to be contrasted to the behavior of n for bosons ($g = 0$) which is $n \sim \epsilon^{-1}$. The weaker dependence $n \sim \epsilon^{-1/2}$ sets in as soon as $g < 0$ and implies a change in the critical dimension in which condensation happens. Assuming the standard density of states

$$\rho(\epsilon) = V\frac{c_d(2m)^{d/2}}{2h^d}\epsilon^{d/2-1}$$

for free nonrelativistic particles in a volume V in d dimensions, with c_d is the d-dimensional spherical factor $c_d = 2\pi^{d/2}/\Gamma(d/2)$, we see that the integrand in (5) for N_{max} near $\epsilon = 0$ behaves as $\sim \epsilon^{(d-3)/2}$ and the integral will be finite as long as

$$\frac{d-3}{2} > -1 \quad \Rightarrow \quad d > 1$$

So we will have particle condensation at $d = 2$, i.e., one dimension lower than for standard Bose-Einstein condensation.

4 Critical temperature for g-inclusion particle condensation

To set the stage, let us first review the usual $g = 0$ Bose-Einstein condensation where $\mu_{max} = 0 \;\Rightarrow\; z_{max} = 1$. The occupation number for $\mu = \mu_{max} = 0$ is

$$n = \frac{1}{\mathrm{e}^{\beta\epsilon} - 1}$$

so

$$\begin{aligned} N_{max} = \int_0^\infty n(\epsilon)\rho(\epsilon)d\epsilon = V\frac{c_d(2m)^{d/2}}{2h^d}\int_0^\infty \frac{\epsilon^{d/2-1}}{\mathrm{e}^{\beta\epsilon}-1}d\epsilon &= \infty \qquad \text{if } d \le 2 \\ &= \frac{V}{\lambda^d}\zeta(d/2) \qquad \text{if } d > 2 \end{aligned}$$

where we introduced the thermal wavelength $\lambda = \frac{h}{\sqrt{2\pi mkT}}$ and used

$$\int_0^\infty \frac{u^{d/2-1}}{\mathrm{e}^u - 1}\,du = \Gamma(d/2)\mathrm{Li}_{d/2}(1) = \Gamma(d/2)\zeta(d/2)$$

The critical temperature T_c (for $d > 2$) is defined as the temperature at which $N = N_{max}$, that is,

$$N = \frac{V}{\lambda_c^d}\zeta(d/2)$$

which gives

$$T_c = \frac{h^2}{2\pi mk}\left(\frac{\rho}{\zeta(d/2)}\right)^{2/d} \tag{6}$$

where $\rho = N/V$ is the boson density. Clearly, if

$$T \le T_c \quad \Rightarrow \quad N_{max} = \frac{V}{\lambda^d}\zeta(d/2) \le N$$

So $N_{cond} = N - N_{max}$ particles will condense in the ground state and there is a macroscopic Bose-Einstein condensate of a fraction of particles N_{cond}/N

$$\frac{N_{cond}}{N} = 1 - \frac{V}{N\lambda^d}\zeta(d/2) = 1 - \left(\frac{T}{T_c}\right)^{d/2}$$

Let us now turn to g-inclusion ($g < 0$). At the maximal fugacity $z_{max} = (-g)^{-g}(1-g)^{g-1}$ (3) gives

$$\mathrm{e}^{\beta\epsilon} = \frac{(-\frac{1}{g}+n)^g(\frac{1}{1-g}+n)^{1-g}}{n} = \left(\frac{1}{ng} - 1\right)^g\left(\frac{1}{n(1-g)} + 1\right)^{1-g}$$

Performing a change of integration variable in the integral $N = \int_0^\infty n(\epsilon)\rho(\epsilon)d\epsilon$ from ϵ to $t = 1/n$

$$\beta\epsilon = g\ln\left(1 - \frac{t}{g}\right) + (1-g)\ln\left(1 + \frac{t}{1-g}\right)$$

$$\beta d\epsilon = \frac{t}{(t-g)(t+1-g)}dt$$

we obtain

$$N_{max} = \frac{V}{\lambda^d}\tilde{\zeta}(g, d/2)$$

where we defined

$$\tilde{\zeta}(g,s) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, \frac{\left[g\ln\left(1-\frac{t}{g}\right) + (1-g)\ln\left(1+\frac{t}{1-g}\right)\right]^{s-1}}{(t-g)(t+1-g)} \tag{7}$$

For $g < 0$, the integral $\tilde{\zeta}(g, d/2)$ is always finite as $t \to \infty$, but the integrand behaves as $\sim (t^2)^{d/2-1} = t^{d-2}$ as $t \to 0$, so it will be finite as long as $d > 1$, as found in section (3). From the above, the critical temperature for particle condensation in $d > 1$ is derived as

$$T_c = \frac{h^2}{2\pi mk}\left(\frac{\rho}{\tilde{\zeta}(g, d/2)}\right)^{2/d} \tag{8}$$

Note that for $g > 0$, for which there is no condensation, the integrand in $\tilde{\zeta}(g, d/2)$ develops poles within the integration domain.

The function $\tilde{\zeta}(g, s)$ defined in (7) (not to be confused with the Hurwitz zeta function $\zeta(s, a)$) is a generalization of Riemann's zeta function, reducing for $g = 0$ to the standard zeta function $\zeta(s)$

$$\tilde{\zeta}(0,s) = \frac{1}{\Gamma(s)}\int_0^\infty dt\frac{[\ln(t+1)]^{s-1}}{t(t+1)} = \zeta(s)$$

This also recovers the bosonic limit $g = 0$ as

$$\int_0^\infty n(\epsilon)\rho(\epsilon)d\epsilon = \frac{V}{\lambda^d}\zeta(0, d/2) = \frac{V}{\lambda^d}\zeta(d/2) \quad \text{if } d > 2$$

$\tilde{\zeta}(g, d/2)$ can actually be explicitly evaluated for even dimensions $d = 2, 4, \ldots$ in terms of polylogarithms. Specifically,

$$\tilde{\zeta}(g, 2/2) = -g\mathrm{Li}_1\big(\frac{1}{g}\big) + (1-g)\mathrm{Li}_1\big(\frac{1}{1-g}\big) = \ln\big(1 - \frac{1}{g}\big)$$
$$\tilde{\zeta}(g, 4/2) = -g\mathrm{Li}_2\big(\frac{1}{g}\big) + (1-g)\mathrm{Li}_2\big(\frac{1}{1-g}\big)$$
$$\tilde{\zeta}(g, 6/2) = -g\bigg(\mathrm{Li}_3\big(\frac{1}{g}\big) - (g-1)\mathrm{S}_{1,2}\big(\frac{1}{g}\big)\bigg) - (g \to 1-g)$$

$$\tilde{\zeta}(g,8/2) = -g\left(\mathrm{Li}_4\big(\frac{1}{g}\big) + (g-1)\left(\frac{1}{2}\mathrm{Li}_2\big(\frac{1}{g}\big)^2 - \mathrm{S}_{2,2}\big(\frac{1}{g}\big) - (g-1)\mathrm{S}_{1,3}\big(\frac{1}{g}\big)\right)\right) - (g \to 1-g)$$

$$\tilde{\zeta}(g,10/2) = g^2\left((2g-3)\mathrm{Li}_5\big(\frac{1}{g}\big) + (g-1)\left(\mathrm{Li}_4\big(\frac{1}{g}\big)\mathrm{Li}_1\big(\frac{1}{g}\big) - \mathrm{S}_{3,2}\big(\frac{1}{g}\big) + (g-1)\mathrm{S}_{1,4}\big(\frac{1}{g}\big)\right)\right) - (g \to 1-g)$$

where Li_n and S_{n_1,n_2} stand for the polylogarithm and generalized Nielsen polylogarithm functions respectively. Note that $\tilde{\zeta}(g,2/2) = \ln(1-1/g)$ indicates that in the Bose limit $g \to 0^-$ the integral diverges logarithmically, recovering the fact that Bose-Einstein condensation does not happen in two dimensions only marginally. What happens is that the critical temperature T_c increases as g approaches 0 from below and diverges at $g=0$.

We conclude by giving an alternative expression for $\tilde{\zeta}(g,s)$ among the several rewritings of the original integral (7), with a somewhat simpler integrand[2], changing variable $t=(g-1)gu/(1+gu)$

$$\tilde{\zeta}(g,s) = \frac{1}{\Gamma(s)}\int_0^{-\frac{1}{g}} \frac{[g\ln(1+u) - \ln(1+gu)]^{s-1}}{1+u}\,du \tag{9}$$

This expression has the advantage of being well-defined for all values of g (except in the interval $[0,1]$) and thus provides an analytic continuation of $\tilde{\zeta}(g,d/2)$ to $g>1$. Note also that the change of variables $v=-u/(1+u)$ leads to

$$\tilde{\zeta}(g,s) = -\frac{1}{\Gamma(s)}\int_0^{-\frac{1}{1-g}} \frac{[(1-g)\ln(1+v) - \ln(1+(1-g)v)]^{s-1}}{1+v}\,dv \tag{10}$$

demonstrating that $\tilde{\zeta}(g,s)$ satisfies the relation

$$\tilde{\zeta}(g,s) = -\tilde{\zeta}(1-g,s) \tag{11}$$

This can be explicitly verified in the expressions for $\tilde{\zeta}(g,2/2)$ up to $\tilde{\zeta}(g,10/2)$. This is rooted at the relation (4) between the grand partition functions $y(z,g)$ and $y(-z,1-g)$, and relates the physical sector $y>1$ of $g<0$ inclusion statistics to the unphysical[3] sector $y<1$ of $g>0$. As such, it is not a physically relevant connection, although it may have possible physical implications in the right context.

5 Conclusions

The possibility to observe particle condensation in dimensions lower than 3 is inherently interesting and, if an appropriate realization of inclusion statistics is achieved, experimentally testable. Even in 3 dimensions, inclusion statistics would have observable physical

[2]The same type of manipulation would lead to the integral (7) being as well analytically continued to

$$\tilde{\zeta}(g,s) = -\frac{1}{g}\frac{1}{\Gamma(s)}\int_0^\infty du\,\frac{\left[(1-g)\ln(1+u) + g\ln(1+u(1-\frac{1}{g}))\right]^{s-1}}{(1+u)(1+u(1-\frac{1}{g}))}$$

which is now properly defined for all g except in the interval $[0,1]$ and satisfies to (11).

[3] The analytical continuation above has to be understood in this way. For actual $g>0$ exclusion statistics, as discussed in section (2), no particle condensation occurs at any d.

consequences, as it would raise the critical temperature, other parameters remaining the same[4]. One can compare the critical temperature (8) for ($g < 0$) inclusion condensation to the critical temperature (6) for the usual ($g = 0$) Bose condensation for a same species of atoms at the same density. Denoting by $r(g)$ the ratio of these two temperatures, one gets

$$\begin{aligned} r(-5) &= 8.12894 \\ r(-4) &= 7.10599 \\ r(-3) &= 6.00133 \\ r(-2) &= 4.77881 \\ r(-1) &= 3.35538 \end{aligned}$$

This renders the observation of condensation, which is quite challenging and nontrivial for bosons, substantially easier. For example, already for $g = -1$ inclusion, we obtain a more than threefold increase in the critical condensation temperature compared to that for the usual Bose-Einstein condensation.

The appearance in the critical temperature for inclusion statistics of a generalization of the Riemann zeta function is intriguing and may have some interesting implications (the standard zeta function has actually appeared in a phenomenological description of the FQHE [10]). Also interesting is the appearance of unphysical yet exact symmetries in this thermodynamics. The relation (4) between g and $1 - g$, mapping physical to unphysical sectors of inclusion and exclusion statistics, has already been noted. An additional relation is the duality

$$\frac{1}{y(z,g)} + \frac{1}{y(z^{-1/g}, 1/g)} = 1$$

which maps statistics of the same kind (inclusion or exclusion) but inverts g and rescales either the temperature or the energy. For exclusion statistics ($g > 0$) this mapping inverts the energy scale and constitutes a generalized particle-hole duality (note that for $g = 1$ it becomes a self-duality between ordinary fermions and holes). For $g < 0$, on the other hand, it does not invert the spectrum and can be interpreted as a temperature rescaling. However, it maps the physical ($y < y_c$) sector of one statistics to the unphysical ($y > y_c$) sector of the other, as can be seen, e.g., by talking $z = 0$, in which case $y(0, g) = 1$ maps to $y(0, g^{-1}) = \infty$. So it is also unphysical in this respect. However, physical and "unphysical" sectors of a system quite often map to alternative versions of the system that are not perturbatively related, and the possibility remains that such a connection is also at play with inclusion statistics.

The most important and physically relevant question remains the realization of inclusion statistics in physical systems. It is often the case that nonstandard statistics is a manifestation of interactions between particles with ordinary statistics. The Calogero model is the canonical example, in which particles exhibit properties consistent with

[4]In the sodium experiment [2] the gas is made of 5×10^5 atoms at densities 10^{14} per cm^3 for a critical temperature of $2\mu K$.

free particles obeying generalized statistics [11]. It is plausible that a similar description involving *attractive* two-body potentials would give rise to inclusion particles. The condensation and other thermodynamic properties of such systems could in principle be derived independently, but the statistical interpretation would be a more compelling and generic approach, capturing the essential features of a class of such systems. Interacting cold atoms, the workhouse of low-temperature condensation physics, may offer the most promising possibility.

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